Turkish Journal of Analysis and Number Theory, 2020, Vol. 8, No. 5, 91-96
Available online at http://pubs.sciepub.com/tjant/8/5/2
Published by Science and Education Publishing
DOI:10.12691/tjant-8-5-2

Laplace Differential Transform Method for Solving Nonlinear Nonhomogeneous Partial Differential Equations

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Received September 06, 2020; Revised October 07, 2020; Accepted October 15, 2020

Abstract In this paper, the Laplace Differential Transform Method (LDTM) was utilized to solve some nonlinear nonhomogeneous partial differential equations. This technique is the combined form of the Laplace transform method with the Differential Transform Method (DTM). The combined method is efficient in handling nonlinear nonhomogeneous partial differential equations with variable coefficients. Laplace transform is introduced to overcome the inadequacy resulted from unsatisfied boundary condition in using DTM. Illustrative examples were examined to demonstrate the effectiveness of Laplace differential transform method. Results revealed that the LDTM is well appropriate for use in solving such problems.

Keywords: *Nonhomogeneous PDE, Nonlinear PDE, Laplace Differential Transform Method, Laplace Transform, Differential Transform Method*

Cite This Article: Daniel Deborah O., and Ayodele Moyosola, "Laplace Differential Transform Method for Solving Nonlinear Nonhomogeneous Partial Differential Equations." *Turkish Journal of Analysis and Number Theory*, vol. 8, no. 5 (2020): 91-96. doi: 10.12691/tjant-8-5-2.

1. Introduction

Many natural events in applied sciences and engineering are modeled with linear and nonlinear partial differential equations as a result of the nature of phenomena happening around the world. Although, most of these problems involving linear or nonlinear partial differential equations do not have analytical solution, hence, numerical or approximate methods have efficiently been used to solve these equations.

Pertubation method [1] is well known for solving nonlinear partial differential equations but it is only used based on the existence of small parameter. Hence, many methods were presented to eliminate the small parameter, which include: Adominan Decomposition Method [2,3], Variational Iteration Method [4,5,6,7], Homotopy Perturbation Method [8,9,10], Differential transform method [11,12,13,14], Homotopy Analysis Method [15,16] and Parameter Expansion Method [17,18]. But, most of these methods works fine for small range, owning to the satisfied boundary conditions, and the remaining unsatisfied conditions are left out of the final results.

However, in recent research, Laplace transform is being introduced to overcome inadequacy resulted from unsatisfied conditions.

Laplace Homotopy Perturbation Method (LHPM) [19,20,21], which combines the Laplace transform and Homotopy Pertubation method(HPM), is effectively employed to solve one-dimensional non-homogeneous partial differential equation; Laplace Adominan Decomposition Method (LADM) [22,23,24,25], which combines the Laplace transform and Adomian Decomposition Method (ADM), is used for solving nonlinear Volterra integro-differential equations; Laplace Differential transform method (LDTM) [26], which combines Laplace transform and Differential Transform Method (DTM), is being used to solve linear non-homogeneous partial differential equations with variable coefficient.

Therefore, in this paper, nonlinear nonhomogeneous partial differential equations with variable coefficient are solved using coupled Laplace transform and Differential Transform Method, with the aim to show the effectiveness of this method in proffering reliable solution to nonlinear PDEs.

2. Overview of Laplace Differential Transform Method

In this section, basic idea of Laplace Differential Transform Method (LDTM) is discussed. But, the basic idea of Laplace transform and Differential transform method (DTM) are introduced first before considering the combined form.

2.1. Overview of Laplace Transform

Laplace transform is the integral transformation of a given derivative function with real variable t into a complex function with variable s. It is used to convert differential equations into algebraic equations.

The Laplace transform for a function $f(t)$ is defined in [27]

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt$$

The Laplace transform for a function $f(t)$ exist if the transformation integral exist. Therefore,

$$\int_0^{\infty} |f(t)|e^{-\sigma t}dt < \infty.$$

For some real positive σ. The integral will converge for $\sigma > \alpha$, if $|f(t)|$, the magnitude of $f(t)$ is less than $Me^{\alpha t}$ i.e. $|f(t)| < Me^{\alpha t}$, σ is the abscissa of absolute convergence.

The basic operations of the dimensional transform which are useful in the transformation in this paper are summarized as follows:

Original Function	Transformed Function
$f(t) = t^n, \quad 1,2,3...$	$F(s) = \dfrac{n!}{s^{n+1}}$
$f(t) = \sin at$	$F(s) = \dfrac{a}{s^2 + a^2}$
$f(t) = \cos at$	$F(s) = \dfrac{s}{s^2 + a^2}$
$f(t) = \sinh at$	$F(s) = \dfrac{a}{s^2 - a^2}$
$f(t) = \cosh at$	$F(s) = \dfrac{s}{s^2 - a^2}$.

2.2. Overview of Differential Transform Method (DTM)

The differential transform method construct a semi-analytical numerical techniques that makes use of Taylor series for the solution of differential equations in the form of polynomials. [28].

The differential transformation of the k th derivative of the function $\overline{w}(x,s)$ is defined as follows:

$$W(k) = \frac{1}{k!}\left[\frac{d^k \overline{w}(x,s)}{dx^k}\right]_{x=x_0} \tag{1}$$

Where $\overline{w}(x,s)$ is the original function and $W(k)$ is the transformed function. Differential inverse transformation of $W(k)$ is defined as follows:

$$\overline{w}(x,s) = \sum_{k=0}^{\infty} W(k)(x - x_0)^k. \tag{2}$$

The basic operations of the dimensional transform which are useful in the transformation in this paper are summarized as follows: [28]

Original Function	Transformed Function
$f(x,s) = \dfrac{d^n \overline{w}(x,s)}{dy^n}$	$\overline{F}(k) = \dfrac{(k+n)!}{k!}W(k+n)$
$f(x,s) = \lambda \overline{w}(x,s)$	$\overline{F}(k) = \lambda W(k)$
$f(x) = x^n$	$F(k) = \delta(k-n)$
$f(x,s) = \prod_{i=1}^{n} \overline{w}_i(x,s)$	

$$\overline{F}(k) = \sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_n=0}^{k-l_1\cdots l_{n-1}} W(l_1)\cdots W(l_{n-1}) \cdot W(k - l_1 - \cdots - l_n).$$

2.3. Basic Idea of the Laplace Differential Transform Method (LDTM)

To illustrate the basic idea of this method, we consider the general form of one-dimensional nonlinear second order nonhomogeneous partial differential equations with variable coefficients of the form:

$$\begin{aligned}&\frac{d^2 w(x,t)}{dt^2} + a_n(x)Rw(x,t) + b_n(x)Sw(x,t)\\ &= f(x,t) \qquad t > 0, x > 0, n \in \mathbb{N}\end{aligned} \tag{3}$$

Where $\frac{d^2}{dt^2}$ is the linear differential operator of order 2, $a_n(x)$ and $b_n(x)$ are the variable coefficients, $n \in \mathbb{N}$, R is the linear operator, S is the nonlinear operator and $f(x,t)$ is the source function.

Equation (3) is subject to

$$w(x,0) = g_1(x), \quad w_t(x,0) = g_2(x) \tag{4}$$

$$w(0,t) = h_1(t), \quad w_x(0,t) = h_2(t) \tag{5}$$

The method involves applying a Laplace transform to equation (3) - (5) and the use of the linearity property of Laplace transform

$$\mathcal{L}\left\{\frac{d^2 w(x,t)}{dt^2}\right\} + \mathcal{L}\{a_n(x)Rw(x,t)\} + \mathcal{L}\{b_n(x)Sw(x,t)\} = \mathcal{L}\{f(x,t)\}$$

$$\begin{aligned}&s^2\overline{w}(x,s) - s\overline{w}(x,0) - \overline{w}_t(x,0)\\ &+ a_n(x)R\overline{w}(x,s) + b_n(x)S\overline{w}(x,s) = \overline{f}(x,s)\end{aligned} \tag{6}$$

Where $\overline{w}(x,s)$ is the Laplace transform of $w(x,t)$.

Applying the initial conditions in equation (4) into equation (6) yields

$$\begin{aligned}&s^2\overline{w}(x,s) - sg_1(x) - g_2(x) + a_n(x)R\overline{w}(x,s)\\ &+ b_n(x)S\overline{w}(x,s) = \overline{f}(x,s)\end{aligned} \tag{7}$$

Subject to

$$\overline{w}(0,s) = \overline{h}_1(s), \quad \frac{d\overline{w}(0,s)}{dx} = \overline{h}_2(s) \tag{8}$$

Which is second order initial value problem.

Accordingly, using differential transform method, the solution of equations (7) and (8) can be written as:

$$\overline{w}(x,s)=\sum_{k=0}^{\infty}W(k)x^k \tag{9}$$

Where $W(k)$ is the differential transform of $\overline{w}(x,s)$ and $W(k)$ is a function of the parameter s. After determining $\overline{w}(x,s)$, inverse Laplace transform is applied to equation (9) to get $w(x,t)$.

3. Results and Discussion

Here, the effectiveness of the technique is demonstrated. The advantage of this method is in the ability of applying Laplace transform and differential transform method for obtaining exact solutions of nonlinear nonhomogeneous PDEs. Hence, the effectiveness of the LDTM is demonstrated with the following examples:

3.1. Example 1

Consider the following nonlinear nonhomogeneous equation given by

$$w_t+ww_x=4t+t^3+\frac{9}{4}xt,\ \ t>0,x\in\mathbb{R} \tag{10}$$

Subject to

$$w(x,0)=\frac{3}{2}x \tag{11}$$

$$w(0,t)=2t^2 \tag{12}$$

Applying Laplace transform to equation (10) in view of initial condition (11)

$$s\overline{w}-\frac{3}{2}x+\overline{w}\frac{d\overline{w}}{dx}=\frac{4}{s^2}+\frac{6}{s^4}+\frac{9x}{4s^2} \tag{13}$$

Subject to

$$\overline{w}(0,s)=\frac{4}{s^3} \tag{14}$$

Applying differential transform to (13) - (14) yields

$$\begin{aligned}&sW(k)-\frac{3}{2}\delta(k-1)+\sum_{l=0}k(l+1)W(l+1)W(k-l)\\&=\frac{4}{s^2}\delta k+\frac{6}{s^4}\delta(k)+\frac{9}{4s^2}\delta(k-1):\ \ k\geq 0\end{aligned} \tag{15}$$

and

$$W(0)=\frac{4}{s^3} \tag{16}$$

Using the recurrence equation (15) and the initial condition (16) yields the following outcomes:

$$W(1)=\frac{3}{2s},\qquad W(k)=0 \text{ for } k\geq 2$$

Hence,

$$\overline{w}(x,s)=W(0)+W(1)x$$

$$\overline{w}(x,s)=\frac{4}{s^3}+\frac{3}{2s}x \tag{17}$$

Inverse Laplace transform of (17) gives

$$w(x,t)=2t^2+\frac{3}{2}x$$

Which is the exact solution to (10) - (12)

3.2. Example 2

Consider the following nonlinear nonhomogeneous equation with variable coefficient given by:

$$\begin{aligned}&w_t+xww_x+w_{xx}=2t+2x^2\\&-\frac{1}{3}xt^3+\frac{2}{3}x^5t^3,\ \ t>0,x\in\mathbb{R}\end{aligned} \tag{18}$$

Subject to

$$w(x,0)=-x \tag{19}$$

and

$$w(0,t)=0,\qquad w_x(0,t)=-1 \tag{20}$$

Applying Laplace transform to equation (18) in view of the initial condition (19)

$$s\overline{w}+x+x\overline{w}\frac{d\overline{w}}{dx}+\frac{d^2\overline{w}}{dx^2}=\frac{2}{s^2}+\frac{2x^2}{s}-\frac{2x}{s^4}+\frac{4x^5}{s^4} \tag{21}$$

Subject to

$$\overline{w}(0,s),\qquad \frac{d\overline{w}}{dx}(0,s)=-\frac{1}{s} \tag{22}$$

Applying the differential transform to (21) - (22) yields

$$\begin{aligned}&sW(k)+\delta(k-1)\\&+\sum_{l_1=0}^{k}\sum_{l_2=0}^{l_1}(l_1+1)W(l_1+1)W(l_2)\delta(k-l_1-l_2-1)\\&+(k+1)(k+2)W(k+2)=\frac{2}{s^2}\delta(k)+\frac{2}{s}\delta(k-2)\\&-\frac{2}{s^4}\delta(k-1)+\frac{4}{s^4}\delta(k-5):\qquad k\geq 0\end{aligned} \tag{23}$$

and

$$W(0)=0,\qquad W(1)=-\frac{1}{s} \tag{24}$$

By the recurrence equation (23) and the initial conditions (24), we have the following results

$$W(2)=\frac{2}{s^2},\qquad W(k)=0 \text{ for } k\geq 3$$

Hence,

$$\overline{w}(x,s)=W(1)x+W(2)x^2$$

$$\overline{w}(x,s)=-\frac{1}{s}x+\frac{2}{s^2}x^2 \tag{25}$$

Inverse Laplace transform of equation (25) gives

$$w(x,t) = -x + 2x^2 t$$

Which is the exact solution to (18) - (20).

3.3. Example 3

Consider the following nonlinear nonhomogeneous PDE with variable coefficient:

$$\begin{aligned} w_{tt} + \frac{1}{2}x^2 w w_x + x^2 w_{xx} &= \frac{x^2 t^4}{6} \\ &+ \left(t - \frac{\sqrt{2}}{2}\sin\sqrt{2}t\right)x^4, \qquad t > 0, x > 0 \end{aligned} \tag{26}$$

Subject to

$$w(x,0) = x, \qquad w_t(x,0) = 2x^2 \tag{27}$$

and

$$w(0,t) = 2t^2, \qquad w_x(0,t) = t \tag{28}$$

In lieu of the initial conditions (27), the Laplace transform of (26) gives

$$\begin{aligned} & s^2\overline{w} - sx - 2x^2 + \frac{1}{2}x^2\overline{w}\frac{d\overline{w}}{dx} + x^2\frac{d^2\overline{w}}{dx^2} \\ &= \frac{x^2}{6s^3} + \frac{2x^4}{s^2(s^2+2)} \end{aligned} \tag{29}$$

Subject to

$$\overline{w}(0,s) = \frac{4}{s^3}, \qquad \frac{d\overline{w}}{dx}(0,s) = \frac{1}{s^2} \tag{30}$$

Applying differential transform to (29) - (30) gives

$$\begin{aligned} & s^2 W(k) - s\delta(k-1) - 2\delta(k-2) \\ & + \frac{1}{2}\sum_{l_1=0}^{k}\sum_{l_2=0}^{l_1}(l_1+1)W(l_1+1)W(l_2)\delta(k-l_1-l_2-2) \\ & + \sum_{l=0} kW(l+2)\delta(k-l-2) = \frac{3}{s^2}\delta(k-1) \\ & + \frac{2}{s^2(s^2+2)}\delta(k-4), \qquad k \ge 2 \end{aligned} \tag{31}$$

and

$$W(0) = \frac{4}{s^3}, \qquad W(1) = \frac{1}{s^2} \tag{32}$$

By the recurred equation (31) and the initial condition (32) altogether yield

$$W(2) = \frac{2}{s^2+2}, \qquad W(k) = 0, \text{ for } k \ge 3$$

Hence,

$$\overline{w}(x,s) = W(0) + W(1)x + W(2)x^2$$

$$\overline{w}(x,s) = \frac{4}{s^3} + \frac{1}{s^2}x + \frac{2}{s^2+2}x^2 \tag{33}$$

The inverse Laplace transform of equation (33) yields

$$w(x,t) = 2t^2 + xt + \sqrt{2}x\sin\sqrt{2}t$$

Which is the exact solution to (26) - (28).

3.4. Example 4

Consider the following nonlinear nonhomogeneous equation with variable coefficient:

$$w_{tt} + 2x^2 w w_{xx} = x + x^2\sin\sqrt{2}t \tag{34}$$

Subject to

$$w(x,0) = x, \qquad w_t(x,0) = 8x^2 \tag{35}$$

and

$$w(0,t) = 0, \qquad w_x(0,t) = 1 + \frac{1}{2}t^3 \tag{36}$$

Laplace transform of equation (34) using (35)

$$s^2\overline{w} - sx - 8x^2 + 2x^2\overline{w}\frac{d\overline{w}}{dx} = \frac{x}{s^2} + \frac{x^2}{s^2+2} \tag{37}$$

Subject to

$$\overline{w}(0,s) = 0, \qquad \frac{d\overline{w}(0,s)}{dx} = \frac{1}{s} + \frac{3}{s^4} \tag{38}$$

Applying differential transform to equations (37) - (38) gives

$$\begin{aligned} & s^2 W(k) - s\delta(k-1) - 8\delta(k-2) \\ & + \sum_{l_1=0}^{k}\sum_{l_2=0}^{l_1}(l_1+1)W(l_1+1)W(l_2)\delta(k-l_1-l_2-2) \\ & = \frac{1}{s^2}\delta(k-1) + \frac{1}{s^2+2}\delta(k-2), \qquad k \ge 2 \end{aligned} \tag{39}$$

and

$$W(0) = 0, \qquad W(1) = \frac{1}{s} + \frac{3}{s^4} \tag{40}$$

By the recurrence equation (39) and the initial condition (40), we have

$$W(2) = \frac{1}{(s^2+2)^2} + \frac{8}{s^2+2}, \qquad W(k) = 0, \text{ for } k \ge 3$$

Hence,

$$\overline{w}(x,s) = W(1)x + W(2)x^2$$

$$\overline{w}(x,s) = \left(\frac{1}{s} + \frac{3}{s^4}\right)x + \left(\frac{1}{(s^2+2)^2} + \frac{8}{s^2+2}\right)x^2 \tag{41}$$

The inverse Laplace of equation (41) yields

$$w(x,t) = x + \frac{1}{2}xt^3 + \left(\frac{33\sqrt{2}}{8}\sin\sqrt{2}t - \frac{t}{4}\cos\sqrt{2}t\right)x^2$$

Which is the exact solution to (34) - (36).

3.5. Example 5

Consider the following nonlinear nonhomogeneous equation with variable coefficient

$$w_{tt} + x^3 w^2 - w_{xx} = x^3 \sinh t \tag{42}$$

Subject to

$$w(x,0) = b\cos x + 2\beta x^7, \qquad w_t(x,0) = 0 \tag{43}$$

and

$$w(0,t) = 0, \qquad w_x(0,t) = \beta(\sinh t - t) \tag{44}$$

For $x \in \left[0, \frac{\pi}{2}\right], \beta, b$ are constants.

Laplace transform of equation (42) using equation (43)

$$s^2\overline{w} - bs\cos x + x^3\overline{w}^2 - \frac{d^2\overline{w}}{dx^2} = \frac{x^3}{s^2-1} \tag{45}$$

Subject to

$$\overline{w}(0,s) = 0, \qquad \frac{d\overline{w}}{dx}(0,s) = \beta\left(\frac{1}{s^2-1} - \frac{1}{s^2}\right) \tag{46}$$

Applying the differential transform to (45) - (46) yield

$$\begin{aligned} & s^2 W(k) - CsG(k) \\ & + \sum_{l_1=0}^{k}\sum_{l_2=0}^{l_1} W(l_1)W(l_2)\delta(k-l_1-l_2-3) \\ & -(k+1)(k+2)W(k+2) \\ & = \frac{1}{s^2-1}\delta(k-3), \qquad k \ge 0 \end{aligned} \tag{47}$$

and

$$W(0) = 0, \qquad W(1) = \beta\left(\frac{1}{s^2-1} - \frac{1}{s^2}\right) \tag{48}$$

Where $G(k)$ in (47) is the coefficient of the Taylor series. By the recurrence equation (47), and the initial conditions (48), yields

$$W(2) = 0,\ \ W(3) = -\frac{\beta}{3s^3},\ \ W(5) = \frac{2\beta}{s},\ \ W(k) = 4, 6, \cdots$$

Hence,

$$\overline{w}(x,s) = W(1)x + W(3)x^3 + W(5)x^5$$

$$\overline{w}(x,s) = \beta\left(\frac{1}{s^2-1} - \frac{1}{s^2}\right)x - \frac{\beta}{3s^3}x^3 + \frac{2\beta}{s}x^5 \tag{49}$$

The inverse Laplace of equation (49) is

$$w(x,t) = \beta(\sinh t - t)x - \frac{1}{6}\beta t^2 x^3 + 2\beta x^5$$

4. Conclusion

The applicability of the combined form of Laplace transform method and Differential Transform Method (DTM) is demonstrated in this paper. This method solves nonlinear nonhomogeneous partial differential equations with variable coefficients. Laplace transform is introduced to overcome the inadequacy resulted from unsatisfied boundary condition using DTM, Consequently, the combined methods efficiently gives the exact solution with little computational work compared to differential transform method. Laplace Differential Transform Method (LDTM) can be applied to solve other nonlinear nonhomogeneous partial differential equations.

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