

New Meromorphic Solutions of the Cubic Nonlinear Schrödinger Equation by Using the Complex Method

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Abstract We show abundant simply periodic solutions, trigonometric solutions, hyperbolic function solutions and Weierstrass elliptic solutions of the reduction of the cubic nonlinear Schrödinger equation by the complex method with Painlevé analysis, and some solutions appear to be new. At last, we give some computer simulations to illustrate our main results.

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1. Introduction

In this paper, we consider the cubic nonlinear Schrödinger equation (NLSE)

$$i \frac{\partial w}{\partial t} + a \frac{\partial^2 w}{\partial x^2} - b \frac{\partial^4 w}{\partial x^4} + c |w|^2 w = 0, \quad (1)$$

where $w = w(x, t)$ is a complex envelop amplitude, t is time, x is distance, a, b are velocity dispersions, c is the coefficient of the cubic nonlinearity. Eq.(1) has been discussed in [1,2,3] by using the tanh method and the sine-cosine method, in [4] by using exp-function method, in [5] by several integration tools. In [6], Hong et al. researched on the general perturbed nonlinear Schrödinger equation by using the homotopy perturbation method. In [7], some analytical solutions were derived for the relevant case of $\alpha = 1$ of the time-dependent Schrödinger equation with the Riesz space-fractional derivative.

Recently, W. Yuan et al. [8] studied meromorphic solutions for some nonlinear differential equations based on the complex method and the Nevanlinna value distribution theory. Therefore, it is of interest to know whether the complex method can be applied to NLSEs. By using the traveling wave transformation $w(x, t) = u(x)e^{iat}$ ($i^2 = -1$), Eq.(1) reduces to

$$bu^{(4)} - au'' + \alpha u - cu^3 = 0. \quad (2)$$

We study the meromorphic solutions of Eq.(2) on the complex plane, therefore, let $u := u(z), z \in \mathbb{C}$.

2. Preliminaries

Consider the following algebraic ordinary differential equation

$$P(w, w', \dots, w^{(m)}) = 0, \quad (3)$$

where P is a polynomial in $w(z)$ and its derivatives with constant coefficients.

If there are exactly p distinct formal meromorphic Laurent series

$$w(z) = \sum_{k=-q}^{\infty} c_k z^k \quad (q > 0, c_{-q} \neq 0) \quad (4)$$

satisfies Eq.(3), we say Eq.(3) satisfies $\langle p, q \rangle$ condition [8].

If we only determine p distinct principle parts

$$w(z) = \sum_{k=-q}^{-1} c_k z^k \quad (q > 0, c_{-q} \neq 0), \text{ we say Eq.(3) satisfies}$$

weak $\langle p, q \rangle$ condition.

Weierstrass elliptic function $\wp(z) := \wp(z, g_2, g_3)$ satisfies

$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$ where the invariants $g_2 = 60s_4, g_3 = 140s_6$ and discriminant $\Delta(g_2, g_3) = g_2^3 - 27g_3^2 \neq 0$. Furthermore, $\wp'(-z) = -\wp'(z), 2\wp''(z) = 12\wp^2(z) - g_2, \wp'''(z) = 12\wp(z)\wp'(z), \dots$, any k th derivatives of \wp can be deduced by these identities, and \wp has the Laurent series expansion

$$\wp(z) = \frac{1}{z^2} + \frac{g_2 z^2}{20} + \frac{g_3 z^4}{28} + O(|z|^6).$$

Defined $f(z) \in W$ [9] if $f(z)$ is an elliptic function, or a rational function of $e^{\alpha z}$ ($\alpha \in \mathbb{C}$), or a rational function of z .

Lemma 2.1. [10] Let $p, q, l, m, n \in \mathbb{N}$, $\deg P(w, w', \dots, w^{(m)}) < n$. Suppose that equation (3) satisfies the $\langle p, q \rangle$ condition, then all meromorphic solutions w belong to the class W . Furthermore, each elliptic solution with pole at $z = 0$ can be written as

$$w(z) = \sum_{i=1}^{l-1} \sum_{j=2}^q \frac{(-1)^j c_{-ij}}{(j-1)!} \frac{d^{j-2}}{dz^{j-2}} \left(\frac{1}{4} \left[\frac{\wp'(z) + B_i}{\wp(z) - A_i} \right]^2 - \wp(z) \right) + \sum_{i=1}^{l-1} \frac{c_{-i1}}{2} \frac{\wp'(z) + B_i}{\wp(z) - A_i} + \sum_{j=2}^q \frac{(-1)^j c_{-lj}}{(j-1)!} \frac{d^{j-2}}{dz^{j-2}} \wp(z) + c_0, \tag{5}$$

where c_{-ij} are given by (4), $B_i^2 = 4A_i^3 - g_2 A_i - g_3$ and $\sum_{i=1}^l c_{-i1} = 0, c_0 \in \mathbb{C}$.

Each rational function solution $w := R(z)$ is of the form

$$R(z) = \sum_{i=1}^l \sum_{j=1}^q \frac{c_{ij}}{(z - z_i)^j} + c_0, \tag{6}$$

with $l (\leq p)$ distinct poles of multiplicity q .

Each simply periodic solution is a rational function $R(\xi)$ of $\xi = e^{\alpha z}$ ($\alpha \in \mathbb{C}$). $R(\xi)$ has $l (\leq p)$ distinct poles of multiplicity q , and is of the form

$$R(\xi) = \sum_{i=1}^l \sum_{j=1}^q \frac{c_{ij}}{(\xi - \xi_i)^j} + c_0. \tag{7}$$

Definition 2.2. [10] Let w be a meromorphic solution of a m -th order algebraic differential equation $E(z, w) = 0$. We call the involved term of $E(z, w) = 0$ which determining the multiplicity q in w as the dominant term. The dominant part of $E(z, w) = 0$ is consists of all dominant terms, and is denoted by $\hat{E} = \hat{E}(z, w)$. The multiplicity of pole of each term in $\hat{E}(z, w(z))$ is the same integer denoted by $D(q)$. The multiplicity of pole of each monomial $M_r[z]$ in $E(z, w) - \hat{E}(z, w)$ is denoted by $D_r(q)$.

Definition 2.3. [11] For any meromorphic function v , the derivative operator of dominant part $\hat{E}(z, w(z))$ with respect to w is defined by

$$\hat{E}'(z, w)v := \lim_{\lambda \rightarrow 0} \frac{\hat{E}(z, w + \lambda v) - \hat{E}(z, w)}{\lambda}. \tag{8}$$

The roots of the following equation

$$P(i) = \lim_{\chi \rightarrow 0} \chi^{-i+D(q)} \hat{E}'(\chi, c_{-q} \chi^{-q}) \chi^{i-q} = 0 \tag{9}$$

is called the Fuchs index of $E(z, w) = 0$.

The complex method [8] can be presented in the following five steps:

- 1) Substituting the transform $T : u(x, y, t) \rightarrow w(z), (x, y, t) \rightarrow z$ into a given PDE.
- 2) Substituting (4) into Eq.(3) to determine the (weak) $\langle p, q \rangle$ condition holds.
- 3) By indeterminant relations (5-7) we find the elliptic, rational and simply periodic solutions $w(z)$ of Eq.(3) with pole at $z = 0$, respectively.
- 4) By Lemma 2.1 we obtain all meromorphic solutions $w(z - z_0)$.
- 5) Substituting the inverse transform T^{-1} into these meromorphic solutions $w(z - z_0)$, then we get the exact traveling solutions $u(x, t)$ of the original given PDE.

3. Main Results

Let $u(z)$ be a meromorphic solution of Eq.(2), and suppose that $u(z)$ has a movable pole at $z = 0$, then in a neighborhood of zero, the Laurent series of w is of the form of $\sum_{k=-q}^{\infty} c_k z^k$ ($q > 0, c_{-q} \neq 0$). Substituting this

Laurent series into Eq.(2), then

$$bc_{-q}(-q)(-q-1)(-q-2)(-q-3)z^{-q-4} + \dots - cc_{-q}^3 z^{-3q} = 0,$$

vanishing the coefficients of the lowest power $z^{-q-4} = z^{3q}$, we have $p = 2, q = 2$, and

$$u(z) = \pm(2\sqrt{30} \sqrt{\frac{b}{c}} z^{-2} - \frac{a}{\sqrt{30b}\sqrt{c}} z^2 + \dots).$$

From (2), we know that $\hat{E} = bu^{(4)} - cu^3$, therefore, for any meromorphic function v ,

$$\begin{aligned} & \hat{E}'(z, u)v \\ &= \lim_{\lambda \rightarrow 0} \frac{b(u + \lambda v)^{(4)} - c(u + \lambda v)^3 - (bu^{(4)} - cu^3)}{\lambda} \tag{10} \\ &= (b \frac{\partial^4}{\partial z^4} - 3cu^2)v. \end{aligned}$$

Hence, the Fuchs index equation of Eq.(2) reads

$$\begin{aligned} P(i) &= \lim_{\chi \rightarrow 0} \chi^{-i+6} (b \frac{\partial^4}{\partial z^4} - 3cc_{-2}^2 \chi^{-4}) \chi^{i-2} \\ &= \lim_{\chi \rightarrow 0} \chi^{-i+6} (b \frac{\partial^4}{\partial z^4} \chi^{i-2} - 3cc_{-2}^2 \chi^{i-6}) \\ &= b(i-2)(i-3)(i-4)(i-5) - 3cc_{-2}^2. \end{aligned}$$

It is easily to know that $P(8) = 0$. By the Painlevé analysis [11] we know that there is an arbitrary coefficient

c_i for some i in the Laurent series $\sum_{k=-2}^{\infty} c_k z^k$. Therefore, Eq.(2) satisfies the weak $\langle p, q \rangle = \langle 2, 2 \rangle$ condition, and then we will build meromorphic solutions for Eq.(2) by Lemma 2.1.

By (6), we infer the indetermiant rational solutions of Eq.(2):

$$u_r(z) = \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{z-z_0} + c_0.$$

Substituting $u_r(z)$ into Eq.(2), combining similar terms, then vanishing all coefficients to zero, we build the following rational solutions:

$$u_r(z) = \pm \frac{60b}{\sqrt{30bc}(z-z_0)^2}, \tag{12}$$

where $a = 0$, $\alpha = 0$, and z_0 is an arbitrary number.

Noting that $c_{-1} = 0$, by Lemma 2.1, we infer that the indetermiant of elliptic solution of Eq.(2) with pole at zero:

$$u_e(z) = c_{-2}\wp(z, g_2, g_3) + c_0.$$

Substituting $u_e(z)$ into Eq.(2), combining similar terms, then vanishing all coefficients to zero, then we build the following system of algebraic equations:

$$\begin{cases} 32400b^3g_2 + 180ba^2 - 1800b^2\alpha = 0, \\ -900b^2ag_2 + 21600b^3g_3 - a^3 + 30ba\alpha = 0. \end{cases} \tag{13}$$

Solving (13), we have $g_2 = -\frac{a^2}{180b^2} + \frac{\alpha}{18b}$,

$$g_3 = -\frac{a^3}{5400b^3} + \frac{a\alpha}{1080b^2}.$$

Therefore, we obtain the following elliptic function solutions of Eq.(2) with pole at $z = z_0 \in \mathbb{C}$:

$$u_e(z) = \pm \frac{60b\wp(z-z_0, g_2, g_3) - a}{\sqrt{30bc}}, \tag{14}$$

where

$$\begin{cases} g_2 = -\frac{a^2}{180b^2} + \frac{\alpha}{18b}, \\ g_3 = -\frac{a^3}{5400b^3} + \frac{a\alpha}{1080b^2}, \\ \Delta = -\frac{\begin{pmatrix} 32a^6 - 420a^4\alpha b \\ +2175a^2\alpha^2b^2 - 5000\alpha^3b^3 \end{pmatrix}}{29160000b^6} \neq 0, \end{cases} \tag{15}$$

and $z_0 \in \mathbb{C}$ is an arbitrary number.

Assuming that $\Delta = 0$, we have

$$(4a^2 - 25ab)(8a^4 - 55a^2ab + 200\alpha^2b^2) = 0,$$

then $\alpha = \frac{4a^2}{25b}$, or $\alpha = \frac{(11 \pm 3\sqrt{15}i)a^2}{80b}$. Furthermore, we

have

$$\begin{cases} \alpha = \frac{4a^2}{25b}, \\ g_2 = \frac{a^2}{300b^2}, \\ g_3 = -\frac{a^3}{27000b^3}, \end{cases} \tag{16}$$

then the elliptic solutions (14) can be degenerated to the following trigonometric function solutions:

$$u(z) = \pm \frac{a\sqrt{30}}{10\sqrt{bc}} \csc^2\left(\frac{1}{10}\sqrt{\frac{-5a}{b}}(z-z_0)\right), \tag{17}$$

$$u(z) = \pm \frac{a\sqrt{30}}{10\sqrt{bc}} \sec^2\left(\frac{1}{10}\sqrt{\frac{-5a}{b}}(z-z_0)\right), \tag{18}$$

where $\alpha = \frac{4a^2}{25b}$.

By (7), we infer the indetermiant simply periodic solutions of Eq.(2):

$$u_s(\xi) = \frac{c_{-2}}{(e^{\beta z} - \xi_1)^2} + \frac{c_{-1}}{e^{\beta z} - \xi_1} + c_0 (\xi_1 \in \mathbb{C}).$$

Put $u_s(\xi)$ into Eq.(2), vanishing all coefficients to zero, we build following simply periodic solutions:

$$\begin{cases} u_s(z) = \pm \left(\frac{c_{-2}}{(e^{\beta z} - \xi_1)^2} + \frac{c_{-1}}{e^{\beta z} - \xi_1} + c_0 \right), \\ \alpha = \frac{4a^2}{25b}, \beta = \frac{\sqrt{a}}{\sqrt{5b}}, c_{-2} = 2\sqrt{30bc}\beta^2 e^{2\beta z} / c, \\ c_{-1} = -2\sqrt{30bc}\beta^2 e^{\beta z} / c, c_0 = 0; \\ \alpha = \frac{4a^2}{25b}, \beta = -\frac{\sqrt{a}}{\sqrt{5b}}, c_{-2} = 2\sqrt{30bc}\beta^2 e^{2\beta z} / c, \\ c_{-1} = -2\sqrt{30bc}\beta^2 e^{\beta z} / c, c_0 = 0; \\ \alpha = \frac{4a^2}{25b}, \beta = \frac{\sqrt{a}}{\sqrt{5b}}, c_{-2} = -2\sqrt{30bc}\beta^2 e^{2\beta z} / c, \\ c_{-1} = -2\sqrt{30bc}\beta^2 e^{\beta z} / c, c_0 = 0; \\ \alpha = \frac{4a^2}{25b}, \beta = -\frac{\sqrt{a}}{\sqrt{5b}}, c_{-2} = -2\sqrt{30bc}\beta^2 e^{2\beta z} / c, \\ c_{-1} = -2\sqrt{30bc}\beta^2 e^{\beta z} / c, c_0 = 0; \\ a \neq 0, \xi_1 \in \mathbb{C}. \end{cases} \tag{19}$$

If $a = 0$, then $\beta = 0$, (19) will be reduce to a constant.

Furthermore, we investigate the special cases of (19) with poles at $z = 0 (\xi = 0)$ and $z = \frac{i\pi}{\alpha} (\xi = -1)$. By $u_s(z)$

and (19), we can obtain the following hyperbolic function solutions:

$$u(z) = \frac{2\sqrt{30}a}{5\sqrt{bc}} \frac{e^{\frac{\sqrt{a}}{\sqrt{5b}}z}}{(e^{\frac{\sqrt{a}}{\sqrt{5b}}z} - 1)^2} \tag{20}$$

$$= \frac{1}{10} \frac{\sqrt{30}a}{\sqrt{bc}} \operatorname{csch}^2\left(\frac{1}{2} \frac{\sqrt{a}}{\sqrt{5b}} z\right),$$

$$u(z) = \frac{2\sqrt{30}a}{5\sqrt{bc}} \frac{e^{\frac{\sqrt{a}}{\sqrt{5b}}z}}{(e^{\frac{\sqrt{a}}{\sqrt{5b}}z} + 1)^2} \tag{21}$$

$$= \frac{1}{10} \frac{\sqrt{30}a}{\sqrt{bc}} \operatorname{sech}^2\left(\frac{1}{2} \frac{\sqrt{a}}{\sqrt{5b}} z\right),$$

$$u(z) = \frac{2\sqrt{30}a}{5\sqrt{bc}} \frac{e^{-\frac{\sqrt{a}}{\sqrt{5b}}z}}{(e^{-\frac{\sqrt{a}}{\sqrt{5b}}z} - 1)^2} \tag{22}$$

$$= \frac{1}{10} \frac{\sqrt{30}a}{\sqrt{bc}} \operatorname{csch}^2\left(-\frac{1}{2} \frac{\sqrt{a}}{\sqrt{5b}} z\right),$$

$$u(z) = \frac{2\sqrt{30}a}{5\sqrt{bc}} \frac{e^{-\frac{\sqrt{a}}{\sqrt{5b}}z}}{(e^{-\frac{\sqrt{a}}{\sqrt{5b}}z} + 1)^2} \tag{23}$$

$$= \frac{1}{10} \frac{\sqrt{30}a}{\sqrt{bc}} \operatorname{sech}^2\left(-\frac{1}{2} \frac{\sqrt{a}}{\sqrt{5b}} z\right),$$

where $\alpha = \frac{4a^2}{25b}$, and z can be replaced by $z - z_0$, $z_0 \in \mathbb{C}$ is an arbitrary number.

It follows easily that the solitons (22) (23) can be reduced to real functions with initial condition $ab < 0$ and $z = x \in \mathbb{R}$ as follows:

$$u(z) = \frac{1}{10} \frac{\sqrt{30}a}{\sqrt{bc}} \operatorname{csc}^2\left(\frac{1}{2} \sqrt{-\frac{a}{5b}} x\right), \tag{24}$$

$$u(z) = \frac{1}{10} \frac{\sqrt{30}a}{\sqrt{bc}} \operatorname{sec}^2\left(\frac{1}{2} \sqrt{-\frac{a}{5b}} x\right). \tag{25}$$

Noting that Eq.(2), if $u(z)$ is a solution, then $-u(z)$ is also.

Remark 1. $w(x,t) = u(x)e^{iat}$ are the traveling wave solutions satisfy Eq.(1). The solutions (20)-(25) are also appear in [2] and [3].p690, and solutions (12) (14) (16) (17) (18) (19) are appear to be new comparing to [1,2] and other open literatures.

4. Computer Simulations

In this section, we give some computer simulations to illustrate the main results. If b, c are fixed, the amplitude

and the width of the solitons (21) (25) can be manipulated using the dispersion coefficients a . We fix $b = 1, c = 1, x \in [-8, 8]$ in the solitons (21) (25), see Figure 1 and Figure 2 for the values of $a = 1, a = 5, a = 10, a = 15$ for bright solitons and $a = -1, a = -5, a = -10, a = -15$ for dark solitons. If bc is big the amplitude are small, and if b is small the solitons are thin, while increasing the dispersion coefficient a , their widths are also increasing if the solitons spread. The figure of Weierstrass function solution (14) be shown as Figure 3 for the values of $a = 1, b = 1, c = 1$ and $\alpha = 1, \alpha = 5, \alpha = 10$.

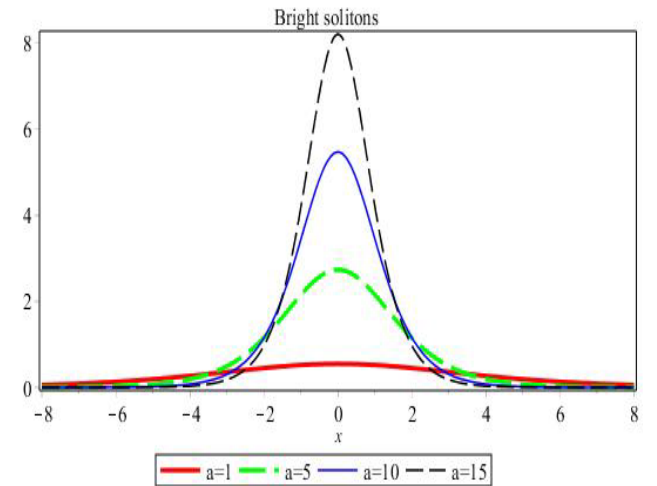


Figure 1. Bright solitons for Eq.(2) with $b = 1, c = 1$ (continuous curves) when solitons propagate

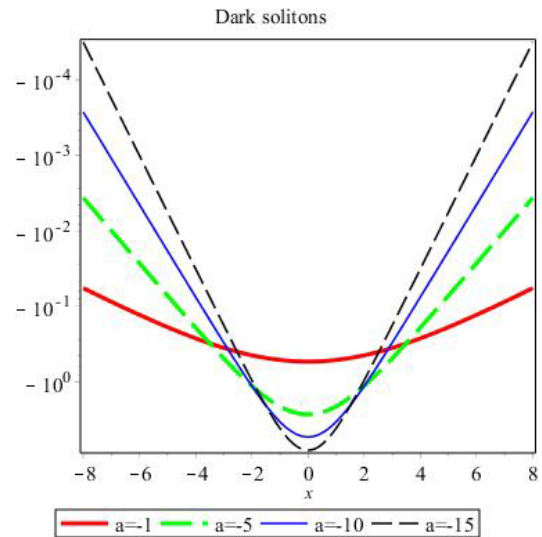


Figure 2. Dark solitons for Eq.(2) with $b = 1, c = 1$ (continuous curves) when solitons propagate.

5. Conclusions

This work is a successive application of the complex method for constructing exact solutions on the cubic nonlinear Schrödinger equation, elliptic function solutions, simply periodic solutions, trigonometric function solutions and hyperbolic function solutions were investigated. The results show that the complex method is a powerful and systematic tool for constructing meromorphic solutions

for some certain complex ordinary differential equations, especially for solitons and periodic solutions.

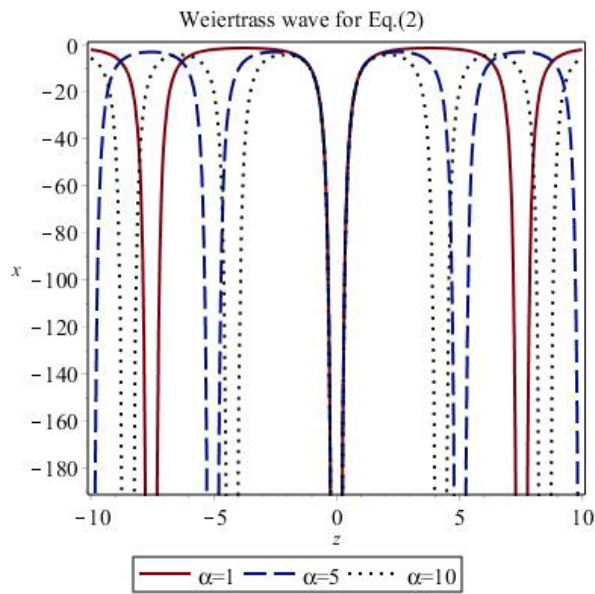


Figure 3. 2D plot of Weierstrass function solution (14) for Eq.(2) with $a = 1, b = 1, c = 1$ (continuous curves) when wave propagate

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