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An Elementary Proof of the Twin Prime Conjecture

B. Gensel*

Berndt Gensel, Carinthia University of Applied Sciences, Austria
*Corresponding author: b.gensel@fh-kaernten.at

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Abstract It is well known that every prime number $p \geq 5$ has the form $6k-1$ or $6k+1$. We will call k the **generator** of p. Twin primes are distinghuished due to a **common generator** for each pair. Therefore it makes sense to search for the Twin Primes on the level of their generators. This paper present a new approach to prove the **Twin Prime Conjecture** by a sieve method to extract all Twin Primes on the level of the Twin Prime Generators. We define the ω_{p_n}--numbers x as numbers for which holds that $6x-1$ and $6x+1$ are coprime to the prime p_n. By dint of the average distance $\bar{\delta}(p_n)$ between the ω_{p_n}--numbers we can prove the **Twin Prime Conjecture** indirectly.

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Notations

We will use the following notations:
$\mathbb{N}$ the set of the positve integers,
$\mathbb{P}$ the set of the primes, $\mathbb{P}^*$ primes ≥ 5,
$\mathbb{P}_- = \{p \in \mathbb{P}^* | \ p \equiv -1 \pmod 6\}$
and $\mathbb{E}_- = \{n \in \mathbb{N} | 6n-1 \in \mathbb{P}_-\}$
$\mathbb{P}_+ = \{p \in \mathbb{P}^* | \ p \equiv +1 \pmod 6\}$
and $\mathbb{E}_+ = \{n \in \mathbb{N} | 6n+1 \in \mathbb{P}_+\}$
and $\mathbb{E} = \mathbb{E}_- \cap \mathbb{E}_+$.

1. Introduction

The question on the infinity of the twin primes keeps busy many mathematicians for a long time. 1919 V. Brun [3] had proved that the series of the inverted twin primes converges while he had tried to prove the Twin Prime Conjecture. Several authors worked on bounds for the length of prime gaps (see f.i. [4,5,6]). 2014 Y. Zhang [7] obtained a great attention with his proof that there are infinitely many consecutive primes with a distance of 70,000,000 at most. With the project "PolyMath8" this bound could be lessened down to 246 respectively to 12 assuming the validity of the Elliott-Halberstam Conjecture [8].

We present in this paper another approach as in the most works on this topic. We tranfer the looking for twin primes to the level of their generators because each twin prime has a common generator.

2. Twin Prime Generators

It is well known that every prime number $p \geq 5$ has the form $6k-1$ or $6k+1$. We will call k the **generator** of p. Twin primes are distinghuished due to a common generator for each pair. Therefore it makes sense to search for the twin primes on the level of their generators.

Let be

$$\kappa(p) = \begin{cases} \dfrac{p+1}{6} & \text{for } p \in \mathbb{P}_- \\ \dfrac{p-1}{6} & \text{for } p \in \mathbb{P}_+ \end{cases} \tag{2.1}$$

an over $\mathbb{P}^*$ defined function, the **generator** of the pair $(6\kappa(p)-1, 6\kappa(p)+1)$.

A number x is a member of $\mathbb{E}$ if $6x-1$ as well as $6x+1$ are primes. This is true if the following statement holds.

Theorem 1. *A number x is a member of $\mathbb{E}$ if and only if there is **no** $p \in \mathbb{P}^*$ with $p < 6x-1$ where one of the following congruences holds*:

$$x \equiv -\kappa(p) \pmod p \tag{2.2}$$

$$x \equiv +\kappa(p) \pmod p \tag{2.3}$$

Proof.

A. $p \in \mathbb{P}_-$, therefore is $p = 6\kappa(p)-1$:

If (2.2) is true then there is an $n \in \mathbb{N}$ with

$$
\begin{aligned}
&x = -\kappa(p) + n\cdot(6\kappa(p)-1)\\
&6x = -6\kappa(p) + 6n\cdot(6\kappa(p)-1)\\
&6x+1 = -6\kappa(p) + 6n\cdot(6\kappa(p)-1)+1\\
&\quad = (6n-1)(6\kappa(p)-1)\\
&\Rightarrow 6x+1 \equiv 0 \pmod p \Rightarrow x \notin \mathbb{E}.
\end{aligned}
$$

For (2.3) the proof will be done with $6x-1$:

$$
\begin{aligned}
6x-1 &= 6\kappa(p) + 6n\cdot(6\kappa(p)-1)-1\\
&= (6n+1)(6\kappa(p)-1)\\
&\Rightarrow 6x-1 \equiv 0 \pmod p \Rightarrow x \notin \mathbb{E}.
\end{aligned}
$$

B. $p \in \mathbb{P}_+$, therefore is $p = 6\kappa(p)+1$: We go the same way with (2.2) and $6x-1$ as well as (2.3) and $6x+1$:

$$
\begin{aligned}
&6x-1 = (6n-1)(6\kappa(p)+1) \Rightarrow 6x-1 \equiv 0 \pmod p\\
&6x+1 = (6n+1)(6\kappa(p)+1) \Rightarrow 6x+1 \equiv 0 \pmod p.
\end{aligned}
$$

With these it's shown that $x \notin \mathbb{E}$ if the congruences (2.2) or (2.3) are valid. They cannot be true both because they exclude each other.

If on the other hand $x \notin \mathbb{E}$, then is $6x-1$ or $6x+1$ no prime. Let be $6x-1 \equiv 0 \pmod p$ and $p \in \mathbb{P}_-$. Then we have

$$
\begin{aligned}
&6x-1 \equiv p \pmod p \equiv (6\kappa(p)-1) \pmod p\\
&6x \equiv 6\kappa(p) \pmod p\\
&x \equiv \kappa(p) \pmod p.
\end{aligned}
$$

For $p \in \mathbb{P}_+$ we have

$$
\begin{aligned}
&6x-1 \equiv -p \pmod p \equiv -(6\kappa(p)+1) \pmod p\\
&6x \equiv -6\kappa(p) \pmod p\\
&x \equiv -\kappa(p) \pmod p.
\end{aligned}
$$

The other both cases we can handle in the same way. Therefore either (2.2) or (2.3) is valid if $x \notin \mathbb{E}$.

If we consider that the least proper divisor of a number $6x-1$ or $6x+1$ is less or equal to $\sqrt{6x+1}$ than p in the congruences (2.2) and (2.3) can be further limited by

$$\hat{p}(x) = \max(p \in \mathbb{P}^* \mid p \le \sqrt{6x+1}).$$

Henceforth we will use the letter p for a general prime number and p_n if we describe an element of a sequence of primes. With p_n as the n-the prime number[1] and $\pi(z)$ as the number of primes $\le z$ we have with

$$
\begin{aligned}
&x \equiv -\kappa(p_n) \pmod{p_n}\\
&or \qquad\qquad (2.4)\\
&x \equiv +\kappa(p_n) \pmod{p_n}
\end{aligned}
$$

for $3 \le n \le \pi(\hat{p}(x))$ a proofable system of criteria to exclude a number $x \ge 4$ as not being a member of $\mathbb{E}$.

[1] It is $p_1 = 2$.

3. The Twin Sieve

The congruences in (2.4) can be combined in the following way:

$$x^2 \equiv \kappa(p_n)^2 \pmod{p_n} \text{ for } 3 \le n \le \pi(\hat{p}(x)) \qquad (3.1)$$

because if $x \equiv \pm\kappa(p_n) \pmod{p_n}$ then there is a number t with $x = \pm\kappa(p_n) + tp_n$. Squared this produces $x^2 = \kappa(p_n)^2 + p_n(t^2 p_n \pm 2t\kappa(p_n))$ and we get $x^2 \equiv \kappa(p_n)^2 \pmod{p_n}$. This results in a system of sieves with sieve functions $\psi(x, p_n)$ for which hold for $3 \le n \le \pi(\hat{p}(x))$

$$
\begin{aligned}
&x^2 - \kappa(p_n)^2 \equiv \psi(x, p_n) \pmod{p_n} \text{ respectively}\\
&\psi(x, p_n) = \left(x^2 - \kappa(p_n)^2\right) \mathrm{Mod} p_n.
\end{aligned} \qquad (3.2)
$$

Obviously $\psi(x, p)$ is a periodical function in x with a period length of p. We'll call the sieve represented by $\psi(x, p_n)$ as S_n. For the system of the sieves $S_3 \times ... \times S_n$ we'll build the *aggregate* sieve functions

$$
\begin{aligned}
&\Psi(x, p_n) = \prod_{i=3}^{n} \frac{\psi(x, p_i)}{p_i}\\
&and \qquad\qquad (3.3)\\
&\hat{\Psi}(x) = \Psi(x, \hat{p}(x)).
\end{aligned}
$$

Because the value set of $\psi(x, p)$ consists of positive integers between 0 and $p-1$, $\Psi(x, p)$ and $\hat{\Psi}(x)$ have rational values between 0 and <1.

A number x will be “sieved” by S_n if and only if $\psi(x, p_n) = 0$. With (3.3) in this case also is $\Psi(x, p_n) = 0$. In contrast to the sieve of ERATOSTHENES in our sieve the exclusion of a number x will be not controlled by $x \text{ Mod } p = 0$, but by $\left(x^2 - \kappa(p)^2\right) \text{Mod } p = 0$.

Let be

$$O_n = \min(x \in \mathbb{N} \mid \hat{p}(x) = p_n). \qquad (3.4)$$

For $x \ge O_n$ “works" the sieve S_n, i.e. O_n is the **origin** of the sieve S_n. Every sieve has up from O_n in every ψ --period just $p_n - 2$ positions with $\psi(x, p_n) \ne 0$ and two positions with $\psi(x, p_n) = 0$, once if (2.2) and on the other hand if (2.3) is valid. We speak about a - and b -bars of the sieve S_n. From (2.2) and (2.3) it is easy to see that the distance between an a - and a b -bar is $2\kappa(p_n)$.

It is $p_n \le \hat{p}(x) \le \sqrt{6x+1}$ and therefore $p_n^2 \le 6x+1$. Then

$$O_n = \frac{p_n^2 - 1}{6} \qquad (3.5)$$

is the least number which meets this relation. It is easy to prove that for every prime p holds that p^2-1 is an integer divisible by 6.

Theorem 2. *Every sieve S_n with $n \geq 3$ starts at its origin O_n with a sieve bar and we have $\psi(O_n, p_n) = 0$.*

Proof. We substitute p_n by $6\kappa(p_n) \pm 1$. With this and (3.5) holds

$$\begin{aligned} O_n &= \frac{(6\kappa(p_n)\pm 1)^2 - 1}{6} \\ &= \frac{6\kappa(p_n)\left(6\kappa(p_n)\pm 2\right)}{6} \\ &= \kappa(p_n)\left(6\kappa(p_n)\pm 1\right) \pm \kappa(p_n) = \kappa(p_n)\cdot p_n \pm \kappa(p_n) \\ &\equiv \pm\kappa(p_n)\left(\mathrm{mod} p_n\right) \to \psi(O_n, p_n) = 0. \end{aligned}$$

We see that S_n starts for $p_n \in \mathbb{P}_-$ with an a-bar (2.2) and in the other case with a b-bar(2.3).

For every $x \geq O_n$ the local position in the sieve S_n relative to the phase start [2] can be determined by the *position function* $\tau(x, p_n)$:

$$\begin{aligned} &x + \kappa(p_n) \equiv \tau(x, p_n)\left(\mathrm{mod}\ p_n\right) \text{ respectively} \\ &\tau(x, p_n) = \left(x + \kappa(p_n)\right)\mathrm{Mod}\ p_n. \end{aligned} \tag{3.6}$$

Between the sieve function $\psi(x, p)$ and the position function $\tau(x, p)$ there is the following relationship:

$$\begin{aligned} \psi(x, p) &= \tau(x, p)\cdot\left(x - \kappa(p)\right)\mathrm{Mod}\ p \\ &= \tau(x, p)\cdot\left(\tau(x, p) - 2\kappa(p)\right)\mathrm{Mod}\ p. \end{aligned} \tag{3.7}$$

Obviously is $\psi(x, p) = 0$ if and only if $\tau(x, p) = 0$ (a-bar) or $\tau(x, p) = 2\kappa(p)$ (b-bar).

4. The Permeability of the Sieves $S_3 \times \ldots \times S_n$

For every x in the interval

$$\mathcal{A}_n := \left[O_n, O_{n+1} - 1\right] \tag{4.1}$$

$\hat{p}(x)$ persists constant on the value p_n. The length of this interval [3] will be denoted as d_n. It is depending on the distance between successive primes. Since they can only be even, we have with $a = 2,4,6,\ldots$

$$\begin{aligned} d_n &= \frac{(p_n + a)^2 - 1}{6} - \frac{p_n^2 - 1}{6} = \frac{2ap_n + a^2}{6} \\ &= \frac{a}{3}(p_n + \frac{a}{2}) \geq \frac{2}{3}(p_n + 1). \end{aligned} \tag{4.2}$$

On the other hand it results because of $p_{n+1} < 2p_n$ ([2], p. 188)

$$\begin{aligned} d_n &= \frac{p_{n+1}^2 - 1}{6} - \frac{p_n^2 - 1}{6} = \frac{p_{n+1}^2 - p_n^2}{6} \\ &= \frac{(p_{n+1} + p_n)(p_{n+1} - p_n)}{6} < \frac{3p_n \cdot p_n}{6} = \frac{p_n^2}{2}. \end{aligned} \tag{4.3}$$

The congruences from (3.6)

$$x + \kappa(p_i) \equiv \tau(x, p_i)\left(\mathrm{mod}\ p_i\right), 3 \leq i \leq n. \tag{4.4}$$

meet the requirements of the Chinese Remainder Theorem (see [1], p. 89). Therefore it is modulo $5\cdot 7\cdot\ldots\cdot p_n$ uniquely resolvable. With

$$p_n\#_5 := \prod_{i=3}^{n} p_i = 5\cdot 7\cdot\ldots\cdot p_n \tag{4.5}$$

it's $\left(\mathrm{mod}\ p_n\#_5\right)$ [4] uniquely resolvable. Therefore the sieves $S_3 \times \ldots \times S_n$ have the period length $p_n\#_5$ and for the aggregate sieve function holds:

$$\Psi(x + a\cdot p_n\#_5, p_n) = \Psi(x, p_n) \mid a \in \mathbb{N}.$$

Definition 4.1. A positive integer x will be called an "ω_p--number" if both $6x-1$ and $6x+1$ are coprime [5] to p. In this case is $\Psi(x, p) > 0$.

Let be

$$\mathcal{P}_n := \left[O_n, O_n + p_n\#_5 - 1\right]$$

the interval of the period of the sieves $S_3 \times \ldots \times S_n$. We'll denote it henceforth as **period section**. Evidently is $\mathcal{A}_n \subset \mathcal{P}_n$ for all $n \geq 3$.

The values of the function $\tau(x, p)$ are the numbers $0,1,\ldots,p-1$. Two of them result in the exlcuding of x and $p-2$ don't. Therefore by working of the sieves $S_3 \times \ldots \times S_n$ we have

$$\varphi(p_n) = \prod_{i=3}^{n}(p_i - 2) \tag{4.6}$$

ω_{p_n}--numbers in $\mathcal{P}_n$. If a lot of them are in $\mathcal{A}_n$, they are members of $\mathbb{E}$ because the sieves $S_3 \times \ldots \times S_n$ here are working only. The relation between (4.6) and the period length of (4.4) results in

$$\eta(p_n) = \frac{\varphi(p_n)}{p_n\#_5} = \prod_{i=3}^{n}\frac{p_i - 2}{p_i}, \tag{4.7}$$

as a measure of the mean "permeability" of working of the sieves $S_3 \times \ldots \times S_n$ or as the density of the ω_{p_n}--numbers in $\mathcal{P}_n$. Obviously $\eta(p)$ is a strong monotonously decreasing function. Its inversion

$$\bar{\delta}(p) = \frac{1}{\eta(p)} \tag{4.8}$$

---

[2] For $p_n \in \mathbb{P}_-$ the phase start is O_n and else it is $O_n - 2\kappa(p_n)$.

[3] Really is $\mathcal{A}_n := \left[O_n, O_{n+1} - 1\right] \cap \mathbb{N}$. Henceforth all intervals will be defined as sections of the number line.

[4] It is $p_n\#_5 = \dfrac{p_n\#}{6}$, with the primorial $p_n\#$

[5] Then is $\gcd(36x^2 - 1, p\#_5) = 1$.

discribes the **average distance** between the ω_p--numbers in their period section.

Theorem 3. *The density of the* ω_p *--numbers in their period section is lower bounded by*

$$\eta(p) > \frac{3}{p} \text{ for } p \in \mathbb{P}^*.$$

Proof. Let be $\mathbb{Q}_p = \{q \in \mathbb{P}^* | \ q \le p\}$ and

$$\mathbb{U}_p = \{u \equiv 1(\mathrm{mod}\, 2) | 5 \le u \le p\}.$$

Because all primes > 2 are odd numbers it holds $\mathbb{Q}_p \subset \mathbb{U}_p$ for $p > 7$ [6]. All factors of $\eta(p)$ are less than 1. It results

$$\eta(p) > \prod_{u \in \mathbb{U}}^{p} \frac{u-2}{u} = \frac{3}{5} \cdot \frac{5}{7} \cdot \frac{7}{9} \cdot \ldots \cdot \frac{p-4}{p-2} \cdot \frac{p-2}{p} = \frac{3}{p}.$$

By inversion of this relationship, we obtain for the average distance

$$\bar{\delta}(p) < \frac{p}{3}. \qquad (4.9)$$

Under consideration of (4.2) we obtain furthermore[7]

$$2\bar{\delta}(p_n) < \frac{2p_n}{3} < \frac{2}{3}(p_n + 1) \le d_n \to \bar{\delta}(p_n) < \frac{d_n}{2}. \quad (4.10)$$

This means that the avarage distance between ω_{p_n} -numbers remains ever less than the half of the length of $\mathcal{A}_n$, the interval where ω_{p_n} -numbers are twin prime generators.

5. The Sieve Process on Average

The intervals $\mathcal{A}_n, n \ge 3$ defined by (4.1) cover the positive integers ≥ 4 gapless and densely. It is

$$\mathbb{N} = \{1,2,3\} \cup \bigcup_{n=3}^{\infty} \mathcal{A}_n \text{ and } \bigcap_{n=3}^{\infty} \mathcal{A}_n = \varnothing.$$

They are the beginnings of the period sections $\mathcal{P}_n$ of the ω_{p_n} --numbers. Hereafter let's say **A**-sections to the intervals $\mathcal{A}_n$. Every ω_{p_n} --number which lies in an A-section is a twin prime generator (see above). In contrast to the A-sections the period sections $\mathcal{P}_n$ overlap each other very densely. So the period section $\mathcal{P}_9$ reachs over 1739 A-sections up to the beginning of the period section $\mathcal{P}_{1748}$ and the next $\mathcal{P}_{10}$ over 7863 A-sections up to the beginning of $\mathcal{P}_{7873}$.

Theorem 4. *Each origin* O_n *cannot be located at the beginning* $O_m + a \cdot p_m\#_5 | \ a \in \mathbb{N}$ *of any period of the sieves* $S_3 \times \ldots \times S_m$ *for* $m < n$. *Therefore it holds for* $m < n$

$$O_n \not\equiv O_m \left(\mathrm{mod}\ p_m\#_5\right).$$

Proof. The equation

$$\frac{p_m^2}{6} + a \cdot p_m \#_5 = \frac{p_n^2 - 1}{6} \text{ and thus } p_m^2 + a \cdot p_m \# = p_n^2$$

is for no primes $p_m < p_n$ solvable, because of

$$\gcd(p_m, p_n) = 1.$$

Vice versa holds that every period section $\mathcal{P}_{n+1}$ starts always inside of the previous period section $\mathcal{P}_n$ nearby to its origin because (see (4.3) also)

$$O_{n+1} = O_n + d_n \ and\ d_n < \frac{p_n^2}{2} \ll \frac{p_m\#_5}{2}.$$

Let be

$$\mathcal{P}_n^k := \left[O_n + k \cdot p_n \#_5, O_n + (k+1) \cdot p_n \#_5 - 1\right]$$

$$and\ \mathcal{A}_n^+ := O_n + p_{n+1} \#_5, O_{n+1} - 1 + p_{n+1} \#_5.$$

With these we can show the recursive structure of the period sections

$$\mathcal{P}_{n+1} = \mathcal{P}_n \setminus \mathcal{A}_n \cup \left(\bigcup_{k=1}^{p_{n+1}-1} \mathcal{P}_n^k \right) \cup \mathcal{A}_n^+. \qquad (5.1)$$

We can clearly see that the period section $\mathcal{P}_n$ overlaps $\mathcal{P}_{n+1}$ up to the end of $\mathcal{P}_n \setminus \mathcal{A}_n$ and $\mathcal{P}_{n+1}$ has much space for A-sections $\mathcal{A}_t$ with $t \ge n+1$.

The one consequence of this dense overlapping of the period sections is that a plurality of the ω_{p_m} --gaps from the period section $\mathcal{P}_m$ persist constant as also ω_{p_n} --gaps for $n > m$ but in a shifted position relative to their origin O_n (see Theorem 4 and (5.1)).

On the other hand this dense overlapping guarantees that extreme anomalies of the distribution of the ω_{p_n} --numbers cannot occur.

For the quantity of the ω_{p_n} --numbers in $\mathcal{P}_n$ is corresponding with (4.6)

$$\varphi(p_n) = \varphi(p_{n-1}) \cdot (p_n - 2).$$

In $\mathcal{P}_n$ the $\varphi(p_n)$ ω_{p_n} --numbers are spread [8] over $p_n\#_5$. positions. According to (5.1) we have $p_{n+1} \cdot \varphi(p_n)$ ω_{p_n} --numbers in $\mathcal{P}_{n+1}$. In comparison

[6] For $p \le 7$ is $\mathbb{Q}_p = \mathbb{U}_p$

[7] We can even prove that $\bar{\delta}(p)^2 < p$ and $\bar{\delta}(p_n)^2 < d_n$ for $p_n > 200 (n > 45)$.

[8] It is easy to prove that the ω_p--numbers in their period section are symmetrically distributed around $\frac{p\#_5}{2}$ and $p\#_5$. Nevertheless the distribution is non-uniform.

between them and the $\omega_{p_{n+1}}$ --numbers resulting from the working of the sieve S_{n+1} we see

$$\begin{aligned} & p_{n+1}\cdot\varphi(p_n)-\varphi(p_{n+1}) \\ & = p_{n+1}\cdot\varphi(p_n)-(p_{n+1}-2)\cdot\varphi(p_n)=2\varphi(p_n). \end{aligned} \quad (5.2)$$

We loose by the working of S_{n+1} in the period section $\mathcal{P}_{n+1}$ just $2\varphi(p_n)$ potential generators of twin primes. In other words, the sieve S_{n+1} has $2\varphi(p_n)$ \`beating bars" in $\mathcal{P}_{n+1}$. At these positions x holds

$$\Psi(x,p_n)>0 \text{ and } \psi(x,p_{n+1})=0. \quad (5.3)$$

Only the **beating bars** let grow the gaps by exclusion of the ω_{p_n} --number between **two** ω_{p_n} --gaps to **one** $\omega_{p_{n+1}}$ --gap. By the working of the sieve S_{n+1} we obtain the following sieve balance "*on average*":

The distances between the ω_{p_n} --numbers persist unchanged at $\bar{\delta}(p_n)$ *on average* except of those $2\varphi(p_n)$ ω_{p_n} --numbers which are met by the beating bars of the sieve S_{n+1}. Thereby a distance D occurs between the adjacent $\omega_{p_{n+1}}$ --numbers on average:

$$\begin{aligned} p_{n+1}\#_5 &= \text{changed } + \text{unchanged} \\ &= 2\varphi(p_n)\cdot D+\left(\varphi(p_{n+1})-2\varphi(p_n)\right)\cdot\bar{\delta}(p_n) \\ &= 2\varphi(p_n)\cdot D+\begin{pmatrix}\varphi(p_n)(p_{n+1}-2)\\ -2\varphi(p_n)\end{pmatrix}\cdot\frac{p_n\#_5}{\varphi(p_n)} \\ &= 2\varphi(p_n)\cdot D+p_{n+1}\#_5-4p_n\#_5 \end{aligned}$$

and therefore

$$0=2\varphi(p_n)\cdot D-4p_n\#_5$$

and hence

$$D=\frac{2p_n\#_5}{\varphi(p_n)}=2\bar{\delta}(p_n).$$

Therefore even the gaps between the $\omega_{p_{n+1}}$ --numbers ($\omega_{p_{n+1}}$ --gaps) which result from the beating bars persist less than d_{n+1} on average because

$$D=2\bar{\delta}(p_n)<\frac{2p_n}{3}<\frac{2p_{n+1}}{3}<\frac{2}{3}(p_{n+1}+1)\le d_{n+1}. \quad (5.4)$$

6. Proof of the Twin Prime Conjecture

The proof will be done indirectly. We assume that there is only a finite number of twin primes and therefore there is only a finite number of twin prime generators. Let be y_o the greatest one. It lies in the A-section $\mathcal{A}_{n_o}$ with $n_o=\pi\left(\hat{p}(y_o)\right)$, the beginning of the period section $\mathcal{P}_{n_o}$. In the subsequent A-sections $\mathcal{A}_t$ with $t>n_o$ consequently there cannot be any twin prime generators and therefore no ω_{p_t} --numbers. But then we have ω_{p_t} --gaps with lengths $>d_t$ in **all** (infinitely many) period sections $\mathcal{P}_t$ for $t>n_o$.

Because

- all period sections $\mathcal{P}_t$ are very densely overlapped and therefore extreme anomalies of the distribution of the ω_{p_t} --numbers cannot occur,
- the average distances between the ω_{p_t} --numbers are **less** than $\frac{d_t}{2}$,
- and even the ω_{p_t} --gaps which are generated by beating bars of the sieves S_t are **less** than d_t on average,

therefore it is not possible to have for **all** $t>n_o$ only period sections $\mathcal{P}_t$ with ω_{p_t} --gaps at their beginnings which are all **greater** than d_t.

Therefore the proof assumtion cannot be valid and thus the Twin Prime Conjecture must be true.

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