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The Collatz Conjecture and Linear Indefinite Equation

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Abstract For the collatz conjecture, we define an iterative formula of odd integers according to the basic theorem of arithmetic, and give the concept of iterative exponent. On this basis, a continuous iterative general formula for odd numbers is derived. With the formula, the equation of cyclic iteration is deduced and get the result of the equation without a positive integer solution except 1. On the other hand, the general formula can be converted to linear indefinite equation. The solution process of this equation reveals that odd numbers are impossible to tend to infinity through iterative operations. Extending the result to even numbers, it can be determined that all positive integers can return 1 by a limited number iterations.

Mathematics Subject Classification (2000): 11T30, 26A18

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1. Introduction

The Collatz Conjecture is an unsolved problem in number theory at present. Its summary is that to take a integer greater than 1, if it is even, divide it by 2, if it is odd, multiply it by 3 and plus 1. With such a transformation, we get a new positive integer. Repeating such transformations (iterations) will produce a string of natural numbers and eventually get 1.

As of 2017, the computer verified all natural numbers less than 2^{64}. Without exception, all test values greater than 1 eventually return to 1.

People speculate that whether all natural numbers have such a property [1,2,3,4].

The conjecture has also been called the 3*x*+1 Conjecture, Kakutanis problem, the Syracuse problem, and Ulams problem.

This paper gives the equivalent definition of the problem based on the fundamental theorem of arithmetic, and analyzes the problem from an odd number point of view.

All alphanumeric characters shown below are positive integers unless otherwise noted.

2. Iteration of Positive Odd Number

According to the fundamental theorem of arithmetic, the number of prime factors with value equal to 2 in each positive integer is unique. If x is a positive odd number then $3x + 1$ is an even number greater than 2. Divide by all factors which equals 2 in $3x + 1$, the quotient is an odd number not less than 1.

Definition 1.

Let x be a positive odd number and $(T = \frac{3x+1}{2^\alpha})$ (α is a positive integer). If T is an odd number not less than 1, then the operation is referred to as an *iteration* of x. Denote $T(x)$ as the iteration result of x, α as the *iterative exponent*.

Definition 2. If the iteration result of x is still x, then x is called the *origin*.

Theorem 1. *Of all the odd numbers, only 1 is the origin.*

Proof. Let odd number x be the origin. By the definition 1, we have

$$T(x) = \frac{3x+1}{2^\alpha} = x. \tag{2.1}$$

Since α is a positive integer, the solution of formula (2.1) is as follows

$$x = \frac{1}{2^\alpha - 3}\begin{cases} = -1, & \alpha = 1. \\ = 1, & \alpha = 2. \\ \le 1/5, & \alpha > 2. \end{cases} \tag{2.2}$$

According to the definition 1, x can only be a positive odd number. Among the above three results, since only $\alpha = 2$ conforms to the definition, only 1 is the origin.

Let $T^{(k)}(x)$ denote the result of the k-th iteration of x.

If $T^{(k)}(x) = 1\,(x > 1)$, i.e. after a limited number iterations for x, the end result is to return to the origin, then x is called *convergence*.

Let α_i be the iterative exponent of the *i*-th $(i \in \mathbb{Z}^+)$ iteration of x.

Let $g(k) = \sum_{i=1}^{k} \alpha_i,\ g(0) = 0.$

Theorem 2. *The k-th iteration formula of x is as follows.*

$$T^{(k)}(x)=\frac{3^k x+\sum_{i=1}^{k}2^{g(i-1)}3^{k-i}}{2^{g(k)}}. \qquad (2.3)$$

Proof. For $k=1$. Since $3^{k-1}2^{g(1-k)}=1$, so, according to (2.3), we have

$$T^{(1)}(x)=\frac{3x+1}{2^{\alpha_1}}. \qquad (2.4)$$

If (2.3) holds for k, then for $k+1$ we have

$$T^{(k+1)}(x)=\frac{3T^{(k)}(x)+1}{2^{\alpha_{k+1}}}=\frac{3\dfrac{3^k x+3\sum_{i=1}^{k}2^{g(i-1)}3^{k-i}}{2^{g(k)}}+1}{2^{\alpha_{k+1}}}$$

$$=\frac{3^{k+1}x+\sum_{i=1}^{k+1}2^{g(i-1)}3^{k+1-i}}{2^{g(k+1)}}.$$

3. Possibility of $T^{(k)}(x)=x$

If $T^{(k)}(x)=x(x>1)$, i.e. x get itself again through continuous iterations, then x is called *cyclic*.

Theorem 3. The condition that the following equation has a positive integer solution is $s(i)=2i(i=0,1,2,3,......,k)$.

$$f(k)=\frac{\sum_{i=1}^{k}2^{s(i-1)}3^{k-i}}{2^{s(k)}-3^k}.(k\in Z^+) \qquad (3.1)$$

(where: $s(k)$ is integer, $s(k+1)>s(k), s(0)=0$)

Proof. For $k=1$, $f(1)=\dfrac{1}{2^{s(1)}-3}$. It is the same as equation (2.2), so substituting $s(1)=2$ into this equation, we have the positive integer solution $f(1)=1$.

If (3.1) holds for k, then for $k+1$, the equation can be written as

$$f(k+1)=\frac{2^{2k+2}-3^{k+1}}{2^{s(k+1)}-3^{k+1}}..$$

It is known that $s(k+1)>s(k)$ and $s(k)=2k$, let $s(k+1)=2k+c$ (c is a positive integer). We can refer to the following equation

$$h(k)=\frac{2^{2k+2}-3^{k+1}}{2^{2k+c}-3^{k+1}}. \qquad (3.2)$$

1. If $c=1$, the above function can be written as

$$h(k)=\frac{2(2^{2k+1}-3^{k+1})+3^{k+1}}{2^{2k+1}-3^{k+1}}$$

$$=2+\frac{3^{k+1}}{2^{2k+1}-3^{k+1}}.$$

and

$$\frac{3^{(k+1)+1}}{2^{2(k+1)+1}-3^{(k+1)+1}}\div\frac{3^{k+1}}{2^{2k+1}-3^{k+1}}=\frac{3}{4+\dfrac{3^{k+1}}{2^{2k+1}-3^{k+1}}}.$$

It shows that, for all $k>1, h(k)>2$ and $h(k)$ is monotonically decreasing. The calculation result shows that $h(2)$ and $h(3)$ are not integers, $h(4)\doteq 2.9$.

So, in this case, $h(k)$ cannot be a positive integer.

2. If $c=2$, substituting it into (3.2), then we get an identity that $h(k)\equiv 1$;

3. If $c>2$ then, for all $k>1$, $h(k)<1$.

Summarizing the above analysis, the theorem is true.

If the result of successive k iterations of x can constitute a loop, i.e. $T^{(k)}(x)=x$, then according to theorem 2, we have

$$T^{(k)}(x)=\frac{3^k x+\sum_{i=1}^{k}2^{g(i-1)}3^{k-i}}{2^{g(k)}}=x.$$

From the formula above we get

$$x=\frac{\sum_{i=1}^{k}2^{g(i-1)}3^{k-i}}{2^{g(k)}-3^k}. \qquad (3.3)$$

Let $x=f(k)$ and $g(k)=s(k)$. According to theorem 3, equation (3.3) has no other positive integer solutions except for $x=1$. Therefore, continuous iterations of x is impossible to constitute a loop.

4. Possibility of $\lim_{k\to\infty}T^{(k)}(x)=\infty$

If $\lim_{k\to\infty}T^{(k)}(x)=\infty$, i.e. x can be iterated infinitely and the result tends to infinity, then x is called *divergent*.

Let $[A]$ be the integer part of the real number A.

Theorem 4. *When $k\to\infty$, if x is finite, then $T^{(k)}(x)$ cannot be infinite.*

Proof. Formula (2.3) can be written as follows

$$2^{g(k)}T^{(k)}(x)=3^k x+\sum_{i=1}^{k}2^{g(i-1)}3^{k-i}. \qquad (4.1)$$

Let $A_k=2^{g(k)}, B_k=3^k, C_k=\sum_{i=1}^{k}2^{g(i-1)}3^{k-i}$.

See the following linear equation

$$A_kY=B_kX+C_k. \qquad (4.2)$$

Since A_k, B_k and C_k are both positive integers and $(A_k,B_k)=1$, this equation has an infinite number of integer solutions:

$$\begin{cases}Y=Y_0+B_kt,\\ X=X_0+A_kt.\end{cases}\quad (t\in\mathbb{Z}) \qquad (4.3)$$

The process of determining X_0 and Y_0 is as follows [5]:

a. If $B_k > A_k$, we can construct the following equations

$$\begin{cases} A_k\left[\dfrac{B_k}{A_k}\right] - B_k = -r_1, \\ A_k\left(\left[\dfrac{B_k}{A_k}\right]+1\right) - B_k = r_2, \end{cases} \tag{4.4}$$

Since $(r_1, r_2) = 1$, so, there are integers μ and ν such that the following formula holds

$$\nu r_2 - \mu r_1 = C_k. \tag{4.5}$$

According to (4.4) and (4.5), a special solution for the equation is

$$\begin{cases} X_0 = \nu + \mu, \\ Y_0 = \nu\left(\left[\dfrac{B_k}{A_k}\right]+1\right) + \mu\left[\dfrac{B_k}{A_k}\right]. \end{cases} \tag{4.6}$$

and

$$\nu = \frac{C_k + \mu(A_k - r_2)}{r_2}. \tag{4.7}$$

Thus, we obtain

$$\begin{cases} X_0 = \dfrac{C_k + \mu A_k}{r_2}, \\ Y_0 = X_0\left(\left[\dfrac{B_k}{A_k}\right]+1\right) - \mu. \end{cases} \tag{4.8}$$

For all $k > 4$,

$$C_k > 0.8 \times 3^k + \sum_{i=5}^{k} 2^{g(i-1)} 3^{k-i} > 0.8 B_k. \tag{4.9}$$

Since $r_2 < A_k$, so, $X_0 > \dfrac{C_k}{A_k} + \mu$. When X_0 is a finite value, then $\dfrac{C_k}{A_k}$ is also a finite value. In this case, according to (4.9), $\dfrac{B_k}{A_k}$ can only be a finite value. It means that if X is finite, then Y is also finite.

b. If $A_k > B_k$, we can construct the following equations in the same way as above

$$\begin{cases} A_k - B_k\left[\dfrac{A_k}{B_k}\right] = r_1, \\ A_k - B_k\left(\left[\dfrac{A_k}{B_k}\right]+1\right) = -r_2, \\ \nu r_1 - \mu r_2 = C_k. \end{cases} \tag{4.10}$$

According to (4.10), a special solution for the equation is

$$\begin{cases} Y_0 = \mu + \nu, \\ X_0 = Y_0\left[\dfrac{A_k}{B_k}\right] + \mu. \end{cases}$$

Since $Y_0 = \dfrac{X_0 - \mu}{[A_k / B_k]}$, so, if X_0 is finite, then Y_0 cannot be infinite. It means that if X is finite, then Y cannot be infinite.

Let $x = X$ and $T^{(k)}(x) = Y$. Thus, the equation (4.1) is equal to (4.2). The above analysis shows that when $k \to \infty$, if x is finite, then $T^{(k)}(x)$ cannot be infinite.

5. Conclusion

Analysis of the above two sections shows that all positive odd numbers are neither *divergent* nor *cyclic*. So, for an odd integer x greater than 1, there is a large integer N such that, for all $k \geq 1$, $T^{(k)}(x) < N$. After consecutive N iterations, we can get N odd numbers less than N. Since loops do not occur during successive iterations, the values of those integers greater than 1 are different from each other, and the number of these integers is less than $N/2$. According to theorem 1, the iteration result for 1 is still 1. Thus, the other iteration results are all 1, and the final iteration result can only be 1. So, *all positive odd numbers are convergent*.

Since each positive even number can be converted to an odd number after divided by factors equal to 2, the above conclusion also applies to even numbers. So *all positive integers are convergent.*

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