

Mean Values of Arithmetic Functions under Congruences with the Euler Function

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Abstract We examine the average order of some arithmetic functions written as sums over Euler function in arithmetic progression and in general over $f(\varphi(d)) \equiv b \pmod{p}$ such that p is a prime number, b an integer and f is a polynomial function with integer coefficients and a degree $d \geq 2$ that is not constant modulo p . Our results are based on various estimates of rational exponential sums with the Euler Function in arithmetic progression which are due to William Banks and Igor E. Shparlinski.

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1. Introduction

In this paper, p always denotes a prime number fixed throughout, φ the Euler function and (m, n) the greatest common divisor. If x is a real number, we have $\lfloor x \rfloor = \max\{n \in \mathbb{Z}, n \leq x\}$. We recall that the notation $U \ll V$ is equivalent to the statement $U = O(V)$ for positive functions U and V where it means that there exists a constant $c > 0$ such that $U(x) \leq cV(x)$ and the implied constants in the symbols are absolute. We also use the expression $U = o(V)$ that is equivalent to $U/V \rightarrow 0$. Furthermore, we have the abbreviation $e_p(x) = e(2\pi ix/p)$ for any real number x and we will note by $\omega = \log N / \log p$.

While the behavior of a number theoretic function $g(n)$ for large n is often difficult to determine because the function values can fluctuate considerably as n increases, it is more fruitful to study partial sums and seek asymptotic formulas of the form

$$\sum_{n \leq x} g(n) = G(x) + O(h(x)),$$

where $G(x)$ is a known function of x and $O(h(x))$ represents the error, a function of smaller order than $G(x)$ for all x in some prescribed range. Some of these arithmetic functions are called multiplicative when they satisfy $f(1) = 1$ and $f(ab) = f(a)f(b)$ whenever a and

b are coprimes. In this work, we will focus on the following arithmetic functions:

- d : The number of non-negative divisors function.
- σ_s : The sum of the s -th powers of all the non-negative divisors function, for $s \in \mathbb{R}$. In particular, $\sigma_0 = d$.
- φ : The Euler function.
- J_k : The Jordan totient function counting the number of k -tuples. In particular, $J_1 = \varphi$.
- P : The Pillai function known also as the gcd -sum function.

Problems involving exponential sums over some arithmetic functions have been considered in many papers as in [2,3,7] where they obtained non trivial bounds in certain ranges for rational exponential sums of the form $\sum_{n \leq N} e_p(ag(n))$ such that $(a, p) = 1$. In particular, Banks and Shparlinski [2] gave upper bounds for the number of solutions to congruences with the Euler functions and the Carmichael function and also non trivial bounds for the exponential sums involving φ . They expected that their methods could be also applied to exponential sums with p replaced by an arbitrary positive integer although certain arguments would be more complicated. Analogous results for similar arithmetic functions such as the aliquot divisor function $s = \sigma - id$ in [3]. Later on, Kerr [5] solved the problem given by Shparlinski in [[8], Problem 27] about bounding the rational exponential sums over the divisor function for specifically odd integer $m \geq 3$ and coprime with a .

In this work, we inquire about calculating the mean values of the already listed arithmetic functions for large values of N where they will be written as Dirichlet convolutions under some constraints involving the Euler function. Indeed, we will be focusing on the following

$$\tilde{\tau}(n) = \sum_{d|n} 1, \quad \bar{\tau}(n) = \sum_{f(\varphi(d)) \equiv b \pmod p} 1,$$

$$\tilde{\sigma}_s(n) = \sum_{d|n} \left(\frac{n}{d}\right)^s,$$

$$\bar{\sigma}_s(n) = \sum_{f(\varphi(d)) \equiv b \pmod p} \left(\frac{n}{d}\right)^s,$$

$$\tilde{J}_k(n) = \sum_{d|n} \left(\frac{n}{d}\right)^k \mu(d),$$

$$\bar{J}_k(n) = \sum_{d|n} \left(\frac{n}{d}\right)^k \mu(d)$$

$$\tilde{\varphi}(n) = \sum_{d|n} \frac{n}{d} \mu(d),$$

$$\bar{\varphi}(n) = \sum_{f(\varphi(d)) \equiv b \pmod p} \frac{n}{d} \mu(d)$$

$$\tilde{P}(n) = \sum_{d|n} \frac{n}{d} \tau\left(\frac{n}{d}\right) \mu(d),$$

$$\bar{P}(n) = \sum_{f(\varphi(d)) \equiv b \pmod p} \frac{n}{d} \tau\left(\frac{n}{d}\right) \mu(d).$$

The main goal of this paper is to give an average order of some chosen arithmetic functions under constraints involving the Euler function. In particular our work is organized as follows: Section 2 is devoted to give some auxiliary lemmas that are necessary for the proofs of our theorems. Section 3 and Section 4 establish the mean values of the defined arithmetic functions in the above first while satisfying the conditions $\varphi(d) \equiv b \pmod p$ then $f(\varphi(d)) \equiv b \pmod p$.

2. Technical Lemmas

We start by providing a rigorous definition of the zeta function ζ for $Re(s) > 1$

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}.$$

The Euler-Mascheroni's constant is usually known as

$$\gamma = \lim_{N \rightarrow +\infty} \left(\sum_{n \leq N} \frac{1}{n} - \log N \right) = 0.5772\dots$$

In order to be able to state our main results, we need the following mean values of the form n^s .

Lemma 2.1 ([1], Theorem 3.2). *If $N \geq 1$, we have*

$$(1) \sum_{n \leq N} \frac{1}{n} = \log N + \gamma + O\left(\frac{1}{N}\right).$$

$$(2) \sum_{n \leq N} \frac{1}{n^s} = \frac{N^{1-s}}{1-s} + \zeta(s) + O(N^{-s}), \text{ if } s > 0, s \neq 1.$$

$$(3) \sum_{n > N} \frac{1}{n^s} = O(N^{1-s}), \text{ if } s > 1.$$

$$(4) \sum_{n \leq N} n^s = \frac{N^{s+1}}{s+1} + O(N^s), \text{ if } s \geq 0.$$

A crucial part in our proofs is the estimation of exponential sums. One plain formula that helps to detect the congruences is the following simple but important observation

$$\frac{1}{b} \sum_{j=0}^{b-1} e\left(\frac{jn}{b}\right) = \begin{cases} 1, & \text{if } b | n, \\ 0, & \text{otherwise.} \end{cases} \quad (2.1)$$

The key input to the present paper is the application of the following bounds of the exponential sums given by Banks and Shparlinski [2].

Theorem A [2], Theorem 5.1]: The following bound holds, for $p \geq \log^8 N$,

$$\max_{(a,p)=1} \left| \sum_{n \leq N} e_p(a\varphi(n)) \right| \ll N \left(\log^4 N p^{-1/2} + \omega^{-2\omega/5+o(\omega)} \right). \quad (2.2)$$

It is to be noted that Banks and Shparlinski bounds are non trivial for a wide range of values of p starting with $p \geq \log^8 N$. They remarked that although it might be possible to improve on this power of $\log N$, for very small values of p relative to N , it is simply not possible to obtain nontrivial bounds.

Now, we define the following set $\mathcal{F}_{d,p}$ of polynomials with integers coefficients of degree $d \geq 2$ whose leading coefficient is relatively prime to p that is

$$\mathcal{F}_{d,p} = \left\{ f(x) = a_d x^d + \dots + a_1 x + a_0 \in \mathbb{Z}[x] / a_d \not\equiv 0 \pmod p \right\}.$$

Theorem B [2], Theorem 5.2]: *For any $\varepsilon > 0$ and $d \geq 2$, the following bound holds for $p \geq \log N$*

$$\max_{f \in \mathcal{F}_{d,p}} \left| \sum_{n \leq N} e_p(af(\varphi(n))) \right| \ll N \left(p^{-1/4+\varepsilon} + \omega^{-\omega/2+o(\omega)} \right). \quad (2.3)$$

3. Sums with Congruences with φ

3.1. Average Order of the Divisor Function

Theorem 3.1. Let $p \geq \log^9 N$ and $b \in \mathbb{Z}$. Then, for a sufficiently large N , we have

$$\sum_{n \leq N} \tilde{\tau}(n) = \frac{N}{p} (\log N + 2\gamma - 1) + O\left(\frac{N}{p} \left(\frac{\log^5 N}{\sqrt{p}} + \omega^{-\omega/5+o(\omega)}\right)\right).$$

Proof. Given N large enough, we have

$$\sum_{n \leq N} \tilde{\tau}(n) = \sum_{n \leq N} \sum_{\substack{d \leq n \\ \varphi(d) \equiv b \pmod{p}}} 1 = \sum_{\substack{dh \leq N \\ \varphi(d) \equiv b \pmod{p}}} 1.$$

Next, we use Dirichlet's hyperbola formula and we get

$$\begin{aligned} \sum_{n \leq N} \tilde{\tau}(n) &= \sum_{\substack{d \leq \sqrt{N} \\ \varphi(d) \equiv b \pmod{p}}} \sum_{\substack{h \leq \frac{N}{d} \\ \varphi(h) \equiv b \pmod{p}}} 1 + \sum_{\substack{h \leq \sqrt{N} \\ \varphi(h) \equiv b \pmod{p}}} \sum_{\substack{d \leq \frac{N}{h} \\ \varphi(d) \equiv b \pmod{p}}} 1 \\ &\quad - \sum_{\substack{h \leq \sqrt{N} \\ \varphi(h) \equiv b \pmod{p}}} \sum_{\substack{d \leq \sqrt{N} \\ \varphi(d) \equiv b \pmod{p}}} 1 \end{aligned}$$

$$= S_1 + S_2 - S_3.$$

Each summand will be treated separately in the sequel. First, we write thanks to the orthogonality formula (2.1)

$$\begin{aligned} S_1 &= \sum_{\substack{d \leq \sqrt{N} \\ \varphi(d) \equiv b \pmod{p}}} \sum_{\substack{h \leq \frac{N}{d} \\ \varphi(h) \equiv b \pmod{p}}} 1 = \sum_{\substack{d \leq \sqrt{N} \\ \varphi(d) \equiv b \pmod{p}}} \left[\frac{N}{d} + O(1) \right] \\ &= N \sum_{\substack{d \leq \sqrt{N} \\ \varphi(d) \equiv b \pmod{p}}} \frac{1}{d} + O\left(\sum_{\substack{d \leq \sqrt{N} \\ \varphi(d) \equiv b \pmod{p}}} 1\right) \\ &= \frac{N}{p} \sum_{d \leq \sqrt{N}} \frac{1}{d} + \frac{N}{p} \sum_{j=1}^{p-1} e_p(-jb) \sum_{d \leq \sqrt{N}} \frac{e_p(j\varphi(d))}{d} + O(\sqrt{N}). \end{aligned} \tag{3.1}$$

The first sum in the right hand side of (3.1) can be estimated using Lemma 2.1 giving

$$\frac{N}{p} \left(\frac{\log N}{2} + \gamma \right) + O\left(\frac{\sqrt{N}}{p}\right). \tag{3.2}$$

Afterwards, we use Abel's summation formula and we get

$$\begin{aligned} &\sum_{d \leq \sqrt{N}} \frac{e_p(j\varphi(d))}{d} \\ &= \frac{1}{\sqrt{N}} \sum_{d \leq \sqrt{N}} e_p(j\varphi(d)) + \int_2^{\sqrt{N}} \sum_{d \leq u} e_p(j\varphi(d)) \frac{du}{u^2}. \end{aligned} \tag{3.3}$$

Due to Theorem A, we have for all $j \in \{1, \dots, p-1\}$

$$\begin{aligned} \text{while we mean by } \tilde{\omega} &= \frac{\log u}{\log p} \\ \frac{1}{u^2} \sum_{d \leq u} e_p(j\varphi(d)) &\ll \frac{1}{u^2} \left[u \frac{\log^4 u}{\sqrt{p}} + u \tilde{\omega}^{-2/5\tilde{\omega}+o(\tilde{\omega})} \right] \\ &\ll \frac{\log^4 u}{u\sqrt{p}} + \left(\frac{\log pu}{\log u} \right)^{2/5}. \end{aligned} \tag{3.4}$$

Hence, we obtain

$$\begin{aligned} &\int_2^{\sqrt{N}} \sum_{d \leq u} e_p(j\varphi(d)) \frac{du}{u^2} \\ &\ll \int_2^{\sqrt{N}} \left[\frac{\log^4 u}{\sqrt{p}} + \left(\frac{\log p}{\log u} \right)^{2/5} \right] \frac{du}{u} \\ &\ll \frac{\log^5 N}{\sqrt{p}} + (\log N)^{3/5} (\log p)^{2/5}. \end{aligned} \tag{3.5}$$

Now, we apply Theorem A again and we get

$$\frac{1}{\sqrt{N}} \sum_{d \leq \sqrt{N}} e_p(j\varphi(d)) \ll \frac{\log^4 N}{\sqrt{p}} + \omega^{-\omega/5+o(\omega)}. \tag{3.6}$$

Summing up the identities (3.5) and (3.6) lead to

$$\begin{aligned} &\sum_{d \leq \sqrt{N}} \frac{e_p(j\varphi(d))}{d} \\ &\ll \frac{\log^5 N}{\sqrt{p}} + (\log N)^{3/5} (\log p)^{2/5} + \omega^{-\omega/5+o(\omega)} \\ &\ll \frac{\log^5 N}{\sqrt{p}} + \omega^{-\omega/5+o(\omega)}. \end{aligned} \tag{3.7}$$

Taking the identities (3.2) and (3.7) and going back to (3.1) therefore yield

$$\begin{aligned} S_1 &= \frac{N \log N}{2p} + \frac{\gamma}{p} N \\ &\quad + O\left(\frac{N \log^5 N}{p\sqrt{p}} + N \frac{\log^4 N}{p\sqrt{p}} + \frac{N}{p} \omega^{-\omega/5+o(\omega)}\right), \\ &= \frac{N \log N}{2p} + \frac{\gamma}{p} N + O\left(\frac{N}{p} \left(\frac{\log^5 N}{\sqrt{p}} + \omega^{-\omega/5+o(\omega)}\right)\right). \end{aligned} \tag{3.8}$$

It is to be said that this choice of $p \geq \log^9 N$ is to balance the first term inside the parentheses in the O -term.

Next, we turn our attention to bounding the S_2 . We write again due to the orthogonality formula,

$$\begin{aligned} S_2 &= \frac{1}{p} \sum_{j=0}^{p-1} e_p(-jb) \sum_{h \leq \sqrt{N}} \sum_{\substack{d \leq \frac{N}{h} \\ \varphi(d) \equiv b \pmod{p}}} e_p(j\varphi(d)) \\ &= \frac{1}{p} \sum_{h \leq \sqrt{N}} \sum_{\substack{d \leq \frac{N}{h} \\ \varphi(d) \equiv b \pmod{p}}} 1 + \frac{1}{p} \sum_{j=1}^{p-1} e_p(-jb) \sum_{h \leq \sqrt{N}} \sum_{\substack{d \leq \frac{N}{h} \\ \varphi(d) \equiv b \pmod{p}}} e_p(j\varphi(d)). \end{aligned}$$

When using Lemma 2.1, it is easy to see that

$$\begin{aligned} \frac{1}{p} \sum_{h \leq \sqrt{N}} \sum_{d \leq \frac{N}{h}} 1 &= \frac{1}{p} \sum_{h \leq \sqrt{N}} \left[\frac{N}{h} + O(1) \right] \\ &= \frac{N}{2p} \log N + \frac{\gamma}{p} N + O\left(\frac{\sqrt{N}}{p}\right). \end{aligned} \tag{3.9}$$

While calling Theorem A and denoting $\tilde{\omega} = \frac{\log N/h}{\log p}$, we get for every $j = 1, \dots, p-1$

$$\begin{aligned} &\sum_{h \leq \sqrt{N}} \sum_{d \leq \frac{N}{h}} e_p(j\varphi(d)) \\ &\ll \sum_{h \leq \sqrt{N}} \frac{N}{h} \left(\frac{\log^4 N/h}{\sqrt{p}} + \tilde{\omega}^{-2\tilde{\omega}/5+o(\tilde{\omega})} \right) \\ &\ll N \sum_{h \leq \sqrt{N}} \frac{1}{h} \left(\frac{\log^4 N}{\sqrt{p}} + \left(\frac{\log p}{\log N/h} \right)^{2/5} \right) \\ &\ll \frac{N \log^4 N}{\sqrt{p}} \sum_{h \leq \sqrt{N}} \frac{1}{h} \\ &\quad + N(\log p)^{2/5} \sum_{h \leq \sqrt{N}} \frac{(\log N/h)^{-2/5}}{h}. \end{aligned}$$

Afterwards by writing $(\log N/h)^{-2/5}$ as $(\log N)^{-2/5} (1 - \frac{\log h}{\log N})^{-2/5}$ and since $h \leq \sqrt{N}$, then necessarily we have $(1 - \frac{\log h}{\log N})^{-2/5} \leq (1/2)^{-2/5}$.

Consequently, the upper bound becomes while using Lemma 2.1

$$\begin{aligned} &\sum_{h \leq \sqrt{N}} \sum_{d \leq \frac{N}{h}} e_p(j\varphi(d)) \\ &\ll \left(\frac{N \log^4 N}{\sqrt{p}} + N(\log p)^{2/5} (\log N)^{-2/5} \right) \cdot \sum_{h \leq \sqrt{N}} \frac{1}{h} \\ &\ll \left(\frac{N \log^4 N}{\sqrt{p}} + N(\log p)^{2/5} (\log N)^{-2/5} \right) \cdot O(\log N) \\ &\ll \frac{N \log^5 N}{\sqrt{p}} + N(\log p)^{2/5} (\log N)^{3/5}. \end{aligned} \tag{3.10}$$

Moreover, by choosing $p \geq \log^9 N$, it follows that $\frac{\log^5 N}{\sqrt{p}} \leq (\log N)^{3/5}$.

Consequently, by putting (3.9) and (3.10) jointly gives

$$S_2 = \frac{N \log N}{2p} + \frac{\gamma N}{p} + O\left(\frac{N}{p} (\log p)^{2/5} (\log N)^{3/5}\right). \tag{3.11}$$

Finally, for the last sum, we obtain in view of (2.1) and the first assertion of Lemma 2.1,

$$\begin{aligned} S_3 &= \sum_{h \leq \sqrt{N}} 1 \sum_{\substack{d \leq \frac{N}{h} \\ \varphi(d) \equiv b \pmod{p}}} 1 \\ &= \left[\sqrt{N} + O(1) \right] \sum_{\substack{d \leq \frac{N}{h} \\ \varphi(d) \equiv b \pmod{p}}} 1 \\ &= \sqrt{N} \sum_{\substack{d \leq \frac{N}{h} \\ \varphi(d) \equiv b \pmod{p}}} 1 + O\left(\sum_{d \leq \frac{N}{h}} 1 \right) \\ &= \frac{\sqrt{N}}{p} \sum_{d \leq \frac{N}{h}} 1 \\ &\quad + \frac{\sqrt{N}}{p} \sum_{j=1}^{p-1} e_p(-jb) \sum_{d \leq \frac{N}{h}} e_p(j\varphi(d)) + O(\sqrt{N}) \\ &= \frac{\sqrt{N}}{p} \left[\sqrt{N} + O(1) \right] \\ &\quad + O\left(\frac{N}{p} \left(\frac{\log^4 N}{\sqrt{p}} + \omega^{-\omega/5+o(\omega)} \right) \right) + O(\sqrt{N}) \\ &= \frac{N}{p} + O\left(\frac{N}{p} \left(\frac{\log^4 N}{\sqrt{p}} + \omega^{-\omega/5+o(\omega)} \right) \right). \end{aligned} \tag{3.12}$$

We reach the desired result by assembling (3.8), (3.11) and (3.12).

3.2. Average Order of the Function $\tilde{\sigma}_s$

The case $s=0$ was already studied in the last paragraph since $\sigma_0 = d$.

Theorem 3.2. *Let $p \geq \log^9 N$ and $b \in \mathbb{Z}$. For s a real number and for a sufficiently large N , we state that*

For $s > 0$:

$$\sum_{n \leq N} \tilde{\sigma}_s(n) = \begin{cases} \frac{\zeta(2)/2 + \beta}{p} N^2 - \frac{N}{p} + O(N \log N), & \text{if } s = 1, \\ \left(\frac{\zeta(s+1) + \beta}{p(s+1)} + \frac{\beta}{p} \right) N^{s+1} \\ \quad + O\left(\frac{N}{p} \left(\frac{\log^4(N)}{\sqrt{p}} + \left(\frac{\log p}{\log \log p} \right)^{2/5} \right) \right), & \text{if } s > 0, \\ \quad + N^{\max(1,s)} \end{cases}$$

such that $\beta = \sum_{j=1}^{p-1} e_p(-jb) \int_2^{+\infty} \sum_{d \leq u} e_p(j\varphi(d)) \frac{du}{u^{s+2}}$.

For $s < 0$,

$$\sum_{n \leq N} \tilde{\sigma}_s(n) = \begin{cases} \frac{\zeta(1-s)}{p} N + O\left(\frac{N^{s+1}}{p} + \frac{N \log^4 N}{p\sqrt{p}} \right), & \text{if } -1 < s < 0, \\ \frac{\zeta(2)}{p} N + O\left(\frac{\log N}{p} \right), & \text{if } s = -1, \\ \frac{\zeta(1-s)}{p} N + O\left(\frac{N \log^4 N}{p\sqrt{p}} \right), & \text{if } s < -1. \end{cases}$$

Proof. We shall treat the subcases $s = 1$, $s > 1$ and $s < 0$ separately.

At first, we might write due to Lemma 2.1 and Abel's summation formula

$$\begin{aligned} \sum_{n \leq N} \tilde{\sigma}_s(n) &= \sum_{n \leq N} \sum_{\substack{d|n \\ \varphi(d) \equiv b \pmod p}} \left(\frac{n}{d}\right)^s \\ &= \sum_{d \leq N} \sum_{\substack{h \leq \frac{N}{d} \\ \varphi(d) \equiv b \pmod p}} h^s \\ &= \frac{N^{s+1}}{s+1} \sum_{\substack{d \leq N \\ \varphi(d) \equiv b \pmod p}} \frac{1}{d^{s+1}} + O\left(N^s \sum_{d \leq N} \frac{1}{d^s}\right). \end{aligned}$$

We will consider separately the last equation in the two positive subcases.

• If $s = 1$: we get by using the orthogonality relation and Lemma 2.1

$$\begin{aligned} \sum_{n \leq N} \tilde{\sigma}_1(n) &= \frac{N^2}{2} \sum_{\substack{d \leq N \\ \varphi(d) \equiv b \pmod p}} \frac{1}{d^2} + O\left(N \sum_{d \leq N} \frac{1}{d}\right) \\ &= \frac{N^2}{2p} \sum_{d \leq N} \frac{1}{d^2} + \frac{N^2}{2p} \sum_{j=1}^{p-1} e_p(-jb) \sum_{d \leq N} \frac{e_p(j\varphi(d))}{d^2} \quad (3.13) \\ &\quad + O(N \log N). \end{aligned}$$

Thus, by Lemma 2.1, the first sum of (3.13) becomes

$$\frac{N^2}{2p} \left(\zeta(2) - \frac{1}{N} + O(N^{-2}) \right) = \frac{\zeta(2)}{2p} N^2 - \frac{N}{2p} + O\left(\frac{1}{p}\right).$$

Consequently, we obtain

$$\begin{aligned} \sum_{n \leq N} \tilde{\sigma}_1(n) &= \frac{\zeta(2)}{2p} N^2 - \frac{N}{2p} + O\left(\frac{1}{p}\right) \\ &\quad + \frac{N^2}{2p} \sum_{j=1}^{p-1} e_p(-ja) \sum_{d \leq N} \frac{e_p(j\varphi(d))}{d^2} \quad (3.14) \\ &\quad + O(N \log N). \end{aligned}$$

• If $s \neq 1$: We set $z = \max(1, s)$ and we write thanks to Lemma

$$\begin{aligned} \sum_{n \leq N} \tilde{\sigma}_s(n) &= \frac{N^{s+1}}{s+1} \sum_{\substack{d \leq N \\ \varphi(d) \equiv b \pmod p}} \frac{1}{d^{s+1}} + O\left(N^s \sum_{d \leq N} \frac{1}{d^s}\right) \\ &= \frac{N^{s+1}}{p(s+1)} \sum_{j=0}^{p-1} e_p(-jb) \sum_{d \leq N} \frac{e_p(j\varphi(d))}{d^{s+1}} \\ &\quad + O\left(N^s \left[\frac{N^{1-s}}{1-s} + \zeta(s) + O(N^{-s}) \right]\right) \\ &= \frac{N^{s+1}}{p(s+1)} \sum_{d \leq N} \frac{1}{d^{s+1}} \\ &\quad + \frac{N^{s+1}}{p(s+1)} \sum_{j=1}^{p-1} e_p(-jb) \sum_{d \leq N} \frac{e_p(j\varphi(d))}{d^{s+1}} + O(N^z) \end{aligned}$$

$$\begin{aligned} &= \frac{N^{s+1}}{p(s+1)} \left[-\frac{N^{-s}}{s} + \zeta(s+1) + O(N^{-s-1}) \right] \\ &\quad + \frac{N^{s+1}}{p(s+1)} \sum_{j=1}^{p-1} e_p(-jb) \sum_{d \leq N} \frac{e_p(j\varphi(d))}{d^{s+1}} + O(N^z) \quad (3.15) \\ &= \frac{\zeta(s+1)}{p(s+1)} N^{s+1} - \frac{N}{ps(s+1)} \\ &\quad + \frac{N^{s+1}}{p(s+1)} \sum_{j=1}^{p-1} e_p(-jb) \sum_{d \leq N} \frac{e_p(j\varphi(d))}{d^{s+1}} + O(N^z). \end{aligned}$$

Thus, we summarize both cases (3.14) and (3.15) by writing as follows

$$\begin{aligned} \sum_{n \leq N} \tilde{\sigma}_s(n) &= \frac{\zeta(s+1)}{p(s+1)} N^{s+1} - \frac{N}{ps(s+1)} + \frac{N^{s+1}}{p(s+1)} T_s \\ &\quad + \begin{cases} O(N \log N), & \text{if } s = 1, \\ O(N^z), & \text{if } s \neq 1. \end{cases} \quad (3.16) \end{aligned}$$

while T_s denotes

$$\sum_{j=1}^{p-1} e_p(-jb) \sum_{d \leq N} \frac{e_p(j\varphi(d))}{d^{s+1}}.$$

Next, we apply Abel's formula to $T_{j,s}$ for every $j = 1, \dots, p-1$, so we obtain

$$\begin{aligned} T_{j,s}(N) &= \sum_{d \leq N} \frac{e_p(j\varphi(d))}{d^{s+1}} \\ &= \frac{1}{N^{s+1}} \sum_{d \leq N} e_p(j\varphi(d)) + (s+1) \int_2^N \sum_{d \leq u} e_p(j\varphi(d)) \frac{du}{u^{s+2}}. \end{aligned}$$

For all $j \in \{1, \dots, p-1\}$, thanks to Theorem A, the latter integral is absolutely convergent and

$$\begin{aligned} &\int_N^{+\infty} \sum_{d \leq t} e_p(j\varphi(d)) \frac{dt}{t^{s+2}} \\ &\ll N^{-s} \left(\frac{\log^4 N}{\sqrt{p}} + \left(\frac{\log p}{\log N} \right)^{2/5} \right). \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\int_2^N \sum_{d \leq t} e_p(j\varphi(d)) \frac{dt}{t^{s+2}} \\ &= \beta_j + O\left(N^{-s} \left(\frac{\log^4 N}{\sqrt{p}} + \left(\frac{\log p}{\log N} \right)^{2/5} \right) \right), \quad (3.17) \end{aligned}$$

such that for every $\beta_j = \int_2^{+\infty} \sum_{d \leq u} e_p(j\varphi(d)) \frac{du}{u^{s+2}}$.

We call Theorem A once again, so that we have

$$\begin{aligned} &\frac{1}{N^{s+1}} \sum_{d \leq N} e_p(j\varphi(d)) \\ &= O\left(N^{-s} \left(\frac{\log^4 N}{\sqrt{p}} + \omega^{-2\omega/5+o(\omega)} \right) \right). \quad (3.18) \end{aligned}$$

Joining (3.17) to (3.18), we find that

$$T_{j,s}(N) = (s+1)\beta_j + O\left(N^{-s}\left(\frac{\log^4 N}{\sqrt{p}} + \left(\frac{\log p}{\log \log p}\right)^{2/5}\right)\right) + O\left(N^{-s}\left(\frac{\log^4 N}{\sqrt{p}} + \omega^{-2\omega/5+o(\omega)}\right)\right).$$

Afterwards, we go back to

$$\frac{N^{s+1}}{(s+1)p}T_s = \frac{\beta}{p}N^{s+1} + O\left(\frac{N}{p}\left(\frac{\log^4 N}{\sqrt{p}} + \left(\frac{\log p}{\log \log p}\right)^{2/5}\right)\right), \tag{3.19}$$

with $\beta = \sum_{j=1}^{p-1} e_p(-jb)\beta_j$.

Hence, taking (3.19) and putting it in (3.16) lead us to the desired asymptotic formula.

Now in order to study the case $s < 0$, we shall set $r = -s$ where $r > 0$.

Then, it follows from the orthogonality formula that

$$\begin{aligned} \sum_{n \leq N} \tilde{\sigma}_s(n) &= \sum_{n \leq N} \sum_{\substack{d|n \\ \varphi(d) \equiv b \pmod{p}}} \left(\frac{d}{n}\right)^r \\ &= \sum_{h \leq N} \frac{1}{h^r} \sum_{\substack{d \leq \frac{N}{h} \\ \varphi(d) \equiv b \pmod{p}}} 1 \\ &= \frac{1}{p} \sum_{h \leq N} \frac{1}{h^r} \left\lfloor \frac{N}{h} \right\rfloor + \frac{1}{p} \sum_{j=1}^{p-1} e_p(-jb) \sum_{h \leq N} \frac{1}{h^r} \sum_{d \leq N/h} e_p(j\varphi(d)) \\ &= U_1 + U_2. \end{aligned}$$

Afterwards, it follows that Lemma 2.1,

$$\begin{aligned} U_1 &= \frac{N}{p} \sum_{h \leq N} \frac{1}{h^{r+1}} + O\left(\frac{1}{p} \sum_{h \leq N} \frac{1}{h^r}\right) \\ &= \frac{N}{p} \left(\zeta(r+1) - \frac{N^{-r}}{r}\right) + O\left(\frac{N^{-r}}{p}\right) \\ &\quad + \begin{cases} O\left(\frac{\log N}{p}\right), & \text{if } r = 1, \\ O\left(\frac{N^{1-r}}{(1-r)p} + \frac{\zeta(r)}{p} + O\left(\frac{N^{-r}}{p}\right)\right), & \text{if } r \neq 1 \end{cases} \end{aligned} \tag{3.20}$$

Whence, we have

$$U_1 = \frac{\zeta(r+1)}{p}N + \begin{cases} O\left(\frac{N^{1-r}}{p}\right), & \text{if } 0 < r < 1, \\ O\left(\frac{\log N}{p}\right), & \text{if } r = 1, \\ O\left(\frac{1}{p}\right), & \text{if } r > 1. \end{cases} \tag{3.21}$$

Furthermore, for U_2 , we denote $\tilde{\omega} = \frac{\log N/h}{\log p}$.

Then, Theorem A and Lemma \ref{lemma} lead for $j \in \{1, \dots, p-1\}$

$$\begin{aligned} U_2 &= \frac{1}{p} \sum_{j=1}^{p-1} e_p(-jb) \sum_{h \leq N} \frac{1}{h^r} \sum_{d \leq N/h} e_p(j\varphi(d)) \\ &\ll \frac{1}{p} \sum_{h \leq N} \frac{1}{h^r} \left[\frac{N}{h} \left(\frac{\log^4 N/h}{\sqrt{p}} + \tilde{\omega}^{-2\tilde{\omega}/5+o(\tilde{\omega})} \right) \right] \\ &\ll \frac{N}{p} \left[\frac{\log^4 N}{\sqrt{p}} + \left(\frac{\log p}{\log N}\right)^{2/5} \right] \\ &\quad \cdot \left[-\frac{N^{-r}}{r} + \zeta(r+1) + O(N^{-r-1}) \right] \\ &\ll \frac{N}{p} \left(\frac{\log^4 N}{\sqrt{p}} + \left(\frac{\log p}{\log N}\right)^{2/5} \right) \ll \frac{N \log^4 N}{p\sqrt{p}}. \end{aligned} \tag{3.22}$$

It is to be noted that we used $\tilde{\omega}^{-2\tilde{\omega}/5+o(\tilde{\omega})} \ll \left(\frac{\log p}{\log N}\right)^{2/5}$ and that with the choice of

$p \geq (\log N)^9$, it is easily seen that $\frac{\log^4 N}{\sqrt{p}} \leq 1$.

Finally, by combining (3.21) and (3.22), the proof is completed.

3.3. Average Order of the Jordan Totient Function

Theorem 3.3. Let $p \geq \log^8 N$ and $b \in \mathbb{Z}$. We assert, for $k \geq 1$ and a sufficiently large N ,

$$\begin{aligned} \sum_{n \leq N} \tilde{J}_k(n) &= \frac{N^{k+1}}{p} \left(\frac{1}{(k+1)\zeta(k+1)} + \theta_{k+1} \right) \\ &\quad + \begin{cases} O(N \log N), & \text{if } k = 1 \\ O(N^k), & \text{if } k > 1, \end{cases} \end{aligned}$$

where θ_{k+1} is defined as

$$\sum_{j=1}^{p-1} e_p(-jb) \int_1^{+\infty} \sum_{d \leq t} \mu(d) e_p(j\varphi(d)) \frac{dt}{t^{k+2}}.$$

Proof. Due to the orthogonality formula and Lemma 2.2, we have

$$\begin{aligned} \sum_{n \leq N} \tilde{J}_k(n) &= \sum_{n \leq N} \sum_{\substack{d|n \\ \varphi(d) \equiv b \pmod{p}}} \mu(d) \frac{n^k}{d^k} \\ &= \sum_{d \leq N} \mu(d) \sum_{\substack{h \leq \frac{N}{d} \\ \varphi(d) \equiv b \pmod{p}}} h^k \\ &= \sum_{d \leq N} \mu(d) \left[\frac{1}{k+1} \left(\frac{N}{d}\right)^{k+1} + O\left(\left(\frac{N}{d}\right)^k\right) \right] \end{aligned}$$

$$\begin{aligned}
 &= \frac{N^{k+1}}{k+1} \sum_{\substack{d \leq N \\ \varphi(d) \equiv b \pmod p}} \frac{\mu(d)}{d^{k+1}} + O\left(N^k \sum_{d \leq N} \frac{1}{d^k}\right) \\
 &= \frac{N^{k+1}}{(k+1)p} \sum_{d \leq N} \frac{\mu(d)}{d^{k+1}} \\
 &+ \frac{N^{k+1}}{(k+1)p} \sum_{j=1}^{p-1} e_p(-jb) \sum_{d \leq N} \mu(d) \frac{e_p(j\varphi(d))}{d^{k+1}} \\
 &+ \begin{cases} O\left(N^k \left[\frac{N^{1-k}}{1-k} + \zeta(k) + O(N^{-k})\right]\right), & \text{if } k > 1 \\ O\left(N \left[\log N + \gamma + O(N^{-1})\right]\right), & \text{if } k = 1 \end{cases} \\
 &= \frac{N^{k+1}}{(k+1)p} \left(\sum_{d \geq 1} \frac{\mu(d)}{d^{k+1}} - \sum_{d > N} \frac{\mu(d)}{d^{k+1}} \right) + B \\
 &+ \begin{cases} O(N^k), & \text{if } k > 1 \\ O(N \log N), & \text{if } k = 1 \end{cases} \\
 &= \frac{N^{k+1}}{(k+1)p} \left(\frac{1}{\zeta(k+1)} + O(N^{-k}) \right) + B \\
 &+ \begin{cases} O(N^k), & \text{if } k > 1, \\ O(N \log N), & \text{if } k = 1, \end{cases} \tag{3.23}
 \end{aligned}$$

such that B denotes

$$\begin{aligned}
 B &= \frac{N^{k+1}}{(k+1)p} \sum_{j=1}^{p-1} e_p(-jb) \sum_{d \leq N} \mu(d) \frac{e_p(j\varphi(d))}{d^{k+1}} \\
 &= \frac{N^{k+1}}{(k+1)p} \sum_{j=1}^{p-1} e_p(-jb) V_{j,k+1}(N). \tag{3.24}
 \end{aligned}$$

Knowing that the last step in (3.23) follows from the fact that the Möbius function is the inverse convolution of 1. Indeed, we formally have for $k \in \mathbb{C}$ such that $\text{Re}(k) > 1$

$$\left(\sum_{d \geq 1} \frac{\mu(d)}{d^k} \right) \cdot \left(\sum_{d \geq 1} \frac{1}{d^k} \right) = 1.$$

Each of the two series being absolutely convergent leads to

$$\sum_{d \geq 1} \frac{\mu(d)}{d^k} = \left(\sum_{d \geq 1} \frac{1}{d^k} \right)^{-1} = \frac{1}{\zeta(k)}. \tag{3.25}$$

Next, we apply Abel's summation again to $V_{j,k+1}$ in order to obtain

$$\begin{aligned}
 V_{j,k+1}(N) &= \frac{1}{N^{k+1}} \sum_{d \leq N} \mu(d) e_p(j\varphi(d)) \\
 &+ (k+1) \int_1^N \left(\sum_{d \leq t} \mu(d) e_p(j\varphi(d)) \right) \frac{dt}{t^{k+2}}.
 \end{aligned}$$

One can easily see that the integral is convergent. Subsequently, we get

$$\int_1^N \left(\sum_{d \leq t} \mu(d) e_p(j\varphi(d)) \right) \frac{dt}{t^{k+2}} = \theta_{j,k+1} + O(N^{-k}), \tag{3.26}$$

with

$$\theta_{j,k+1} = \int_1^{+\infty} \left(\sum_{d \leq t} \mu(d) e_p(j\varphi(d)) \right) \frac{dt}{t^{k+2}},$$

for each $j = 1, \dots, p-1$.

Clearly, we have

$$\frac{1}{N^{k+1}} \sum_{d \leq N} \mu(d) e_p(j\varphi(d)) = O(N^{-k}). \tag{3.27}$$

Considering the identities (3.26) and (3.27) jointly, we find that

$$V_{j,k+1}(N) = (k+1)\theta_{j,k+1} + O(N^{-k}). \tag{3.28}$$

Hence, by going back to (3.26) and putting (3.27) in B ,

$$\begin{aligned}
 B &= \frac{N^{k+1}}{(k+1)p} ((k+1)\theta_{k+1} + O(N^{-k})) \\
 &= \frac{\theta_{k+1}}{p} N^{k+1} + O\left(\frac{N}{p}\right), \tag{3.29}
 \end{aligned}$$

with

$$\theta_{k+1} = \sum_{j=1}^{p-1} e_p(-jb) \theta_{j,k+1}.$$

Finally, gathering (3.23) and (3.29), we reach the desired expression.

3.4. Average Order of the Euler Function

One can easily see that the Euler function is the Jordan totient function for $k=1$. It follows that from Theorem 3.3.

Corollary 3.4. *Let $p \geq \log^8 N$ and $b \in \mathbb{Z}$, then for a sufficiently large N , we have*

$$\sum_{n \leq N} \tilde{\varphi}(n) = \frac{N^2}{p} \left(\frac{3}{\pi^2} + \theta_2 \right) + O(N \log N),$$

where θ_2 defined as in Theorem 3.3 for $k = 1$.

3.5. Average Order of the Pillai Function

Theorem 3.5. *Let $p \geq \log^8 N$ and $b \in \mathbb{Z}$. For a sufficiently large N , we state*

$$\begin{aligned}
 \sum_{n \leq N} \tilde{P}(n) &= \frac{N^2}{p} (\log N + 2\gamma) \left(\theta_2 + \frac{1}{2\zeta(2)} \right) \\
 &- \frac{N^2}{p} \left(\varepsilon + \frac{1}{4} + \frac{\zeta'(2)}{\zeta^2(2)} \right) + O\left(\frac{N^{3/2}}{p}\right),
 \end{aligned}$$

such that θ_2 is defined as in Theorem 3.3 for $k = 1$

$$\varepsilon = \sum_{j=1}^{p-1} e_p(-jb) \int_1^{+\infty} \sum_{d \leq t} \mu(d) e_p(j\varphi(d)) \frac{\log t}{t^3} dt.$$

Proof.

$$\begin{aligned} \sum_{n \leq N} \tilde{P}(n) &= \sum_{n \leq N} \sum_{d|n} \frac{n}{d} \tau\left(\frac{n}{d}\right) \mu(d) \\ &= \sum_{hd \leq N} h \tau(h) \mu(d) \tag{3.30} \\ &= \sum_{\substack{d \leq N \\ \varphi(d) \equiv b \pmod p}} \mu(d) \sum_{\substack{h \leq \frac{N}{d} \\ \varphi(d) \equiv b \pmod p}} h \tau(h). \end{aligned}$$

We need to evaluate the second sum in the product in (3.30). Thanks to Abel's formula, we have

$$\sum_{h \leq v} h \tau(h) = v \sum_{h \leq v} \tau(h) - \int_1^v \sum_{h \leq t} \tau(t) dt.$$

Observing that

$$\sum_{n \leq N} \tau(n) = N \log N + (2\gamma - 1)N + O(\sqrt{N}),$$

we get

$$\begin{aligned} \sum_{h \leq v} h \tau(h) &= v(v \log v + (2\gamma - 1)v + O(\sqrt{v})) \\ &\quad - \int_1^v (t \log t + (2\gamma - 1)t + O(\sqrt{t})) dt \tag{3.31} \\ &= \frac{v^2}{2} \log v + (\gamma - \frac{1}{4})v^2 + O(v^{3/2}). \end{aligned}$$

Therefore, by putting the last equation in (3.30) for $v = \frac{N}{d}$, we obtain

$$\begin{aligned} \sum_{\substack{h \leq \frac{N}{d} \\ \varphi(d) \equiv b \pmod p}} h \tau(h) &= \frac{N^2}{2d^2} \log \frac{N}{d} + (\gamma - \frac{1}{4}) \frac{N^2}{d^2} + O\left(\left(\frac{N}{d}\right)^{3/2}\right) \\ &= \frac{N^2}{2d^2} \left[\log N - \log d + 2\gamma - \frac{1}{2} \right] + O\left(\left(\frac{N}{d}\right)^{3/2}\right). \end{aligned}$$

Now, we use (2.1) and we get

$$\begin{aligned} \sum_{n \leq N} \tilde{P}(n) &= \frac{1}{p} \sum_{d \leq N} \mu(d) \sum_{\substack{h \leq \frac{N}{d} \\ \varphi(d) \equiv b \pmod p}} h \tau(h) \\ &\quad + \frac{1}{p} \sum_{j=1}^{p-1} e_p(-jb) \sum_{d \leq N} \mu(d) e_p(j\varphi(d)) \sum_{\substack{h \leq \frac{N}{d} \\ \varphi(d) \equiv b \pmod p}} h \tau(h) \tag{3.32} \\ &= \frac{1}{p} \sum_{d \leq N} \mu(d) \sum_{\substack{h \leq \frac{N}{d} \\ \varphi(d) \equiv b \pmod p}} h \tau(h) + \frac{1}{p} \sum_{j=1}^{p-1} e_p(-jb) W_j(N) \\ &= \frac{1}{p} (W_1 + W_2). \end{aligned}$$

So on one hand, we have

$$\begin{aligned} W_1 &= \frac{N^2}{2} \sum_{d \leq N} \frac{\mu(d)}{d^2} \left[\log N - \log d + 2\gamma - \frac{1}{2} \right] \\ &\quad + O\left(N^{3/2} \sum_{d \leq N} \frac{\mu(d)}{d^{3/2}}\right) \\ &= \frac{N^2}{2} \log N \sum_{d \leq N} \frac{\mu(d)}{d^2} - \frac{N^2}{2} \sum_{d \leq N} \frac{\log d}{d^2} \mu(d) \\ &\quad + (\gamma - \frac{1}{4}) N^2 \sum_{d \leq N} \frac{\mu(d)}{d^2} + O\left(N^{3/2} \sum_{d \leq N} d^{-3/2}\right) \\ &= \frac{N^2}{2} \left(\log N + 2\gamma - \frac{1}{2} \right) \sum_{d \geq 1} \frac{\mu(d)}{d^2} \\ &\quad + O\left(N^2 (\log N + 2\gamma - \frac{1}{2}) \sum_{d \geq N} \frac{1}{d^2}\right) \\ &\quad - \frac{N^2}{2} \sum_{d \geq 1} \frac{\log d}{d^2} \mu(d) + O\left(N^2 \sum_{d \geq N} \frac{\log d}{d^2}\right) + O(N^{3/2}), \end{aligned}$$

such that the last equation follows from Lemma 2.1.

Going back to (3.25), it gives by differentiation for $Re(k) > 1$,

$$\frac{\zeta'(k)}{\zeta^2(k)} = \sum_{d \geq 1} \frac{\log d}{d^k} \mu(d). \tag{3.33}$$

Afterwards, again by (3.25) and using (3.33), we write

$$\begin{aligned} W_1 &= \frac{N^2}{2\zeta(2)} \left(\log N + 2\gamma - \frac{1}{2} \right) + O\left(N^2 \log NN^{-1}\right) \\ &\quad - \frac{\zeta'(2)}{\zeta^2(2)} \frac{N^2}{2} + O\left(N^2 \frac{\log N}{N}\right) + O(N^{3/2}) \\ &= \frac{N^2}{2\zeta(2)} \left(\log N + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) \tag{3.34} \\ &\quad + O(N \log N) + O(N^{3/2}), \\ &= \frac{N^2}{2\zeta(2)} \left(\log N + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) + O(N^{3/2}), \end{aligned}$$

such that $\frac{\zeta'(2)}{\zeta^2(2)} = \gamma - \log \frac{\mathcal{A}^{12}}{2\pi}$ where $\mathcal{A} \approx 1.2824\dots$ is the

Glaischer-Kinkelin constant (for further details about this see [4,6]).

On the other hand, we have while calling (3.31),

$$\begin{aligned} W_j(N) &= \sum_{d \leq N} \mu(d) e_p(j\varphi(d)) \left(\frac{N^2}{2d^2} \left[\log N - \log d \right] + 2\gamma - \frac{1}{2} \right) + O\left(\frac{N}{d}\right)^{3/2} \tag{3.35} \\ &= \frac{N^2}{2} \left(\log N + 2\gamma - \frac{1}{2} \right) \sum_{d \leq N} \frac{\mu(d)}{d^2} e_p(j\varphi(d)) \\ &\quad - \frac{N^2}{2} \sum_{d \leq N} \mu(d) \frac{\log d}{d^2} e_p(j\varphi(d)) + O\left(N^{3/2} \sum_{d \leq N} \frac{1}{d^{3/2}}\right), \end{aligned}$$

where the error term is equal to $O(N^{3/2})$, thanks to Lemma 2.1.

For the first sum, all we need to do is to appeal the property (3.28) and apply it for $k=1$ so that we get

$$\sum_{d \leq N} \frac{\mu(d)}{d^2} e_p(j\varphi(d)) = 2\theta_{j,2} + O(N^{-1}).$$

Now, for the second sum in (3.35), we use Abel's summation and we obtain

$$\begin{aligned} & \sum_{d \leq N} \mu(d) \frac{\log d}{d^2} e_p(j\varphi(d)) \\ &= \frac{\log N}{N^2} \sum_{d \leq N} \mu(d) e_p(j\varphi(d)) \\ & - \int_1^N \sum_{d \leq t} \mu(d) e_p(j\varphi(d)) \left(\frac{1}{t^3} - 2 \frac{\log t}{t^3} \right) dt. \end{aligned}$$

Since

$$\sum_{d \leq N} \mu(d) e_p(j\varphi(d)) = O(N), \tag{3.36}$$

it follows that

$$\begin{aligned} & \sum_{d \leq N} \mu(d) \frac{\log d}{d^2} e_p(j\varphi(d)) \\ &= - \int_1^N \sum_{d \leq t} \mu(d) e_p(j\varphi(d)) \frac{dt}{t^3} \\ & + 2 \int_1^N \sum_{d \leq t} \mu(d) e_p(j\varphi(d)) \frac{\log t}{t^3} dt \\ & + O\left(\frac{\log N}{N}\right). \end{aligned} \tag{3.37}$$

Appealing the term (3.26) for $k=1$ allows us to find that the first integral in (3.37) is only equal to $\theta_{j,2} + O(N^{-1})$.

As for the second integral in (3.37), we have from (3.36) and since the integral is trivially convergent, it follows that

$$\begin{aligned} & \int_N^{+\infty} \sum_{d \leq t} \mu(d) e_p(j\varphi(d)) \frac{\log t}{t^3} dt \\ &= O\left(\frac{\log N}{N} + \frac{1}{N}\right) \\ &= O\left(\frac{\log N}{N}\right). \end{aligned}$$

Thus,

$$\begin{aligned} & \int_1^N \sum_{d \leq t} \mu(d) e_p(j\varphi(d)) \frac{\log t}{t^3} dt \\ &= \int_1^{+\infty} \sum_{d \leq t} \mu(d) e_p(j\varphi(d)) \frac{\log t}{t^3} dt + O\left(\frac{\log N}{N}\right) \tag{3.38} \\ &= \varepsilon_j + O\left(\frac{\log N}{N}\right). \end{aligned}$$

Collecting (3.37) and (3.38) gives us

$$\begin{aligned} & \sum_{d \leq N} \mu(d) e_p(j\varphi(d)) \frac{\log d}{d^2} \\ &= 2\varepsilon_j - \theta_{j,2} + O\left(\frac{\log N}{N}\right). \end{aligned} \tag{3.39}$$

Hence, we insert (3.39) in (3.35) in order to get

$$\begin{aligned} W_j(N) &= \theta_{j,2} \left(\log N + 2\gamma - \frac{1}{2} \right) N^2 \\ & - \varepsilon_j N^2 + \frac{\theta_{j,2}}{2} N^2 + O(N^{3/2}). \end{aligned}$$

So that we obtain

$$\begin{aligned} W_2 &= \sum_{j=1}^{p-1} e_p(-jb) T_j(N) \\ &= (\theta_2(\log N + 2\gamma) - \varepsilon) N^2 + O(N^{3/2}). \end{aligned} \tag{3.40}$$

At the end, by inserting (3.34) and (3.40) in (3.32), we achieve the desired asymptotic formula.

4. Sums with Congruences with $f(\varphi)$

In this section, we turn our attention to the mean value of the listed functions before with restrictions including the polynomials f from the class $\mathcal{F}_{d,p}$ given at the beginning with $d \geq 2$. In fact, our methods can be used without any further modifications to estimate the similar sums when $\varphi(n) \equiv b \pmod p$. One only needs to use appropriately Theorem B instead of Theorem A when needed.

Theorem 4.1. Let $p \geq \log N$ and $b \in \mathbb{Z}$. Then we have for a sufficiently large N

(1)

$$\begin{aligned} & \sum_{n \leq N} \tilde{\tau}(n) = (\log N + 2\gamma - 1) \frac{N}{p} \\ & + O\left(\frac{N}{p} \left(\log(N) p^{-1/4+\varepsilon} + \omega^{-\omega/4+o(\omega)} \right)\right). \end{aligned}$$

(2)

$$\sum_{n \leq N} \tilde{\sigma}_s(n) = \begin{cases} \frac{\zeta(2)/2 + \tilde{\beta}}{p} N^2 \\ - \frac{N}{p} + O(N \log N), \text{ if } s = 1, \\ \left(\frac{\zeta(s+1)}{p(s+1)} + \frac{\tilde{\beta}}{p} \right) N^{s+1} \\ + O\left(\frac{N}{p} \left(p^{-1/4+\varepsilon} + \left(\frac{\log p}{\log \log p} \right)^{1/2} \right) \right) \\ + O(N^{\max(1,s)}), \text{ if } s > 0, \end{cases}$$

For $s < 0$, we have

$$\sum_{n \leq N} \tilde{\sigma}_s(n) = \begin{cases} \frac{\zeta(1-s)}{p} N + O\left(\frac{N^{s+1}}{p} + p^{-5/4+\varepsilon} N\right), & \text{if } -1 < s < 0, \\ \frac{\zeta(2)}{p} N + O\left(\frac{\log N}{p} + p^{-5/4+\varepsilon} N\right), & \text{if } s = -1, \\ \frac{\zeta(1-s)}{p} N + O\left(p^{-5/4+\varepsilon} N\right), & \text{if } s < -1. \end{cases}$$

(3)

$$\sum_{n \leq N} \tilde{J}_k(n) = \begin{cases} \frac{N^{k+1}}{(k+1)p} \frac{1}{\zeta(k+1)} + \tilde{\theta}_{k+1} + O(N^k), & \text{if } k > 1 \\ \frac{N^2}{2p} \left(\frac{1}{\zeta(2)} + \tilde{\theta}_2\right) + O(N \log N), & \text{if } k = 1. \end{cases}$$

(4) $\sum_{n \leq N} \tilde{\varphi}(n) = \frac{N^2}{p} \left(\frac{3}{\pi^2} + \tilde{\theta}_2\right) + O(N \log N).$

(5) For $p \geq \log N$,

$$\sum_{n \leq N} \tilde{P}(n) = \frac{N^2 \log N}{p} \left(\tilde{\theta}_2 + \frac{1}{2\zeta(2)}\right) + \frac{N^2}{p} \left(\left(2\tilde{\theta}_2 + \frac{1}{\zeta(2)}\right) \gamma - \tilde{\varepsilon} - \frac{1}{4} - \frac{\zeta'(2)}{\zeta^2(2)}\right) + O\left(\frac{N^{3/2}}{p}\right).$$

such that the mentioned constants are defined as follows

$$\tilde{\beta} = \sum_{j=1}^{p-1} e_p(-jb) \int_2^{+\infty} \sum_{d \leq u} e_p(jf(\varphi(d))) \frac{du}{u^{s+2}},$$

$$\tilde{\theta}_k = \sum_{j=1}^{p-1} e_p(-jb) \int_1^{+\infty} \sum_{d \leq t} \mu(d) e_p(jf(\varphi(d))) \frac{dt}{t^{k+2}},$$

$$\tilde{\varepsilon} = \sum_{j=1}^{p-1} e_p(-jb) \int_1^{+\infty} \sum_{d \leq t} \mu(d) e_p(jf(\varphi(d))) \frac{\log t}{t^3} dt.$$

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