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Some Classes of Invariant Submanifolds of LP-Sasakian Manifolds

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Abstract The object of the present paper is to study invariant pseudo parallel submanifolds of a LP-Sasakian manifold and obtain the conditions under which the submanifolds are pseudoparallel, 2-pseudoparallel, generalized pseudoparallel and 2-generalized pseduoparallel. Finally, a non-trivial example is used to demonstrate that the method presented in this paper is effective.

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Keywords: *LP-Sasakian Manifold, Pseudoparallel, 2-pseudoparallel, Ricci-Generalized Pseudoparallel Submanifolds*

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1. Introduction

Invariant submanifolds of a (para) contact metric manifold have been a major area of research for a long time. It helps us to understand several important topics of applied mathematics. For example, in studying non-linear autonomous systems the idea of invariant submanifolds plays an important role [1].

A submanifold of a Riemannian manifold is said to be totally geodesic if every geodesic in that submanifold is also geodesic in the ambient manifold. In [2], Kon proved that invariant submanifolds of a Sasakian manifold are totally geodesic if the second fundamental form of the immersion is covariantly constant. On the other hand, any submanifold of a Kenmotsu manifold is totally geodesic if and only if the second fundamental form of the immersion is covariantly constant [3].

Furthermore, in [4], authors proved that some equivalent conditions of an invariant submanifold of trans-Sasakian to be totally geodesic. Recently, in [5], authors considered invariant submanifolds of (κ,μ)-contact metric manifold and obtained some conditions for an invariant submanifold to be totally geodesic.

In the present paper, we also introduce some new equivalent conditions for an invariant submanifold of a LP-Sasakian manifold to be pseudoparallel.

2. Preliminaries

On the analogy of Sasakian manifolds, Matsumoto introduced the notion of LP-Sasakian manifolds [6]. After then, some properties of the their submanifolds and LP-Sasakian manifolds are also studied many geometers [4,6,8,9,13].

An n-dimensional smooth manifold $\widetilde{M}$ is said to be an Lorentzian para(briefly LP) Sasakian manifold if it admits a (1,1)-type tensor field ϕ, a unit timelike vector field ξ, a 1-form η and a Lorentzian metric tensor g which satisfy;

$$\phi^2X = X+\eta(X)\xi,\ \eta(X) = g(X,\xi),\ \eta(\xi) = -1, \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y)+\eta(X)\eta(Y), \tag{2}$$

$$(\widetilde{\nabla}_X\phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{3}$$

for all $X, Y \in \Gamma(T\widetilde{M})$, where $\widetilde{\nabla}$ and $\Gamma(T\widetilde{M})$denote the Levi-Civita connection and set of the differentiable vector fields on $\widetilde{M}$, respectively.

Moreover, we can easily to see that in an LP-Sasakian manifold,

$$\widetilde{\nabla}_X\xi = \phi X, \tag{4}$$

$$\tilde{R}(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{5}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{6}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y) \tag{7}$$

for all $X, Y \in \Gamma(TM)$, where $\tilde{R}$ and S are the Riemannian curvature tensor and Ricci tensor of $\widetilde{M}$, respectively.

Now, let M be an immersed submanifold of an LP-Sasakian manifold $\widetilde{M}$. Then the Gauss and Weingarten formulae are, respectively, given by

$$\widetilde{\nabla}_XY = \nabla_XY + \sigma(X, Y) \tag{8}$$

and

$$\widetilde{\nabla}_XV = -A_VX + \nabla_X^{\perp}V. \tag{9}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where ∇ is the induced Levi-Civita connection on M, $\nabla^{\perp}$ is the normal connection on the normal bundle $T^{\perp}M$, σ is the second fundamental form of M and A_V is the shape operator with respect to the normal vector field V.

Moreover the shape operator A_V and second fundamental form σ are related by

$$g(\sigma(X, Y), V) = g(A_V X, Y), \quad (10)$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

The covariant derivatives of σ and A_V are defined

$$(\tilde{\nabla}_X \sigma)(Y, Z) = \nabla^{\perp}_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \quad (11)$$

and

$$(\nabla_X A)_V Y = \nabla_X A_V Y - A_{\nabla^{\perp}_X V} Y - A_V \nabla_X Y, \quad (12)$$

for all $X, Y, Z \in \Gamma(TM)$. They are also related

$$g((\tilde{\nabla}_X \sigma)(Y, Z), V) = g((\nabla_X A)_V Y, Z).$$

If $\tilde{\nabla}\sigma = 0$, then the submanifold said to have parallel second fundamental form.

By R, we denote the Riemannian curvature tensor of submanifold M, we have the following Gauss equation;

$$\tilde{R}(X, Y)Z = R(X, Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}Z + (\tilde{\nabla}_X \sigma)(Y, Z) - (\tilde{\nabla}_Y \sigma)(X, Z) \quad (13)$$

for all $X, Y, Z \in \Gamma(TM)$, where if $(\tilde{\nabla}_X \sigma)(Y, Z) - (\tilde{\nabla}_Y \sigma)(X, Z) = 0$, then submanifold is called curvature-invariant submanifold.

For a $(0, k)$-type tensor field T, $k>1$ and a $(0, 2)$-type tensor field A on a Riemannian manifold (M, g), $Q(A,T)$-tensor field is defined by

$$\begin{aligned} &Q(A,T)(X_1, X_2, ..., X_k; X, Y) \\ &= -T((X \wedge_A Y)X_1, X_2, ..., X_k)... \\ &-T(X_1, X_2, ...X_{k-1}, (X \wedge_A Y)X_k), \end{aligned} \quad (14)$$

for all $X_1, X_2, ..., X_k, X, Y \in \Gamma(TM)$, where

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y. \quad (15)$$

In [10], J. Deprez defined the semiparallel immersion in the following way;

$$\begin{aligned} &(\tilde{R}(X, Y) \cdot \sigma)(Z, V) \\ &= R^{\perp}(X, Y)\sigma(Z, V) - \sigma(R(X, Y)Z, V) \\ &-\sigma(Z, R(X, Y)V), \end{aligned} \quad (16)$$

for all $X, Z, U, V \in \Gamma(TM)$, where $R^{\perp}$ denote the Riemannian curvature tensor of the normal bundle $T^{\perp}M$. If $\tilde{R} \cdot \sigma = 0$, then M is said to be semiparallel.

In [11], Authors called pseudoparallel submanifold the satisfying the curvature condition

$$\tilde{R} \cdot \sigma = L_g Q(g, \sigma). \quad (17)$$

On the other hand, C. Murathan, K. Arslan and R. Ezentas defined and studied submanifolds satisfying the condition [12]

$$\tilde{R} \cdot \sigma = L_S Q(S, \sigma). \quad (18)$$

This kind of submanifolds are called generalized Ricci-pseudoparallel, where (18) is defined by

$$\begin{aligned} &R^{\perp}(X,Y)\sigma(U,V) - \sigma(R(X,Y)U,V) - \sigma(U,\sigma(X,Y)V) \\ &= -L_S\{\sigma((X \wedge SY)U, V) + \sigma(U, (X \wedge SY)V)\} \end{aligned}$$

Furthermore, the submanifolds satisfying the condition

$$\tilde{R} \cdot \tilde{\nabla}\sigma = L_{\tilde{\nabla}\sigma} Q(g, \tilde{\nabla}\sigma) \quad (19)$$

are called 2-pseudoparallel [13]. Particularly, $\tilde{R} \cdot \tilde{\nabla}\sigma = 0$, then submanifold is said to be 2-semiparallel.

Also, the submanifolds satisfying the condition

$$\tilde{R} \cdot \tilde{\nabla}\sigma = L_S Q(S, \tilde{\nabla}\sigma) \quad (20)$$

are called 2-generalized Ricci pseudoparallel.

Riemannian curvature tensor satisfies

$$\tilde{R} \cdot \tilde{R} = LQ(g, \tilde{R}). \quad (21)$$

Particularly, if $L = 0$, it is called semisymmetric manifold.

3. Some classes of Invariant Submanifolds of LP-Sasakian Manifolds

Now, let M be an immersed submanifold of an LP-Sasakian manifold manifold $M^n(\phi, \xi, \eta, g)$.

If $\phi(T_x M) \subseteq T_x M$, for each point at $x \in M$, then M is said to be invariant submanifold. We note that all of the properties of an invariant submanifold inherit the ambient manifold.

From (14), we have the following proposition for later use.

Proposition 3.1. *Let M be an invariant submanifold of an LP-Sasakan manifold $M^n(\phi, \xi, \eta, g)$. Then the following equalities hold on M.*

$$\nabla_X \xi = \phi X, \quad (22)$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \quad (23)$$

$$Q\xi = (n-1)\xi, \quad (24)$$

$$S(X,\xi) = (n-1)\eta(X), \quad (25)$$

$$(\nabla_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \quad (26)$$

$$\sigma(\phi X, Y) = \phi\sigma(X,Y) = \sigma(X, \phi Y), \quad (27)$$

$$\sigma(X,\xi) = 0, \quad (28)$$

for all $X, Y \in \Gamma(TM)$.

Next, we will give the main results of this paper.

Theorem 3.2. If an LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$ is a pseudosymmetric, then it is an η-Einstein provided that $L \neq 1$ and $n \neq 3$.

Proof. If $M^n(\phi, \xi, \eta, g)$ is a pseudosymmetric, (21) implies that

$$(\tilde{R}(X, Y) \cdot \tilde{R})(U, V, Z) = LQ(g, \tilde{R})(U, V, Z; X, Y),$$

for all $X, Y, Z, U, V \in \Gamma(T\tilde{M})$. It follows that

$$\begin{aligned} &\tilde{R}(X,Y)\tilde{R}(U,V)Z - \tilde{R}(\tilde{R}(X,Y)U,V)Z \\ &-\tilde{R}(U,(X,Y)V)Z \;\; - \tilde{R}(U,V)\tilde{R}(X,Y)Z \\ &= -L_S \left\{ \begin{aligned} &\tilde{R}((X \wedge_g Y)U, V)Z + \tilde{R}(U, (X \wedge_g Y)V)Z \\ &+\tilde{R}(U,V)(X \wedge_g Y)Z \end{aligned} \right\} \end{aligned} \quad (29)$$

The relation (29) yields for $Y = Z = \xi$,

$$\tilde{R}(U,V)X - g(X,V)U + g(U,X)V = L\left\{\begin{array}{l}\tilde{R}(U,V)X - g(X,V)U + g(X,U)V \\ +g(X,U)\eta(V)\xi - g(X,V)\eta(U)\xi\end{array}\right\}.$$

It follows that

$$\begin{aligned}&g\left(\tilde{R}(U,V)X,Z\right) - g(X,V)g(U,Z) \\ &+g(U,X)g(V,Z) \\ &= L\left\{\begin{array}{l} g\left(\tilde{R}(U,V)X,Z\right) - g(X,V)g(U,Z) \\ +g(X,U)g(V,Z) + g(X,U)\eta(V)\eta(Z) \\ -g(X,V)\eta(U)\eta(Z)\end{array}\right\}\end{aligned} \tag{30}$$

for all $Z \in \Gamma(T\widetilde{M})$. Putting $X = V = e_1, e_2, \dots, e_{n-1}, \xi$ in (30) for orthonormal basis of $\Gamma(T\widetilde{M})$. Then by a straightforward calculations, we obtain

$$(1-L)S(U,Z) = (n-3)\{(1-L)g(U,Z) - L\eta(U)\eta(Z)\},$$

which proves our assertion.

From the Theorem 3.2 and (21), we have following proposition.

Proposition 3.3. *LP-Sasakian manifold $\widetilde{M}^n(\emptyset,\xi,\eta,g)$ is semisymmetric if and only if it has 1-constant sectional curvature.*

Theorem 3.4. *Let M be an invariant pseudoparallel submanifold of an LP-Sasakain manifold $\widetilde{M}^n(\emptyset,\xi,\eta,g)$. Then M is either totally geodesic or $L_g = 1$.*

Proof. Let us assume that M is an invariant pseudoparallel submanifold. Then from (14) and (17), we have

$$(\tilde{R}\cdot\sigma)(X,Y,U,V) = L_g Q(g,\sigma)(U,V;X,Y)$$

for all $X, Y \in \Gamma(TM)$. This implies that

$$\begin{aligned}&R^{\perp}(X,Y)\sigma(U,V) - \sigma(R(X,Y)U,V) \\ &-\sigma(U,R(X,Y)V) \\ &= -L_g\left\{\sigma\left((X\wedge gY)U,V\right) + \sigma\left(U,(X\wedge gY)V\right)\right\} \\ &= -L_g\left\{\begin{array}{l} g(Y,U)\sigma(X,V) - g(X,U)\sigma(Y,V) \\ +g(Y,V)\sigma(U,X) - g(X,V)\sigma(U,Y)\end{array}\right\}.\end{aligned} \tag{31}$$

The relation (31) yields for $V = \xi$,

$$\begin{aligned}&L_g\{\eta(Y)\sigma(U,X) - \eta(X)\sigma(Y,U)\} \\ &= \sigma(R(X,Y)\xi,U) = \sigma(\eta(Y)X - \eta(X)Y,U).\end{aligned}$$

This proves our assertion.

Theorem 3.5. *Let M be an invariant 2-pseudoparallel submanifold of an LP-Sasakain manifold $\widetilde{M}^n(\emptyset,\xi,\eta,g)$. Then M is either totally geodesic or $L_{\tilde{\nabla}\sigma} = 1$.*

Proof. If M is a 2-pseudoparallel, then (14) and (19) imply that

$$\begin{aligned}&\left(\tilde{R}(X,Y)\tilde{\nabla}\sigma\right)(U,V)Z \\ &= L_{\tilde{\nabla}\sigma}Q\left(g,\tilde{\nabla}\sigma\right)(U,V;X,Y)\end{aligned}$$

for all $X, Y, Z, U, V \in \Gamma(TM)$. It follows that

$$\begin{aligned}&R^{\perp}(X,Y)\left(\tilde{\nabla}_U\sigma\right)(V,Z) \\ &-\left(\tilde{\nabla}_{(RX,Y)U}\sigma\right)(V,Z) - (\tilde{\nabla}_U\sigma)(R(X,Y)V,Z) \\ &-(\tilde{\nabla}_U\sigma)(V,R(X,Y)Z) \\ &= -L_{\tilde{\nabla}\sigma}\left\{\begin{array}{l}(\tilde{\nabla}_{(X\wedge_g Y)U}\sigma)(V,Z) \\ +(\tilde{\nabla}_U\sigma)\left(X\wedge_g Y\right)V,Z) \\ +(\tilde{\nabla}_U\sigma)\left(V,\left(X\wedge_g Y\right)Z\right)\end{array}\right\}.\end{aligned} \tag{32}$$

Putting $Y = V = \xi$ in (32) and taking into account of (15), we have

$$\begin{aligned}&R^{\perp}(X,\xi)\left(\tilde{\nabla}_U\sigma\right)(\xi,Z) - \tilde{\nabla}_{R(X,\xi)U}\sigma(\xi,Z) \\ &-\left(\tilde{\nabla}_U\sigma\right)\left(R(X,\xi)\xi,Z\right) - \left(\tilde{\nabla}_U\sigma\right)\left(\xi,R(X,\xi)Z\right) \\ &= -L_{\tilde{\nabla}\sigma}\left\{\begin{array}{l}(\tilde{\nabla}_{(X\wedge_g\xi)U}\sigma)(\xi,Z) \\ +(\tilde{\nabla}_U\sigma)\left(\left(X\wedge_g\xi\right)\xi,Z\right) \\ +(\tilde{\nabla}_U\sigma)\left(\xi,\left(X\wedge_g\xi\right)Z\right)\end{array}\right\}.\end{aligned} \tag{33}$$

Now, Let's calculate each term. Also taking into account that (11) and (28), we obtain

$$\begin{aligned}&R^{\perp}(X,\xi)\left(\tilde{\nabla}_U\sigma\right)(\xi,Z) \\ &= R^{\perp}(X,\xi)\left\{\nabla^{\perp}_U\sigma(\xi,Z) - \sigma(\xi,Z) - \sigma\left(\xi,\nabla_U Z\right)\right\} \\ &= R^{\perp}(X,\xi)\phi\sigma(U,Z).\end{aligned} \tag{34}$$

Making use of (5), we reach at

$$\begin{aligned}&\left(\tilde{\nabla}_{R(X,\xi)U}\sigma\right)(\xi,Z) \\ &= \nabla^{\perp}_{R(X,\xi)U}\sigma(\xi,Z) - \sigma\left(\nabla_{R(X,\xi)U}\xi,Z\right) \\ &\quad -\sigma(\xi,\nabla_{R(X,\xi)U}Z) \\ &= -\sigma(\phi R(X,\xi)U,Z) \\ &= \phi\sigma(g(X,U)\xi - \eta(U)X,Z) \\ &-\eta(U)\phi\sigma(X,Z),\end{aligned} \tag{35}$$

$$\begin{aligned}&\left(\tilde{\nabla}_U\sigma\right)\left(R(X,\xi)\xi,Z\right) = \left(\tilde{\nabla}_U\sigma\right)(-X - \eta(X)\xi,Z) \\ &= \left(\tilde{\nabla}_U\sigma\right)(X,Z) - \left(\tilde{\nabla}_U\sigma\right)\left(\eta(X)\xi,Z\right) \\ &= -\left(\tilde{\nabla}_U\sigma\right)(X,Z) - \nabla^{\perp}_U\sigma\left(\eta(X)\xi,Z\right) \\ &\quad +\sigma\left(\nabla_U\eta(X)\xi,Z\right) + \sigma\left(\eta(X)\xi,\nabla_U Z\right) \\ &= -\left(\tilde{\nabla}_U\sigma\right)(X,Z) + \sigma\left(U\eta(X)\xi + \eta(X)\nabla_U\xi,Z\right) \\ &= -\left(\tilde{\nabla}_U\sigma\right)(X,Z) + \eta(X)\varphi\sigma(U,Z),\end{aligned} \tag{36}$$

$$\begin{aligned}&\left(\tilde{\nabla}_U\sigma\right)\left(\xi,R(X,\xi)Z\right) \\ &= \nabla^{\perp}_U\sigma(\xi,R(X,\xi)Z) - \sigma(\nabla_U\xi,R(X,\xi)Z) \\ &\quad -\sigma(\xi,\nabla_U R(X,\xi)Z) \\ &= -\sigma(\phi U,R(X,\xi)Z) \\ &= -\phi\sigma(U,\eta(Z)X - g(X,Z)\xi) = -\eta(Z)\phi\sigma(U,X),\end{aligned} \tag{37}$$

$$\left(\tilde{\nabla}_{(X\wedge g\xi)U}\sigma\right)(\xi,Z)$$
$$=\nabla^{\perp}_{\left(X\wedge_g Y\right)U}\sigma(\xi,Z)-\sigma(\nabla_{\left(X\wedge_g Y\right)U}\xi,Z)$$
$$-\sigma(\xi,\nabla_{\left(X\wedge_g Y\right)U}Z) \tag{38}$$
$$=-\sigma((X\wedge_g \xi)U,U)$$
$$=-\eta(U)\phi\sigma(X,Z),$$

$$\left(\tilde{\nabla}_{(X\wedge g\xi)}U\sigma)(\xi,Z)\right)$$
$$=\left(\tilde{\nabla}_U\sigma\right)\left(-X-\eta(X)\xi,Z\right)$$
$$=-\left(\tilde{\nabla}_U\sigma\right)(X,Z)-\left(\tilde{\nabla}_U\sigma\right)\left(\eta(X)\xi,Z\right)$$
$$=-\left(\tilde{\nabla}_U\sigma\right)(X,Z)-\nabla^{\perp}_U\sigma\left(\eta(X)\xi,Z\right) \tag{39}$$
$$+\sigma\left(\nabla_U\eta(X)\xi,Z\right)+\sigma\left(\eta(X)\xi,\nabla_U Z\right)$$
$$=-\left(\tilde{\nabla}_U\sigma\right)(X,Z)+\sigma\left(U\eta(X)\xi+\eta(X)\nabla_U\xi,Z\right)$$
$$=-\left(\tilde{\nabla}_U\sigma\right)(X,Z)+\eta(X)\varphi\sigma(U,Z).$$

Finally,

$$\left(\tilde{\nabla}_U\sigma\right)\left(\xi,\left(X\wedge_g \xi\right)Z\right)$$
$$=\nabla^{\perp}_U\sigma(\xi,(X\wedge_g \xi)Z)-\sigma(\xi,(X\wedge_g \xi)Z)$$
$$-\sigma(\xi,(X\wedge_g \xi)Z) \tag{40}$$
$$-\sigma(\varphi U,\eta(Z)X-g(X,Z)\xi)$$
$$=-\eta(Z)\sigma(X,U).$$

If (34), (35), (36), (37), (38), (39) and (40) statements are substituted in (33), we obtain

$$-R^{\perp}(X,\xi)\phi\sigma(U,Z)\eta(U)\phi\sigma(X,Z)+\left(\tilde{\nabla}_U\sigma\right)(X,Z)$$
$$-\eta(X)\phi\sigma(U,Z)+\eta(Z)\phi\sigma(X,U)$$
$$=-L_{\tilde{\nabla}\sigma}\left\{-\eta(U)\varphi\sigma(X,Z)-\left(\tilde{\nabla}_U\sigma\right)(X,Z)\right\} \tag{41}$$
$$+\eta(X)\varphi\sigma(U,Z)-\eta(Z)\varphi\sigma(U,Z).$$

The relation (41) yields for $Z=\xi$,

$$L_{\tilde{\nabla}\sigma}\left\{\left(\tilde{\nabla}_U\sigma\right)(X,\xi)-\sigma\varphi(X,U)\right\}$$
$$=-\sigma\varphi(X,U)+\left(\tilde{\nabla}_U\sigma\right)(X,\xi). \tag{42}$$

On the other hand, by means of (11) and (28), we conclude

$$\left(\tilde{\nabla}_U\sigma\right)(X,\xi)$$
$$=\nabla^{\perp}_U\sigma(X,\xi)-\sigma\left(\nabla_U X,\xi\right)-\sigma(X,\nabla_U\xi) \tag{43}$$
$$=-\sigma(X,\phi U)=-\phi\sigma(X,U).$$

From (42) and (43), we get

$$(L_{\tilde{\nabla}\sigma}-1)\sigma\varphi(X,U)=0$$

which proves our assertion.

Theorem 3.6. *Let M be an invariant Ricci generalized pseudoparallel submanifold of an LP-Sasakian manifold $\widetilde{M}^n$ (φ, η, ξ, g). Then M is either totally geodesic or* $L_S=\dfrac{1}{n-1}$.

Proof. We assume that M is an invariant Ricci-generalized pseudoparallel. Then from (18), we have

$$\left(\tilde{R}(X,Y).\sigma\right)(U,V)=L_S Q(S,\sigma)(U,V;X,Y)$$

for all $X,Y,U,V\in\Gamma(TM)$. (14) and (15) lead to

$$R^{\perp}(X,\ Y)\sigma(U,\ V)-\sigma(R(X,\ Y)U,V)$$
$$-\sigma(U,R(X,\ Y)V) \tag{44}$$
$$=-L_S\left\{\sigma\left(\left(X\wedge SY\right)U,V\right)+\sigma\left(U,\left(X\wedge SY\right)V\right)\right\}.$$

Replacing by $V=\xi$ in (44), we have

$$\sigma(U,R(X,Y)\xi)=L_S\sigma(U,(X\wedge_S Y)\xi)$$
$$\sigma(U,\eta(Y)X-\eta(X)Y)$$
$$=L_S\left\{\sigma(U,S(Y,\xi)X-S(X,\xi)Y)\right\}$$
$$=(n-1)L_S\left\{U,(Y)X-\eta(X)Y\right\}.$$

This completes the proof.

Theorem 3.7. *Let M be an invariant 2-Ricci generalized pseudoparallel submanifold of an LP-Sasakian manifold $\widetilde{M}^n(\varphi,\xi,\eta,g)$ Then M is either totally geodesic or* $L_S=\dfrac{1}{n-1}$.

Proof. Let M be an invariant 2-Ricci generalized pseudoparallel submanifold of an LP-Sasakian manifold $\widetilde{M}^n(\varphi,\xi,\eta,g)$. Then (20) implies that

$$\left(\tilde{R}(X,Y).\tilde{\nabla}\sigma\right)(U,V,Z)=L_S Q\left(S,\tilde{\nabla}\sigma\right)(U,V,Z;X,Y),$$

for all $X,\ Y,\ Z,\ U,\ V\in\Gamma(TM)$. By virtue of (14) and (15), it follows that

$$R^{\perp}(X,Y)\left(\tilde{\nabla}_U\sigma\right)(V,Z)-\left(\tilde{\nabla}_{R(X,Y)U}\sigma\right)(V,Z)$$
$$-(\tilde{\nabla}_U\sigma)(R(X,Y)V,Z)-\left(\tilde{\nabla}_U\sigma\right)(V,R(X,Y)Z)$$
$$=-L_S\left\{\begin{matrix}(\tilde{\nabla}_{(X\wedge SY)}U\sigma)(V,Z)\\+(\tilde{\nabla}_U\sigma)((X\wedge_S Y)V,Z)\\+(\tilde{\nabla}_U\sigma)(V,(X\wedge_S Y)Z)\end{matrix}\right\}.$$

This yields for $X=V=\xi$,

$$R^{\perp}(\xi,Y)(\tilde{\nabla}_U\sigma)(\xi,Z)-(\tilde{\nabla}_{R(\xi,Y)}\sigma)(\xi,Z)$$
$$-(\tilde{\nabla}_U\sigma)(R(\xi,Y)\xi,Z)-\left(\tilde{\nabla}_U\sigma\right)\left(\xi,R(\xi,Y)Z\right) \tag{45}$$
$$=-L_S\left\{\begin{matrix}\left(\tilde{\nabla}_{(\xi\wedge_S Y)U}\sigma\right)(\xi,Z)\\+\left(\tilde{\nabla}_U\sigma\right)\left(\left(\left(\xi\wedge_S Y\right)U\right)\xi,Z\right)\\+\left(\tilde{\nabla}_U\sigma\right)\left(\xi,\left(\xi\wedge_S Y\right)Z\right)\end{matrix}\right\}.$$

Now, let's examine these situation separately. Also, by using of (11), (22) and (28), we obtain

$$R^{\perp}(\xi,Y)\left(\tilde{\nabla}_U\sigma\right)(\xi,Z)$$
$$=R^{\perp}(\xi,Y)\left\{\nabla^{\perp}_U\sigma(\xi,Z)-\sigma(\tilde{\nabla}\xi,Z)-\sigma\left(\xi,\nabla_U Z\right)\right\} \tag{46}$$
$$=R^{\perp}(\xi,Y)\left\{-\sigma(\phi U,Z\right\}$$
$$=-(\xi,Y)\phi\sigma(U,Z).$$

$$(\tilde{\nabla}_{R(\xi,Y)U}\sigma)(\xi,Z)$$
$$=\nabla^{\perp}_{R(\xi,Y)U}\sigma(\xi,Z)-\sigma\left(\nabla_{R(\xi,Y)U}\xi,Z\right)$$
$$=\sigma(\xi,\nabla_{R(\xi,Y)U},Z)=-\sigma(\phi Z,R(\xi,Y)U) \quad (47)$$
$$=-\sigma(\phi Z,g(Y,U)\xi-\eta(U)Y)$$
$$=\eta(U)\phi\sigma(Y,Z).$$

$$\left(\tilde{\nabla}_U\sigma\right)(R(\xi,Y)\xi,Z)=\left(\tilde{\nabla}_U\sigma\right)(\eta(Y)\xi+Y,Z)$$
$$=\left(\tilde{\nabla}_U\sigma\right)(Y,Z)+\left(\tilde{\nabla}_U\sigma\right)(\eta(Y)\xi,Z)$$
$$=\left(\tilde{\nabla}_U\sigma\right)(Y,Z)+\nabla^{\perp}_U\sigma(\eta(Y)\xi,Z) \quad (48)$$
$$-\sigma(\nabla_U\eta(Y)\xi,Z)-\sigma(\eta(Y)\xi,\nabla_U Z)$$
$$=\left(\tilde{\nabla}_U\sigma\right)(Y,Z)-\sigma(U\eta(Y)\xi+\eta(Y)\nabla_U\xi,Z)$$
$$=\left(\tilde{\nabla}_U\sigma\right)(Y,Z)-\eta(Y)\varphi\sigma(U,Z),$$

$$\left(\tilde{\nabla}_U\sigma\right)(\xi,R(\xi,Y)Z)$$
$$=\nabla^{\perp}_U\sigma(\xi,R(\xi,Y)Z)-\sigma(\xi,R(\xi,Y)Z)$$
$$-\sigma(\xi,R(\xi,Y)Z)-\sigma(\phi U,R(\xi,Y)Z) \quad (49)$$
$$=-\phi\sigma(U,g(Y,Z)\xi-\eta(Z)Y)$$
$$=\eta(Z)\phi\sigma(U,Y),$$

$$(\tilde{\nabla}_{(\xi\wedge SY)}U\,\sigma)(\xi,Z)$$
$$=\nabla^{\perp}_{(\xi\wedge_S Y)U}\sigma(\xi,Z)-\sigma(\nabla_{(\xi\wedge_S Y)U}\xi,Z)$$
$$-\sigma(\xi,\nabla_{(\xi\wedge_S Y)U}Z) \quad (50)$$
$$=-\phi\sigma(Z,S(Y,U)\xi-S(\xi,U)Y)$$
$$=(n-1)\eta(U)\phi\sigma(Z,Y),$$

$$\left(\tilde{\nabla}_U\sigma\right)(\xi\wedge_S Y)\xi,Z)$$
$$=\left(\tilde{\nabla}_U\sigma\right)(S(Y,\xi)\xi-S(\xi,\xi)Y,Z)$$
$$=(n-1)\left\{\left(\tilde{\nabla}_U\sigma\right)(Y,Z)+\left(\tilde{\nabla}_U\sigma\right)(\eta(Y)\xi,Z)\right\}$$
$$=(n-1)\begin{Bmatrix}\left(\tilde{\nabla}_U\sigma\right)(Y,Z)-\bar{\sigma}(\eta(Y)\xi,Z)\\-\sigma(\nabla_U\eta(Y)\xi,Z)-\sigma(\eta(Y)\xi,\nabla_U Z)\end{Bmatrix} \quad (51)$$
$$=(n-1)\left\{\left(\tilde{\nabla}_U\sigma\right)(Y,Z)-\sigma(U\eta(Y)\xi+\eta(Y)\nabla_U\xi,Z)\right\}$$
$$=(n-1)\left\{\left(\tilde{\nabla}_U\sigma\right)(Y,Z)-\eta(Y)\varphi\sigma(U,Z)\right\}.$$

Finally,

$$({}_U\sigma)(\xi,(\xi\wedge_S Y)Z)$$
$$=\nabla^{\perp}_U\sigma(\xi,(\xi\wedge_S Y)Z)-\sigma(\nabla_U\xi,(\xi\wedge_S Y)Z)$$
$$-\sigma(\xi,\nabla_U(\xi\wedge_S Y)Z) \quad (52)$$
$$=-\phi\sigma(U,S(Y,Z)\xi-S(\xi,Z)Y)$$
$$=S(\xi,Z)\phi\sigma(U,Y)$$
$$=(n-1)\eta(Z)\phi\sigma(U,Y).$$

Consequently, substituting (46), (47), (48), (49), (50), (51) and (52) into (45), we reach at

$$-R^{\perp}(\xi,Y)\phi\sigma(U,Z)-\eta(U)\phi\sigma(Z,Y)-(\tilde{\nabla}_U\sigma)(Y,Z)$$
$$+\eta(Y)\phi\sigma(Z,U)-\eta(Z)\phi\sigma(U,Y) \quad (53)$$
$$=-(n-1)L_S\begin{Bmatrix}\eta(U)\phi\sigma(Z,Y)+\left(\tilde{\nabla}_U\sigma\right)(Y,Z)\\-\eta(Y)\sigma\varphi(U,Z)+\eta(Z)\sigma\varphi(U,Y)\end{Bmatrix}.$$

Replacing by $Z=\xi$ in (53), we conclude that

$$(n-1)L_S\left\{\left(\tilde{\nabla}_U\sigma\right)(Y,\xi)-\sigma\varphi(U,Y)\right\}$$
$$=\left(\tilde{\nabla}_U\sigma\right)(Y,\xi)-\varphi\sigma(U,Y).$$

By means of (43), we have

$$(n-1)L_S\varphi\sigma(U,Y)=\varphi\sigma(U,Y),$$

which proves our assertion.

Example 3.8. *Let us the 5-dimensional manifold*

$$M^5=\{(x_1,x_2,x_3,x_4,t)\in IR^5 : t\neq 0,\}$$

where (x_i, t) denote the cartesian coordinates in R^5 for $1\le i\le 4$. Then the vector fields

$$e_1=t\frac{\partial}{\partial x_1},e_2=t\frac{\partial}{\partial x_2},e_3=t\frac{\partial}{\partial x_3},$$
$$e_4=t\frac{\partial}{\partial x_4},e_5=-t\frac{\partial}{\partial t}$$

are linearly independent at each point of $\widetilde{M}^5$. By g, we denote the semi-Riemannian metric tensor such that

$$g(e_i,e_i)=1,\quad 1\le i\le 4$$
$$g(e_5,e_5)=-1,$$
$$g(e_i,e_j)=0,\quad 1\le i\ j\le 5.$$

Let η be the 1-form defined by $\eta(X)=g(X,e_5)$ for all X $\epsilon\Gamma(T\widetilde{M})$. Now, we define the tensor field (1,1)-type φ such that

$$\phi e_i=e_i,\ 1\le i\le 4,\ \phi e_5=0.$$

Then we can easily to see that

$$\eta(e_5)=1,\phi^2X=X+\eta(X)\xi,e_5=\xi$$

and

$$g(\phi X,\phi Y)=g(X,Y)+\eta(X)\eta(Y)$$

for all X, Y$\in\Gamma(T\widetilde{M})$. Thus $\widetilde{M}$ (φ,η,ξ,g) defines an almost Lorentzian paracontact metric manifold. By $\tilde{\nabla}$, we denote the Levi-Civita connection on $\widetilde{M}$. Then by direct calculations, we have

$$[e_i,e_5]=e_i,\tilde{\nabla}_{e_i}e_5=e_i,1\le i\le 4,\tilde{\nabla}_{e_i}e_j=0,\ \textit{otherwise},$$

thus one can easily verified

$$\tilde{\nabla}_X e_5=\phi X,$$

this tell us that $\widetilde{M}$ (φ,η,ξ,g) is a LP-Sasakian manifold.

Now, let us a submanifolds M of $\widetilde{M}^5(\phi,\eta,\xi,g)$ is defined by immersion ψ asfollows;

$$\psi(x_1, x_2, x_3, x_4, t) = \left(tx_1, tx_2, tx_3, tx_4, \frac{1}{2}t^2\right),$$

$x_1^2 + x_2^2 = 1$, $x_3^2 + x_4^2 = 1$. Then the tangent space of M is spanned by the vector fields

$$U = x_2e_1 - x_1e_2, V = x_4e_3 - x_3e_4, \xi = e_5.$$

Moreover, we can easily observe that $\phi U = U$ and $\phi V = V$ that is, M is a 3-dimensional invariant submanifold of an LP-Sasakian manifold $\widetilde{M}^5(\varphi, \eta, \xi, g)$. Furthermore, we can easily verifythat

$$\nabla_U \xi = U, \nabla_V \xi = V, \nabla_U V = \nabla_V U = 0.$$

This tell us that M is pseudoparallel, Ricci generalized pseudoparallel submanifold because of it is a totally geodesic submanifold of $\widetilde{M}^5(\phi, \eta, \xi, g)$

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