Turkish Journal of Analysis and Number Theory, 2020, Vol. 8, No. 1, 1-5
Available online at http://pubs.sciepub.com/tjant/8/1/1
Published by Science and Education Publishing
DOI:10.12691/tjant-8-1-1

A Note on Golden Ratio and Higher Order Fibonacci Sequences

Paolo Emilio Ricci*

Mathematics Department, International Telematic University UniNettuno, Roma, Italia
*Corresponding author: paoloemilioricci@gmail.com

Received December 02, 2019; Revised January 10, 2020; Accepted January 19, 2020

Abstract The Lucas formula representing integer powers of the Golden ratio in terms of Fibonacci numbers is derived starting from a general result on matrix powers. The same technique is applied to the Tribonacci sequence, and the extension to higher-order Fibonacci sequences is also considered. It is shown that, by using classical results on matrix theory, the problem can be treated in a general and uniform method.

Keywords: *golden ratio, Fibonacci sequence, Tribonacci numbers, matrix theory*

Cite This Article: Paolo Emilio Ricci, "A Note on Golden RATIO and Higher Order Fibonacci Sequences." *Turkish Journal of Analysis and Number Theory*, vol. 8, no. 1 (2020): 1-5. doi: 10.12691/tjant-8-1-1.

1. Introduction

The Golden ratio *Phi* and Fibonacci's numbers have always fascinated not only mathematicians, but also lovers of nature and fine arts. Countless works have been dedicated to this theme, particularly within the Fibonacci Association, which has contributed to the study of this and other related topics.

It would be impossible to list in the Reference section even the most important articles dedicated to this subject. Although various criticisms [1] have been levelled at M. Livio's book [2], which traces, since ancient times, a systematic use of *Phi* in the construction of buildings, sculpture and paintings, it is undeniable that these ideas are constantly being re-proposed [3].

Links between the Golden ratio and Fibonacci's numbers, usually attributed to J. Kepler, or more recently to É. Lucas, seem to date back to the Indian mathematician Acarya Hemachandra [4], of the 11th Century (see also [5] for historical information about the origin of these numbers, and their connection with the Sanscrit metrics).

The formula that allows the calculation of the powers of *Phi* using the Fibonacci's numbers can be obtained using a companion matrix of the second order. The origin of this formula was discussed in an article by H.W. Gould [6] and the method was later generalized by J. Ivie [7]. One might then wonder what the reason is for returning to this subject. The problem is that, in the above mentioned works, the natural relation between the studied problem and the representation of integral powers of an $r \times r$ matrix by means of a linear combination of the powers of an order not exceeding $r - 1$ is not highlighted. I want to show in what follows how, using this classic result (see e.g F.R. Gantmacher [8]) and previous articles on the subject [9,10], it is possible to recover the known formulas related to the cases of Fibonacci, Tribonacci and, theoretically, also to the case of higher order sequences, even if a complete verification of the general case should be done by using of a computer algebra system. Furthermore, in Sect. 5, another type of representation formula is proposed.

2. Basic Definitions

Definition. Given the $r \times r$, matrix $\mathcal{A} = (a_{ij})$, its *characteristic polynomial* is given by

$$\begin{aligned} P(\lambda) &:= det(\lambda \mathcal{I} - \mathcal{A}) = \\ &= \lambda^r - u_1 \lambda^{r-1} + u_2 \lambda^{r-2} + \ldots + (-1)^r u_r . \end{aligned} \tag{1}$$

and the coefficients

$$\begin{cases} u_1 := tr\,\mathcal{A} = a_{11} + a_{22} + \ldots + a_{rr} \\ u_2 := \displaystyle\sum_{i<j}^{1,r} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ u_3 := \displaystyle\sum_{i<j<k}^{1,r} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix} \\ \ldots \\ u_r := det\,\mathcal{A} \end{cases} \tag{2}$$

are called the *invariants* of $\mathcal{A}$.

2.1. Recalling the $F_{k,n}$ Functions

Consider the $(r+1)$ -terms homogeneous linear bilateral recurrence relation with complex constant coefficients $u_k \quad (k = 1, 2, \ldots, r)$, with $u_r \neq 0$:

$$X_n = u_1 X_{n-1} - u_2 X_{n-2} + \ldots \\ \ldots + (-1)^{r-1} u_r X_{n-r}, \; (n \in \mathbf{Z}). \tag{3}$$

Its solution is given by every bilateral sequence $\{X_n(u_1, u_2, \ldots, u_r)\}_{n \in \mathbf{Z}}$ such that $r+1$ consecutive terms satisfy equation (3).

A basis for the r-dimensional vectorial space $\mathcal{V}_r$ of solutions isgiven by the functions $F_{k,n} = F_{k,n}(u_1, u_2, \ldots, u_r)$, $(k = 1, 2, \ldots, r, \; n \geq -1)$, defined by the initial conditions below:

$$\begin{aligned} &F_{1,-1} = 0 \quad F_{1,0} = 0 \; \ldots \; F_{1,r-2} = 1, \\ &F_{2,-1} = 0 \quad F_{2,0} = 1 \ldots \; F_{2,r-2} = 0, \\ &F_{r,-1} = 1 \quad F_{r,0} = 0 \; \ldots \; F_{r,r-2} = 0. \end{aligned} \tag{4}$$

Since $u_r \neq 0$, the $F_{k,n}$ functions can be defined even if $n < -1$, by means of the positions:

$$\begin{aligned} &F_{k,n}(u_1, \ldots, u_r) \\ &= F_{r-k+1,-n+r-3}\left(\frac{u_{r-1}}{u_r}, \ldots, \frac{u_1}{u_r}, \frac{1}{u_r}\right), \\ &(k = 1, \ldots, r; \; n \in \mathbf{Z}) \end{aligned} \tag{5}$$

Therefore, assuming the initial conditions $X_j, X_{j+1}, \ldots, X_{j+r-1}$, the *general solution* of the recurrence (3), $\forall n \in \mathbf{Z}$, is given by

$$X_{n+j+1} = X_j F_{r,n} + X_{j+1} F_{r-1,n} + \ldots \\ \ldots + X_{j+r-1} F_{1,n}. \quad (n \in \mathbf{Z}). \tag{6}$$

Remark 2.1. The $F_{k,n}$ functions constitute a different basis with respect to che usual one, which uses the roots of the characteristic equation [11]. This basis does not imply the knowledge of roots and does not depend on their multiplicity, so that it is sometimes more convenient.

It has been show by É Lucas [10,12] that all $\{F_{k,n}\}_{n \in \mathbf{Z}}$ functions are expressed through the only bilateral sequence $\{F_{1,n}\}_{n \in \mathbf{Z}}$ the bilateral sequence $\{F_{1,n}\}_{n \in \mathbf{Z}}$, corresponding to the initial conditions:

$$F_{1,-1} = F_{1,0} = \ldots = F_{1,r-3} = 0, F_{1,r-2} = 1. \tag{7}$$

More precisely, the following equations hold

$$\begin{cases} F_{1,n} = u_1 F_{1,n-1} + F_{2,n-1} \\ F_{2,n} = -u_2 F_{1,n-1} + F_{3,n-1} \\ \cdots \\ F_{r-1,n} = (-1)^{r-2} u_{r-1} F_{1,n-1} + F_{r,n-1} \\ F_{r,n} = (-1)^{r-1} u_r F_{1,n-1} \end{cases} \tag{8}$$

Therefore, the bilateral sequence $\{F_{1,n}\}_{n \in \mathbf{Z}}$ called called the *fundamental* solution of (3) (*"function fondamentale"* by É. Lucas [12]).

The functions $F_{1,n}(u_1, \ldots, u_r)$ are called in literature [13] *generalized Lucas polynomials of the second kind* (in r variables), and are related to the multivariate Chebyshev polynomials (see e.g. R. Lidl - C. Wells [14], R. Lidl [15], T. Koornwinder [16,17], M. Bruschi - P.E. Ricci [18], K.B. Dunn - R. Lidl [19], R.J. Beerends [20]).

2.2. Matrix Powers Representation

In preceding articles [9,10], the following result is proved:

Theorem 2.2. – *Given an $r \times r$ matrix $\mathcal{A}$, and denoting by equation (1) its characteristic polynomial, the matrix powers $\mathcal{A}_n$, with integer exponent n, are given by the equation:*

$$\mathcal{A}^n = F_{1,n-1}(u_1, \ldots, u_r)\mathcal{A}^{r-1} \\ + F_{2,n-1}(u_1, \ldots, u_r)\mathcal{A}^{r-2} \\ + \ldots + F_{r,n-1}(u_1, \ldots, u_r)\mathcal{I}, \tag{9}$$

where the functions $F_{k,n}(u_1, \ldots, u_r)$ are defined in Section 2.1.

Moreover, if $\mathcal{A}$ is not singular, i.e. $u_r \neq 0$, equation (9) still works for negative integers n, assuming the definition (5) for the $F_{k,n}$ functions.

2.3. A Special Companion Matrix

In connection with higher order Fibonacci-type sequences, J. Ivie [7] considered the $r \times r$ companion matrix $\mathcal{Q}$:

$$\mathcal{Q} := \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{10}$$

which is a particular case of the general companion matrix associated with higher-order linear recurrence relations.

The invariants of the matrix (10) are given by

$$u_1 = 1, u_2 = -1, u_3 = 1, \cdots, u_r = (-1)^r,$$

so that its characteristic polynomial is

$$det(\lambda\mathcal{I} - \mathcal{Q}) = \lambda^r - \lambda^{r-1} - \lambda^{r-2} - \ldots - \lambda - 1, \tag{11}$$

and its highest eigenvalue is a solution of the equation

$$\lambda^r = \lambda^{r-1} + \lambda^{r-2} + \ldots + \lambda + 1. \tag{12}$$

By Lagrange or Lagrange's or Cauchy's bounds for the roots of polynomials [21] it immediately follows that, in case of the equation (12), the highest eigenvalue is bounded by 2, for every r. Actually: for $r = 2$ the highest positive eigenvalue is the Golden ratio:

$$\Phi = \frac{1+\sqrt{5}}{2} \simeq 1.61803...,$$

for $r = 3$ the Tribonacci constant is:

$$T = \frac{1}{3}\left[1+(19-3\sqrt{33})^{1/3}+(19+3\sqrt{33})^{1/3}\right] \simeq 1.83929...$$

Denoting by $\Lambda_{[r]}$ the highest eigenvalue of equation (12), we have, of course: $\Lambda_{[2]} = \Phi$, $\Lambda_{[3]} = T$. In general, this eigenvalue can be computed numerically and, by using a computer algebra system, it results that apparently the sequence $\{\Lambda_{[r]}\}$ is monotonically increasing, so that it tends to its upper bound:

$$\lim_{r\to\infty} \Lambda_{[r]} = 2. \qquad (13)$$

We find, for example:

$$\Lambda_{[4]} = 1.9276,\ \Lambda_{[5]} = 1.96595,\ \Lambda_{[6]} = 1.98358,$$
$$\Lambda_{[7]} = 1.99196,\ \Lambda_{[8]} = 1.99603,\ \Lambda_{[9]} = 1.99803,$$
$$\Lambda_{[10]} = 1.99902.$$

3. Recalling a Lucas' Formula

Probably it was É. Lucas who first discovered the equation

$$\Phi^n = f_n\Phi + f_{n-1}, \qquad (14)$$

relating the powers of the Golden ratio with the classical sequence of Fibonacci numbers $\{f_n\}$, ($f_{n+1} = f_n + f_{n-1}$, with $f_0 = 0, f_1 = 1$).

Remark 3.1. Note that the definition of the Fibonacci sequence is sometimes started form the initial conditions $f_0 = 1,\quad f_1 = 1$, which simply implies a shift of the index.

Remark 3.2. For $n = 2$ equation (14) gives: $\Phi^2 = \Phi + 1$ (as it was first noticed by J. Kepler). Starting form this equation it is easily seen that the two geometric progressions:

$$1,\Phi,\Phi^2,\Phi^3,\Phi^4,\ldots \quad \text{and}$$
$$1,(1-\Phi),(1-\Phi)^2,(1-\Phi)^3,(1-\Phi)^4,\ldots,$$

are both solution of the characteristic equation, so that by Binet's formula, the link between the Golden ratio Φ and the Fiboncci numbers immediately follows:

$$f_n = \frac{\Phi^n - (1-\Phi)^n}{\sqrt{5}}, \quad (n \geq 0).$$

Another approach for finding the representation (14), is the use of the so-called $\mathcal{Q}_{2\times 2}$ -matrix:

$$\mathcal{Q} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (15)$$

which gives:

$$\mathcal{Q}^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}. \qquad (16)$$

This method is discussed in [6] (see also the references therein), and it is easily recovered in what follows.

The invariants of the matrix (15) are $u_1 = 1, u_2 = -1$, so that, by equation (9) we have:

$$\mathcal{Q}^n = F_{1,n-1}(1,-1)\mathcal{Q} + F_{2,n-1}(1,-1)\mathcal{I}, \qquad (17)$$

but the functions $F_{1,n-1}$ and $F_{2,n-1}$ satisfy the recursion (3), with $r = 2$, and $u_1 = 1$, $u_2 = 0$, i.e. the Fibonacci recursion:

$$F_{h,n}(1,-1) = F_{h,n-1}(1,-1) + F_{h,n-2}(1,-1),\ (h = 1, 2), \qquad (18)$$

with initial values:

$$F_{1,-1}(1,-1) = 0,\ F_{1,0}(1,-1) = 1,$$
$$F_{1,1}(1,-1) = u_1 = 1, \qquad (19)$$

$$F_{2,-1}(1,-1) = 1,\ F_{2,0}(1,-1) = 0,$$
$$F_{2,1}(1,-1) = 1, \qquad (20)$$

and therefore

$$F_{1,n-1}(1,-1) = f_n,\ F_{2,n-1}(1,-1) = f_{n-1}, \qquad (21)$$

(i.e. the Fibonacci numbers). Then, recalling equation (17), we find

$$\mathcal{Q}^n = f_n\mathcal{Q} + f_{n-1}\mathcal{I}, \qquad (22)$$

and equation (14) follows.

4. Extension of Lucas' Formula

Although the proof in Sect. 3 seems to be a little bit involved, it has the advantage to be based on a general result about powers of matrices (Theorem 2.2). Therefore Lucas' formula (14) can be theoretically extended to the general case.

We start with the Tribonacci case, for which the following result holds.

Theorem 4.1. - *Given the Tribonacci sequence of numbers* $\{t_n\}$, $(t_{n+1} = t_n + t_{n-1} + t_{n-2}$, *with* $t_0 = 0,\ t_1 = 0,\ t_2 = 1)$ *and introducing the* $\mathcal{Q}_{3\times 3}$ *-matrix*

$$\mathcal{Q} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad (23)$$

the integral powers of $\mathcal{Q}$ (for $n \geq 2$) are represented by

$$\mathcal{Q}^n = \begin{pmatrix} t_{n+2} & t_{n+1} & t_n \\ t_{n+1}+t_n & t_n + t_{n-1} & t_{n-1}+t_{n-2} \\ t_{n+1} & t_n & t_{n-1} \end{pmatrix}. \qquad (24)$$

Note that the Fibonacci sequence we are considering here is defined in The Encyclopedia of Integer Sequences [22] under A000073: {0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, ...}.

Proof. −*Note that, putting* $r = 3$ *and taking into account that* $u_1 = 1, u_2 = -1, u_3 = 1$, *the recursion (3) for the* $F_{1,n-1}$ *functions is given by*

$$F_{1,n-1}(1,-1,1) = F_{1,n-2}(1,-1,1)+F_{1,n-3}(1,-1,1)+F_{1,n-4}(1,-1,1), \quad (25)$$

with initial conditions $F_{1,-1}=F_{1,0}=0, F_{1,1}=1$, so that $F_{1,n-1}(1,-1,1)=t_{n-1}$. The same result holds for the $F_{3,n-1}$ functions, since we can put the shifted initial conditions $F_{3,0}=F_{3,1}=0, F_{3,2}=1$, so that $F_{3,n-1}(1,-1,1)=t_{n-2}$.

For the $F_{2,n}$ functions we must recall the first equation in (8), which gives:

$$F_{2,n-1}(1,-1,1)=F_{1,n}(1,-1,1)-F_{1,n-1}(1,-1,1),$$

and therefore, $F_{2,n-1}(1,-1,1)=t_n-t_{n-1}$.

As a consequence we find:

$$\mathcal{Q}^n = t_{n-1}\mathcal{Q}^2+(t_n-t_{n-1})\mathcal{Q}+t_{n-2}\mathcal{I}, \quad (26)$$

which gives

$$\mathcal{Q}^n = \begin{pmatrix} t_{n+2} & t_{n+1} & t_n \\ t_{n+3}-t_{n+2} & t_{n+2}-t_{n+1} & t_{n+1}-t_n \\ t_{n+1} & t_n & t_{n-1}.. \end{pmatrix}. \quad (27)$$

This equation is equivalent to (24), owing the Tribonacci recursion.

Remark 4.2. The sum of Tribonacci numbers appearing in the second line of the matrix (24) depends on the initial conditions assumed before. Denoting by t_n^* the Tribonacci numbers defined by $t_{n+1}=t_n+t_{n-1}+t_{n-2}$, with initial conditions: $t_0=0, t_1=1, t_2=0$, equation (24) can be written in a form more similar to the equation (16) of the Fibonacci case, that is:

$$\mathcal{Q}^n = \begin{pmatrix} t_{n+2} & t_{n+1} & t_n \\ t^*_{n+2} & t^*_{n+1} & t^*_n \\ t_{n+1} & t_n & t_{n-1} \end{pmatrix}.$$

The Tribonacci-type sequence $\{t_n^*\}$ appears in The Encyclopedia of Integer Sequences [22] under A001590: {0, 1, 0, 1, 2, 3, 6, 11, 20, 37, 68, 125, 230, ...}.

The general result, extending equation (24), has been proved by J. Ivie [7], by using induction. It is as follows:

Theorem 4.3. *Consider the Fibonacci-type sequence of order r, defined by the recursion:*

$$\varphi_n = \varphi_{n-1}+\varphi_{n-2}+\ldots\varphi_{n-r}, \quad (28)$$

with initial conditions:

$$\varphi_0=\varphi_1=\ldots=\varphi_{n-r-1}=0, \varphi_{n-r}=1. \quad (29)$$

The powers of the $\mathcal{Q}_{r\times r}$ matrix (10) are represented, in terms of the sequence $\{\varphi_n\}$ by means of the equation:

$$\mathcal{Q}^n = \begin{pmatrix} \varphi_{n+r-1} & \varphi_{n+r-2} & \cdots & \varphi_n \\ \sum_{k=0}^{r-2}\varphi_{n+r-2-k} & \sum_{k=0}^{r-2}\varphi_{n+r-3-k} & \cdots & \sum_{k=0}^{r-2}\varphi_{n-k-1} \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_{n+r-2} & \varphi_{n+r-3} & \cdots & \varphi_{n-1} \end{pmatrix}. \quad (30)$$

The same equation can be proved by using by the matrix technique, recalled in the preceding sections, and the $F_{k,n}$ functions computed at the point $(\mathbf{u}) := (1,-1,1,\ldots,(-1)^r)$.

It will be necessary to take into account the recurrence relation (3), the initial conditions (4) and Lucas' formulas (8).

Consequently, we can proclaim that equation (30) is a natural consequence of the representation

$$\mathcal{Q}^n = F_{1,n-1}(\mathbf{u})\mathcal{Q}^{r-1}+F_{2,n-1}(\mathbf{u})\mathcal{Q}^{r-1}+\cdots \cdots+F_{r,n-1}(\mathbf{u})\mathcal{I}, \quad (31)$$

where, according to (8), we have:

$$\begin{cases} F_{2,n-1}(\mathbf{u})=F_{1,n}(\mathbf{u})-F_{1,n-1}(\mathbf{u}) \\ F_{3,n-1}(\mathbf{u})=F_{1,n+1}(\mathbf{u})-F_{1,n}(\mathbf{u})-F_{1,n-1}(\mathbf{u}) \\ \ldots \\ F_{r,n-1}(\mathbf{u})=-F_{1,n-2}(\mathbf{u}), \end{cases} \quad (32)$$

5. A General Result

Consider again the general Fibonacci-type sequence of order r, defined by the recursion (28), and put the initial conditions:

$$\begin{aligned} &(1): \varphi_0=\varphi_1=\ldots=\varphi_{n-r-1}=0, \varphi_{n-r}=1, \\ &(2): \varphi_0=\varphi_1=\ldots=\varphi_{n-r-1}=1, \varphi_{n-r}=0, \\ &\ldots \\ &(\mathrm{r}): \varphi_0=1, \ \varphi_1=\ldots=\varphi_{n-r-1}=\varphi_{n-r}=0, \end{aligned} \quad (33)$$

and denote respectively by

$$\{\varphi_n^{(k)}\}, \ (k=1,2,\ldots,r)$$

the generalized Fibonacci-type sequences starting with the k-th initial conditions in equation (33). Then, equation (30) could be replaced by

$$\mathcal{Q} = \begin{pmatrix} \varphi^{(1)}_{n+r-1} & \varphi^{(1)}_{n+r-2} & \cdots & \varphi^{(1)}_n \\ \varphi^{(2)}_{n+r-1} & \varphi^{(2)}_{n+r-2} & \cdots & \varphi^{(2)}_n \\ \vdots & \vdots & \cdots & \vdots \\ \varphi^{(r-1)}_{n+r-1} & \varphi^{(r-1)}_{n+r-2} & \cdots & \varphi^{(r-1)}_n \\ \varphi^{(1)}_{n+r-2} & \varphi^{(1)}_{n+r-3} & \cdots & \varphi^{(1)}_{n-1} \end{pmatrix}, \quad (34)$$

Equation (34) has been checked for r = 3, 4, 5, but should be proved for general r.

Remark 5.1. The result of this article could be extended to the case of Fibonacci, Tribonacciand higher order Fibonacci polynomials, as it will be done in a later article.

6. Conclusion

By using a classical result about a representation formula for matrix powers [8], and the basic solution of a linear recurrence relation, it has been shown that the Lucas formula for powers of the Golden ratio in terms of

Fibonacci numbers can be easily recovered. Furthermore, the used technique have been extended to the case of the Tribonacci sequence, and can be theoretically applied to the general case of higher order Fibonacci sequences. It can be noted that the considered method could be useful for analyzing general number sequences defined by linear recurrence relations with constant coefficients. This could be possibly done in a subsequent investigation.

References

[1] G. Markowsky, *The Golden Ratio* – Book review, Notices of the A.M.S., 52 (3), 2005, 344-347.

[2] M. Livio, *The Golden Ratio: The story of Phi, the extraordinary mumber of nature, art and beauty, Broadway Books*, New York, 2003.

[3] G.B. Meisner, *The Golden Ratio: The Divine Beauty of Mathematics*, Race Point Publ., New York, 2018.

[4] D. Perkins, *φ, π, e* & , MAAA Press, Washington D.C., 2017.

[5] P. Singh, The so-called Fibonacci numbers in ancient and medieval India, *Historia Math.*, 12, (1985), 229-244.

[6] H.W. Gould, A history of the Fibonacci *Q*matrix and a higher-dimensional problem, *The Fibonacci Quart.*, 19 1981, 250-257.

[7] J. Ivie, A general *Q*-matrix, *Fibonacci Quart.*, 10 (1972), No. 3, 255-261, 264.

[8] F.R. Gantmacher, *The Theory of Matrices*, Chelsea Pub. Co, New York, 1959.

[9] M. Bruschi, P.E. Ricci, An explicit formulafor $f(\mathcal{A})$ and the generating function of the generalized Lucas polynomials, *Siam J.Math. Anal.*, 13, (1982), 162-165.

[10] P.E. Ricci, Sulle potenze di una matrice, *Rend. Mat.* (6) 9 (1976), 179-194.

[11] J.L. Brenner, Linear recurrence relations, *Amer. Math. Monthly*, 61 (1954), 171-173.

[12] É. Lucas, *Théorie des Nombres*, Gauthier-Villars, Paris, 1891.

[13] I.V.V. Raghavacharyulu, A.R. Tekumalla, Solution of the Difference Equations of Generalized Lucas Polynomials, *J. Math. Phys.*, 13 (1972), 321-324.

[14] R. Lidl, C. Wells, Chebyshev polynomials in several variables, *J. Reine Angew. Math.*, 255, (1972), 104-111.

[15] R. Lidl, Tschebyscheffpolynome in mehreren variabelen, *J. Reine Angew. Math.*, 273, (1975), 178-198.

[16] T.H. Koornwinder, Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent partial differential operators, I-II, *Kon. Ned. Akad. Wet.* Ser. A, 77, 46-66.

[17] T.H. Koornwinder, Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent partial differential operators, III-IV, *Indag. Math.*, 36. (1974), 357-381.

[18] M. Bruschi, P.E. Ricci, I polinomi di Lucas e di Tchebycheff in pi`u variabili, *Rend. Mat.*, S. VI, 13, (1980), 507-530.

[19] K.B. Dunn, R. Lidl, Multi-dimensional generalizations of the Chebyshev polynomials, I-II, *Proc. Japan Acad*, 56 (1980), 154-165.

[20] R.J. Beerends, Chebyshev polynomials in several variables and the radial part of the Laplace-Beltrami operator, *Trans. Am. Math. Soc.*, 328, 2, 1991, 779-814.

[21] H.P. Hirst, W.T. Macey, Bounding the Roots of Polynomials, *The College Math. J.*, 28 (4) (1997), 292-295.

[22] N.J.A. Sloane, S. Plouffe, *The Encyclopedia of Integer Sequences*, Academic Press, San Diego, 1995.

[23] E.P. Miles, Jr., Generalized Fibonacci numbers and associated matrices, *Amer. Math.Monthly*, 67 (1960), 745-752.

[24] R.A. Rosenbaum, An application of matrices to linear recursion relations, *Amer. Math. Monthly*, 66 (1959), 792-793.