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On the Hermite-Based Poly-Genocchi Polynomials with a q-Parameter

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Abstract Many kinds of generalizations of these polynomials and numbers have been presented in the literature (see [1,2]). Bayad and Hamahata in [1] introduced and investigated the poly-Bernoulli polynomials and proved some relations for these polynomials. Cenkci and Kamatsu in [3] are defined q-parameter poly-Bernoulli numbers. They proved some relations between these polynomials. Kim et al. in ([4,5]) defined poly-Genocchi polynomials and gave the some properties for the multiple-poly-Bernoulli numbers and multiple-zeta values. We introduce the Hermite-based poly-Genocchi polynomials with a q-parameter. After, we give and investigate some properties and identities for these polynomials. Furthermore, we prove closed formula and two explicit relations.

Keywords: *the Genocchi polynomials and numbers, the 2-variable Hermite-Kampé de Feriét polynomials, the polylogarithm function, the poly-Genocchi polynomials*

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1. Introduction

As usual, throughout this paper, N denotes the set of natural numbers, N_0 denotes the set of nonnegative integers, Z denotes the set of integer numbers, R denotes the set of real numbers and C denotes the complex numbers.

In the usual notations, let $B_n(x)$, $E_n(x)$ and $G_n(x)$ denote respectively, the classical Bernoulli polynomials, the classical Euler polynomials and the classical Genocchi polynomials defined by the following generating functions;

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t - 1}e^{xt}, |t| < 2\pi \tag{1}$$

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t + 1}e^{xt}, |t| < \pi \tag{2}$$

and

$$\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} = \frac{2t}{e^t + 1}e^{xt}, |t| < \pi. \tag{3}$$

Also, let x=0,

$B_n(0) = B_n, E_n(0) = E_n \text{ and } G_n(0) = G_n$. Where B_n, E_n and G_n are respectively, the classical Bernoulli numbers, the classical Euler numbers and the classical Genoccchi numbers.

The 2-variable Hermite-Kampé de Feriét polynomials are defined by (see [6,7,8])

$$\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} = e^{xt+yt^2} \tag{4}$$

Let $k \in Z, k > 1$, the k-th polylogarithm function is defined by (see [1,6,9])

$$Li_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}, z \in C, k > 1 \tag{5}$$

when k=1, $Li_1(z) = -\log(1-z)$. In the case $k \le 0$, $Li_k(z)$ are the rational functions:

$$Li_0(z) = \frac{z}{1-z}, Li_{-1}(z) = \frac{z}{(1-z)^2}, \cdots.$$

Further information about poly-logarithm function and polynomials (see [1,4]).

Cenkci et al. in [3] defined the weighted Stirling numbers of the second kind as

$$\frac{e^{xt}\left(e^t - 1\right)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k,x)\frac{t^n}{n!} \tag{6}$$

Duran et al. in [6] defined the Hermite-based λ-Stirling polynomials of the second kind as

$$\frac{\left(\lambda e^t - 1\right)^m}{m!} e^{xt+yt^j} = \sum_{n=0}^{\infty} S_2^{(\lambda,j)}(n,m,x,y)\frac{t^n}{n!}. \tag{7}$$

The special values of the (6) are given in [6]. Let $n, k \in Z, n \ge 0, k > 0 \text{ and } q \in R \setminus \{0\}$. We define the

Hermite-based poly-Genocchi polynomials with a q-parameter by the following generating functions:

$$\sum_{n=0}^{\infty} {}_H G_{n,q}^{(k)}(x,y)\frac{t^n}{n!} = \frac{2qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{e^{qt}+1} e^{xt+yt^2}. \quad (8)$$

For x=y=0, we get ${}_H G_{n,q}^{(k)}(0,0) = {}_H G_{n,q}^{(k)}$ which is called a new class of the Hermite-based poly-Genocchi number with a q-parameter. Some special cases of ${}_H G_{n,q}^{(k)}(x,y)$ are following remarks.

Remark. For y=0, we have ${}_H G_{n,q}^{(k)}(x,0) = {}_H G_{n,q}^{(k)}(x)$ called the Hermite-based poly-Genocchi polynomials with a q-parameter.

Remark. For q=1, ${}_H G_{n,q}^{(k)}(x,y)$ reduces to the Hermite-based poly-Genocchi polynomials.

Remark. For q=1 and y=0, ${}_H G_{n,q}^{(k)}(x,y)$ reduces to the poly-Genocchi polynomials.

Remark. When q=k=1 and y=0, we obtain the classical Genocchi polynomials.

Srivastava and Srivastava et al. in ([10,11]) investigated some properties and proved some theorems for the Bernoulli, Euler and Genocchi polynomials. D. S. Kim et al. in ([12,13]) introduced the poly-Bernoulli polynomials and gave some recurrences relations and idebtities. Cenkci et al. in [3] gave the poly-Bernoulli polynomials with a q-parameter. Kurt [7] gave the poly-Genocchi polynomials with a q-parameter. Duran et al. in ([6,14]) considered the (p,q)-Hermite polynomials and the (p,q)-Euler polynomials.

2. Main Theorems

In this section, we give some basic identites and relations for the Hermite-based poly-Genocchi polynomial with a q-parameter. Further we give closed formula and explicit relation for these polynomials.

Theorem. The Hermite-based poly-Genocchi polynomials with a q-parameter satisfy following relation:

$${}_H G_{n,q}^{(k)}(x,y) = \sum_{m=0}^{n}\binom{n}{m}{}_H G_{m,q}^{(k)} H_{n-m}(x,y),$$

$${}_H G_{n,q}^{(k)}(x_1+x_2, y_1+y_2) = \sum_{m=0}^{n}\binom{n}{m}{}_H G_{m,q}^{(k)}(x_1,y_1) H_{n-m}(x_2,y_2)$$

and

$${}_H G_{n,q}^{(k)}(x,y) = n!\sum_{m=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\frac{{}_H G_{n-2m,q}^{(k)}(x)}{(n-2m)!m!}y^m.$$

The proof of this theorem is easily obtain from the Hermite-based poly-Genocchi polynomials definition.

Theorem. The following relation holds true:

$$\begin{aligned}&{}_H G_{n,q}^{(k)}(x,y) + \sum_{v=0}^{n}\binom{n}{v}q^{n-v}{}_H G_{v,q}^{(k)}(x,y)\\ &= 2\sum_{n=0}^{\infty}\frac{1}{q^m(m+1)^k}\sum_{r=0}^{m+1}\binom{m+1}{r}(-1)^r H_n(x-qr,y).\end{aligned} \quad (9)$$

Proof. By (4) and (8), we write as

$$\sum_{n=0}^{\infty}{}_H G_{n,q}^{(k)}(x,y)\frac{t^n}{n!}\left(e^{qt}+1\right) = 2qLi_k\left(\frac{1-e^{-qt}}{q}\right)e^{xt+yt^2}$$

L. H. S of this equation is

$$\sum_{n=0}^{\infty}\left\{\begin{aligned}&{}_H G_{n,q}^{(k)}(x,y)\\ &+\sum_{v=0}^{n}\binom{n}{v}q^{n-v}{}_H G_{v,q}^{(k)}(x,y)\end{aligned}\right\}\frac{t^n}{n!} \quad (10)$$

R. H. S of this equation is

$$\begin{aligned}&= 2q\sum_{m=0}^{\infty}\frac{\left(1-e^{-qt}\right)^{m+1}}{q^{m+1}}\frac{1}{(m+1)^k}\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!}\\ &= 2\sum_{m=0}^{\infty}\frac{1}{q^{m+1}(m+1)^k}\sum_{r=0}^{m+1}\binom{m+1}{r}(-1)^r e^{t(x-qr)+yt^2}\\ &= 2\sum_{n=0}^{\infty}\left\{\begin{aligned}&\sum_{m=0}^{\infty}\frac{1}{q^m(m+1)^k}\\ &\sum_{r=0}^{m+1}\binom{m+1}{r}(-1)^r H_n(x-qr,y)\end{aligned}\right\}\frac{t^n}{n!}\end{aligned} \quad (11)$$

From (10) and (11), we obtain (9).

Theorem. There is the following relation between the Hermite-based poly-Genocchi polynomials with a q-parameter and the Euler polynomials:

$$\begin{aligned}&{}_H G_{n,q}^{(k)}(x,y)\\ &= \frac{1}{2}\sum_{m}^{n}\binom{n}{m}_H G_{m,q}^{(k)}(0,y)q^{n-m}\left(\begin{aligned}&E_{n-m}\left(\frac{x}{q}+1\right)\\ &+E_{n-m}\left(\frac{x}{q}\right)\end{aligned}\right)\end{aligned} \quad (12)$$

Proof. By (2) and (8), we write

$$\begin{aligned}&\sum_{n=0}^{\infty}{}_H G_{n,q}^{(k)}(x,y)\frac{t^n}{n!}\\ &= \frac{2qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{e^{qt}+1}e^{yt^2}\frac{e^{qt}+1}{2}\frac{2}{e^{qt}+1}e^{\frac{x}{q}qt}\\ &= \frac{1}{2}\left\{\sum_{m=0}^{\infty}{}_H G_{m,q}^{(k)}(0,y)\frac{t^m}{m!}\sum_{l=0}^{\infty}E_l\left(\frac{x}{q}+1\right)\frac{q^l t^l}{l!}\right.\\ &\left.+\sum_{m=0}^{\infty}{}_H G_{m,q}^{(k)}(0,y)\frac{t^m}{m!}\sum_{l=0}^{\infty}E_l\left(\frac{x}{q}\right)\frac{q^l t^l}{l!}\right\}\end{aligned}$$

By using Cauchy product and comparing the coefficients of $\frac{t^n}{n!}$, we have (12).

Theorem. There is the following relation between the Hermite-based poly-Genocchi polynomials with a q-parameter and the Bernoulli polynomials:

$$ {}_H G_{n-1,q}^{(k)}(x,y) = \frac{1}{n}\sum_m^n \binom{n}{m} {}_H G_{m,q}^{(k)}(0,y) q^{n-m} \begin{pmatrix} B_{n-m}\left(\frac{x}{q}+1\right) \\ -B_{n-m}\left(\frac{x}{q}\right) \end{pmatrix} \quad (13)$$

Proof. From (1) and (8), we write as

$$\sum_{n=0}^{\infty} {}_H G_{n,q}^{(k)}(x,y)\frac{t^n}{n!} = \frac{1}{t}\frac{2qe^{yt^2} Li_k\left(\frac{1-e^{-qt}}{q}\right)}{e^{qt}+1}\frac{e^{qt}-1}{q}\frac{q}{e^{qt}-1}e^{\frac{x}{q}qt}$$

$$= \frac{1}{qt}\left\{ \begin{array}{l} \frac{2qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{e^{qt}+1}e^{yt^2}\frac{qte^{qt\left(\frac{x}{q}+1\right)}}{e^{qt}-1} \\ -\frac{2qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{e^{qt}+1}e^{yt^2}\frac{qte^{qt\left(\frac{x}{q}\right)}}{e^{qt}-1} \end{array} \right\}$$

$$\sum_{n=0}^{\infty} {}_H G_{n,q}^{(k)}(x,y)\frac{t^{n+1}}{n!} = \frac{1}{q}\left\{ \begin{array}{l} \sum_{m=0}^{\infty} {}_H G_{m,q}^{(k)}(0,y)\frac{t^m}{m!}\sum_{l=0}^{\infty} B_l\left(\frac{x}{q}+1\right)\frac{q^l t^l}{l!} \\ -\sum_{m=0}^{\infty} {}_H G_{m,q}^{(k)}(0,y)\frac{t^m}{m!}\sum_{l=0}^{\infty} B_l\left(\frac{x}{q}\right)\frac{q^l t^l}{l!} \end{array} \right\}.$$

By using Cauchy product and comparing the coefficients of $\frac{t^n}{n!}$, we have (13).

Theorem. The following relation holds true:

$$ {}_H G_{n,q}^{(k)}(x,y) = 2\sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^k}\sum_{r=0}^{m+1}\binom{m+1}{r} \times (-1)^{r+s} H_n(x+qs-qr,y) \quad (14)$$

Proof. By (8),

$$\sum_{n=0}^{\infty} {}_H G_{n,q}^{(k)}(x,y)\frac{t^n}{n!} = \frac{2qe^{xt+yt^2}}{e^{qt}+1}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\frac{\left(1-e^{-qt}\right)^{m+1}}{q^{m+1}}$$

$$= 2\sum_{s=0}^{\infty}(-1)^s e^{qts}e^{xt+yt^2}\sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^k} \times \sum_{r=0}^{m+1}\binom{m+1}{r}(-1)^r e^{-qrt}$$

$$= 2\sum_{n=0}^{\infty}\left\{ \begin{array}{l} \sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^k}\sum_{r=0}^{m+1}\binom{m+1}{r} \\ \times(-1)^{r+s} H_n(x+qs-qr,y) \end{array} \right\}\frac{t^n}{n!}.$$

From here, we have (14).

Theorem. There is following relationship between the Hermite-based poly-Genocchi polynomials with a q-parameter and the Stirling numbers of the second kind as:

$$ {}_H G_{n,q}^{(k)}(x,y) + \sum_{m=0}^{n}\binom{n}{m}q^{n-m} {}_H G_{m,q}^{(k)}(x,y) = 2q\sum_{m=0}^{\infty}\frac{m!(-1)^{m+1}}{(m+1)^k}\sum_{r=0}^{n}\binom{n}{r}(-q)^r H_r(x,y) \quad (15)$$
$$\left\{S_2(r,m,1) - S_2(r,m)\right\}$$

Proof. By (16) and (8), we write as:

$$\sum_{n=0}^{\infty} {}_H G_{n,q}^{(k)}(x,y)\frac{t^n}{n!} + e^{qt}\sum_{n=0}^{\infty} {}_H G_{n,q}^{(k)}(x,y)\frac{t^n}{n!} = 2qLi_k\left(\frac{1-e^{-qt}}{q}\right)e^{xt+yt^2}.$$

L. H. S. of this equation is

$$\sum_{n=0}^{\infty}\left\{ \begin{array}{l} {}_H G_{n,q}^{(k)}(x,y) \\ +\sum_{m=0}^{n}\binom{n}{m}q^{n-m} {}_H G_{m,q}^{(k)}(x,y) \end{array} \right\}\frac{t^n}{n!} \quad (16)$$

R. H. S. of this equation

$$= 2q\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(m+1)^k}\left(e^{-qt}-1\right)^{m+1}\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}$$

$$= 2q\sum_{m=0}^{\infty} m!\frac{(-1)^{m+1}}{(m+1)^k}\frac{\left(e^{-qt}-1\right)^m e^{-qt}}{m!}\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}$$

$$-2q\sum_{m=0}^{\infty} m!\frac{(-1)^{m+1}}{(m+1)^k}\frac{\left(e^{-qt}-1\right)^m}{m!}\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}$$

$$= 2q\sum_{m=0}^{\infty}\frac{m!(-1)^{m+1}}{(m+1)^k}\sum_{r=0}^{\infty} S_2(r,m,1)\frac{(-qt)^r}{r!} \times \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}$$

$$-2q\sum_{m=0}^{\infty}\frac{m!(-1)^{m+1}}{(m+1)^k}\sum_{r=0}^{\infty} S_2(r,m)\frac{(-qt)^r}{r!}$$

$$\times\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!}.$$

By using Cauchy product, we have

$$=\sum_{n=0}^{\infty}\left\{\begin{array}{l}2q\sum_{m=0}^{\infty}\frac{m!(-1)^{m+1}}{(m+1)^k}\sum_{r=0}^{n}\binom{n}{r}(-q)^r H_r(x,y)\\ \times\left(S_2(r,m,1)-S_2(r,m)\right)\frac{t^n}{n!}\end{array}\right\}. \quad (17)$$

From (16) and (17), we get (15).

Theorem. The following relation holds true:

$${}_HG_{n+m,q}^{(k)}(x,y)=\left(\begin{array}{l}\sum_{p=0}^{n}\sum_{r=0}^{m}\binom{n}{p}\binom{m}{r}\\ (x-v)^{p+r}{}_HG_{n+m-p-r,q}^{(k)}(0,y)\end{array}\right) \quad (18)$$

Proof. By (8),

$$\sum_{n=0}^{\infty}{}_HG_{n,q}^{(k)}(x,y)\frac{t^n}{n!}=\frac{2qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{e^{qt}+1}e^{xt+yt^2}. \quad (19)$$

We replace t by t+u in (19)

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}{}_HG_{n+m,q}^{(k)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!}$$
$$=\frac{2qLi_k\left(\frac{1-e^{-q(t+u)}}{q}\right)}{1+e^{q(t+u)}}e^{x(t+u)+y(t+u)^2}.$$

From this equation, we write as

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}{}_HG_{n+m,q}^{(k)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!}e^{-x(t+u)}$$
$$=\frac{2qLi_k\left(\frac{1-e^{-q(t+u)}}{q}\right)}{1+e^{q(t+u)}}e^{y(t+u)^2}. \quad (20)$$

In the last equation, we replace x by v, we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}{}_HG_{n+m,q}^{(k)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!}e^{-v(t+u)}$$
$$=\frac{2qLi_k\left(\frac{1-e^{-q(t+u)}}{q}\right)}{1+e^{q(t+u)}}e^{y(t+u)^2}. \quad (21)$$

By (12) and (13), we write

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}{}_HG_{n+m,q}^{(k)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!}$$
$$=e^{(v-x)(t+u)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}{}_HG_{n+m,q}^{(k)}(v,y)\frac{t^n}{n!}\frac{u^m}{m!}. \quad (22)$$

Now, by applying the following known series identity [[19], p. 52, Eq. 1. 6 (2)]

$$\sum_{N=0}^{\infty}f(N)\frac{(x+y)^N}{N!}=\sum_{n,m=0}^{\infty}f(n+m)\frac{x^n}{n!}\frac{y^m}{m!} \quad (23)$$

in the right hand side of (23), we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}{}_HG_{n+m,q}^{(k)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!}$$
$$\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}(x-v)^{p+r}\frac{t^p}{p!}\frac{u^r}{r!} \quad (24)$$
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}{}_HG_{n+m,q}^{(k)}(v,y)\frac{t^n}{n!}\frac{u^m}{m!}$$

Finally, upon first replacing n by n-p and m by m-r by using the Cauchy product in the left hand side of the above equation (24) and comparing the coefficients of $\frac{t^n}{n!}$ and $\frac{u^m}{m!}$ On both sides of the resulting equation, we have (18).

Theorem (Closed Formula) The following relation holds true:

$${}_HG_{n,q}^{(-k)}(x,y)$$
$$=2\sum_{l=0}^{\infty}(-1)^l\sum_{m=0}^{\min(n,k)}(m!)^2S_2^{q^{-1}}(k,m,1)q^n \quad (25)$$
$$\times\left\{\begin{array}{l}S_2^{(1,2)}\left(n,m;\frac{x}{q}+1+l,\frac{y}{q^2}\right)\\ -S_2^{(1,2)}\left(n,m;\frac{x}{q}+l,\frac{y}{q^2}\right)\end{array}\right\}.$$

Proof. By replacing k by (-k) in (8), we get

$$\sum_{n=0}^{\infty}{}_HG_{n,q}^{(-k)}(x,y)\frac{t^n}{n!}=\frac{2qLi_{-k}\left(\frac{1-e^{-qt}}{q}\right)}{e^{qt}+1}e^{xt+yt^2}$$

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}{}_HG_{n,q}^{(-k)}(x,y)\frac{t^n}{n!}\frac{u^k}{k!}$$
$$=\sum_{k=0}^{\infty}\frac{2q}{e^{qt}+1}\sum_{m=0}^{\infty}\left(\frac{1-e^{-qt}}{q}\right)^{m+1}(m+1)^k e^{xt+yt^2}\frac{u^k}{k!}$$
$$=\frac{2q}{e^{qt}+1}e^{xt+yt^2}\left(\frac{1-e^{-qt}}{q}\right)e^u\sum_{m=0}^{\infty}\left(\left(\frac{1-e^{-qt}}{q}\right)e^u\right)^m$$
$$=\frac{2e^{xt+yt^2}}{e^{qt}+1}\left(1-e^{-qt}\right)e^u\frac{e^{qt}}{1-\left(e^{qt}-1\right)\left(q^{-1}e^u-1\right)}$$
$$=\left\{\begin{array}{l}2\sum_{l=0}^{\infty}(-1)^l e^{qlt}\left(e^{qt}-1\right)\\ \sum_{m=0}^{\infty}\left(e^{qt}-1\right)^m e^u\left(q^{-1}e^u-1\right)^m e^{xt+yt^2}\end{array}\right\}$$

$$= 2\sum_{l=0}^{\infty}(-1)^l \sum_{m=0}^{\infty} e^{(x+q+ql)t+yt^2}\left(e^{qt}-1\right)^m e^u\left(q^{-1}e^u-1\right)^m$$

$$-\sum_{m=0}^{\infty} e^{(x+ql)t+yt^2}\left(e^{qt}-1\right)^m e^u\left(q^{-1}e^u-1\right)^m \quad (26)$$

For λ=1 and j=2 in (8), we get

$$\sum_{n=0}^{\infty} S_2^{(1,2)}(n,m;x,y)\frac{t^n}{n!} = \frac{\left(e^t-1\right)^m}{m!}e^{xt+yt^2}. \quad (27)$$

We put the equation (8) and (27) in (26). We have

$$= 2\sum_{l=0}^{\infty}(-1)^l \left\{ \begin{array}{l} \sum_{m=0}^{\infty}\left[m!\sum_{n=0}^{\infty} S_2^{(1,2)}\left(\begin{array}{c} n,m; \\ \frac{x}{q}+1+l, \frac{y}{q^2} \end{array} \right) q^n \frac{t^n}{n!} \right] \\ \left[m!\sum_{k=0}^{\infty}(k,m,1)\frac{u^k}{k!} \right] \\ -\sum_{m=0}^{\infty}\left[m!\sum_{n=0}^{\infty} S_2^{(1,2)}\left(\begin{array}{c} n,m; \\ \frac{x}{q}+l, \frac{y}{q^2} \end{array} \right) q^n \frac{t^n}{n!} \right] \\ \left[m!\sum_{k=0}^{\infty} S_2^{q^{-1}}(k,m,1)\frac{u^k}{k!} \right] \end{array} \right\}$$

From the last equation, comparing the coefficients of $\frac{t^n}{n!}$ and $\frac{u^k}{k!}$, we have (25).

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References

[1] A. Bayad and Y. Hamahata, "Polylogarithms and poly-Bernoulli polynomials," Kyushu. J. Math., 65, pp. 15-34, 2011.

[2] R. Sanchez-Peregrino, "Closed formula for Poly-Bernoulli numbers," Fibonacci Quart., 40, pp. 362-364, 2002.

[3] M. Cenkci and T. Komatsu, "Poly-Bernoulli numbers and polynomials with a q-parameter," J. Number Theory, 152, pp. 38-54, 2015.

[4] T. Kim, S. Y. Jang and J. J. Seo, "A note on poly-Genocchi numbers and polynomials," Appl. Math. Sci., 18(96), 4775-4781, 2014.

[5] T. Kim, "Multiple zeta values and their application," Lecture Notes in Number Theory, 31-95, 1996.

[6] U. Duran, M Acikgoz and S. Araci, "Hermite based poly-Bernoulli polynomials with q-parameter," Advanced Stud. In Comptem., 28(2), pp. 285-296, 2018.

[7] B. Kurt, "Identities and relation on the Poly-Genocchi polynomials with a q-parameter," J. Inequa. Special Func., 9(1), pp. 1-8, 2018.

[8] M. A. Ozarslan, Hermite-based unified Apostol-Bernoulli, Euler and Genocchi polynomials," Advances in Diff. Equa., 2013.2013.1847.

[9] Y. Hamahata, "Poly-Euler polynomials and Arakawa-Kaneko type zeta functions," Functione et. App. Commentarii Mathematica, 51(1), pp. 7-27, 2014.

[10] H. M. Srivastava, "Some generalization and basic (or q-) extensition of the Bernoulli, Euler and Genocchi polynomials," App. Math. Information Sci., 5, pp. 390-444, 2011.

[11] H. M. Srivastava and J. Choi, "Series associated with the zeta and related functions," Kluver Academic Pub., Dordrect, Boston and London, 2001.

[12] D. S. Kim and T. Kim, "Higher-order Frobenius-Euler and Poly-Bernoulli mixed-type polynomials," Advances in Diff. Equa., 2013.2013.251.

[13] D. S. Kim and T. Kim, "Higher order Bernoulli and poly-Bernoulli mixed type polynomials," Georgian Math. J., 22, pp. 265-272, 2015.

[14] U. Duran and M. Acikgoz, "On (p,q)-Euler numbers and polynomials associated with (p,q)-Volkenborn integral," Int. J. of Number Theory, 14(1), pp. 241-253, 2018.

[15] A. Bayad and T. Kim, "Higher recurrences for Apostol-Bernoulli-Euler polynomials," Russ. J. of Math. Phys., 19(1), pp. 1-10, 2012.

[16] U. Duran, M Acikgoz, A. Esi and S. Araci, "A note on the (p,q)-Hermite polynomials," App. Math. And Information Sciences, 12, pp. 227-231, 2018.

[17] H. Jolany, R. B. Corcino and T. Komatsu, "More properties on multi-Poly-Euler polynomials," Bull. Soc. Math. Mex., 21, pp. 149-162, 2015.

[18] D. S. Kim and T. Kim, Hermite and Poly-Bernoulli mixed type polynomials," Advances in Diff. Equa., 2013. 2013. 343.

[19] D. S. Kim, T. Kim, D. V. Dolgy and S. H. Rim, "Some new identities of Bernoulli, Euler and Hermite polynomials arising from umbral calculus," Advances in Diff. Equa., 2013. 2013. 73.

[20] D. S. Kim and T. Kim, "Some identities of Frobenius-Euler polynomials arising from umbral calculus," Advances in Diff. Equa., 2012. 2012. 196.

[21] D. S. Kim and T. Kim, "A note on poly-Bernoulli and higher order poly-Bernoulli polynomials," Russ. J. Math., 22(1), pp. 26-33, 2015.

[22] B. Kurt and Y. Simsek, "On the generalized Apostol-type Frobenius-Euler polynomials," Advances in Diff. Equa., 2013. 2013. 1.

[23] H. M. Srivastava and H. L. Manocho, "A Treatise on generating functions," Halsted Press, John Willey and sums, New York, Chichester, Brisbane and Toronto, 1984.