

Hermite-Hadamard Type Inequalities for Preinvex Functions via Right Riemann-Liouville Fractional Integrals

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Abstract In this paper, with a new approach, some new Hermite-Hadamard type inequalities for preinvex functions are obtained by using only the right Riemann-Liouville fractional integrals. Our results generalize previous studies. Results proved in this paper may stimulate further research in this field.

Keywords: Preinvex functions, Hermite-Hadamard inequalities, right Riemann-Liouville fractional integral

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1. Introduction

Let I be a finite interval of real numbers. A function $f: I \rightarrow \mathbb{R}$ is said to be convex if the inequality

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$$

holds for $a, b \in I$ and $t \in [0,1]$ (see [1]).

Convexity plays a fundamental role in many branches of pure and applied sciences. In recent years, the investigation on extended convex functions has become a deep research area. The applications of extended convex functions in establishing various inequalities have received renewed attention by many researchers. A significant generalization of convex functions is that of invex functions introduced by Hanson [2]. Ben-Israel and Mond [3] introduced the notions of invex sets and preinvex functions. Pini [4] introduced the notion of prequasiinvex functions which is a generalization of invex functions. Weir and Mond [5] and Noor [6] have studied the basic properties of the preinvex functions. For recent applications and generalizations of the preinvex functions, see [7-14] and references therein.

Inequalities have a key role in pure and applied mathematics. A number of studies have shown that the theory of convexity has a closely relationship with the theory of inequalities. One of the most famous inequality in the literature for convex functions is known as Hermite-Hadamard integral inequality. This double inequality is stated as:

Let f be a convex function on $[a, b] \subset \mathbb{R}$. Then the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}, a, b \in I.$$

Hermite-Hadamard inequality for convex functions has attracted many researchers and as gradually a remarkable of generalizations and extensions in various directions have appeared in the literature, one can see [15-24] and references therein.

2. Preliminaries

Let us recall some definitions and known results concerning invexity and preinvexity.

Definition 1. [25] A set $K \subseteq \mathbb{R}$ is said to be invex if there exist a function $\eta: K \times K \rightarrow \mathbb{R}$ such that

$$a + t\eta(b, a) \in K, \forall a, b \in K, t \in [0,1].$$

The invex set K is also called a η -connected set.

Definition 2. [5] Let f be a function on the invex set K . Then, f is said to be preinvex with respect to η , if

$$f(a + t\eta(b, a)) \leq (1-t)f(a) + tf(b),$$

$$\forall a, b \in K, t \in [0,1].$$

It is to be noted that every convex function is preinvex with respect to the map $\eta(b, a) = b - a$, but the converse is not true in general.

In [26] Noor obtained the following Hermite-Hadamard inequalities for the preinvex functions.

Theorem 1. Let $f: K = [a + \eta(b, a)] \rightarrow (0, \infty)$ be a preinvex function on the interval of real numbers K° and $a, b \in K^\circ$ with $a < a + \eta(b, a)$. Then the following inequality holds:

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Following definitions of the left and right side Riemann-Liouville fractional integrals are well known in the literature.

Definition 3. [27] Let $f \in L[a, b]$. The left and right Riemann-Liouville fractional integrals $J_{a^+}^\alpha f$ and $J_{b^-}^\alpha f$ of order $\alpha > 0$ are defined by

$$J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, x > a$$

and

$$J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$.

In [22], Sarikaya et al. gave the fractional analogue of the inequality (1) as follows:

Theorem 2. Let $f: [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] \leq \frac{f(a)+f(b)}{2}$$

with $\alpha > 0$.

In [28], Iscan proved the following Hermite-Hadamard inequalities for preinvex functions via Riemann-Liouville fractional integrals:

Theorem 3. Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta: K \times K \rightarrow \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b, a)$. If $f: [a, a + \eta(b, a)] \rightarrow (0, \infty)$ is a preinvex function, $f \in L[a, a + \eta(b, a)]$, then the following inequalities for fractional integrals hold:

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)} \left[J_{a^+}^\alpha f(a+\eta(b,a)) + J_{(a+\eta(b,a))^-}^\alpha f(a) \right] \leq \frac{f(a)+f(a+\eta(b,a))}{2}$$

with $\alpha > 0$.

3. Main Results

In this section, we will obtain some generalizations of the right side of the Hermite-Hadamard type inequalities for functions whose first derivatives absolute values are preinvex via right Riemann-Liouville fractional integrals.

Lemma 1. Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta: K \times K \rightarrow \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b, a)$. Suppose $f: K \rightarrow \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a + \eta(b, a)])$. Then the following equality for the right Riemann-Liouville fractional integrals holds:

$$\frac{f(a) + \alpha f(a + \eta(b, a))}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{\eta^\alpha(b, a)} J_{(a+\eta(b,a))^-}^\alpha f(a) = \frac{\eta(b, a)}{\alpha + 1} \int_0^1 [(\alpha + 1)t^\alpha - 1] f'(a + t\eta(b, a)) dt.$$

Proof. If we apply the partial integration to the right hand side of the above equality, we have

$$\begin{aligned} & \frac{\eta(b, a)}{\alpha + 1} \int_0^1 [(\alpha + 1)t^\alpha - 1] f'(a + t\eta(b, a)) dt \\ &= \eta(b, a) \left[\int_0^1 t^\alpha f'(a + t\eta(b, a)) dt - \frac{1}{\alpha + 1} \int_0^1 f'(a + t\eta(b, a)) dt \right] \\ &= \eta(b, a) \left[t^\alpha \frac{f(a + t\eta(b, a))}{\eta(b, a)} \Big|_0^1 - \frac{1}{\alpha + 1} \int_0^1 \frac{f(a + t\eta(b, a))}{\eta(b, a)} dt \right] \\ &= f(a + \eta(b, a)) - \frac{\Gamma(\alpha + 1)}{\eta^\alpha(b, a)} J_{(a+\eta(b,a))^-}^\alpha f(a) \\ &= \frac{f(a + \eta(b, a)) - f(a)}{\alpha + 1} \\ &= \frac{f(a) + \alpha f(a + \eta(b, a))}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{\eta^\alpha(b, a)} J_{(a+\eta(b,a))^-}^\alpha f(a). \end{aligned}$$

This completes the proof.

Remark 1. In Lemma 1,

1. If we take $\eta(b, a) = b - a$, then we get the inequality given in [[17], Lemma 3.1].

2. If we take $\eta(b, a) = b - a$ and $\alpha = 1$, then we get the inequality given in [[20], Lemma A].

Theorem 4. Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta: K \times K \rightarrow \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b, a)$. Suppose $f: K \rightarrow \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a + \eta(b, a)])$. If $|f'|$ is preinvex on $[a, a + \eta(b, a)]$, then the following Riemann-Liouville fractional integral inequality holds:

$$\left| \frac{f(a) + \alpha f(a + \eta(b, a))}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{\eta^\alpha(b, a)} J_{(a+\eta(b,a))^-}^\alpha f(a) \right| \leq \frac{\eta(b, a)}{\alpha + 1} \left[\begin{aligned} & M_1(\alpha) |f'(a)| + M_2(\alpha) |f'(b)| \\ & + M_3(\alpha) |f'(a)| + M_4(\alpha) |f'(b)| \end{aligned} \right]$$

Where

$$M_1(\alpha) = \frac{1}{\int_0^1 \frac{1}{\sqrt[\alpha]{\alpha+1}} (1-(\alpha+1)t^\alpha)(1-t) dt}$$

$$= \frac{\alpha \left(2(\alpha+2)(\alpha+1)^{\frac{1}{\alpha}-1} - 1 - (\alpha+1)^{\frac{2}{\alpha}} \right)}{2(\alpha+2)(\alpha+1)^{\frac{2}{\alpha}}},$$

$$M_2(\alpha) = \frac{1}{\int_0^1 \frac{1}{\sqrt[\alpha]{\alpha+1}} (1-(\alpha+1)t^\alpha) t dt} = \frac{\alpha}{2(\alpha+2)(\alpha+1)^{\frac{2}{\alpha}}},$$

$$M_3(\alpha) = \frac{1}{\int_0^1 \frac{1}{\sqrt[\alpha]{\alpha+1}} ((\alpha+1)t^\alpha - 1)(1-t) dt}$$

$$= \frac{\alpha \left(2(\alpha+2)(\alpha+1)^{\frac{1}{\alpha}-1} - 1 - (\alpha+1)^{\frac{2}{\alpha}} \right)}{2(\alpha+2)(\alpha+1)^{\frac{2}{\alpha}}},$$

$$M_4(\alpha) = \frac{1}{\int_0^1 \frac{1}{\sqrt[\alpha]{\alpha+1}} ((\alpha+1)t^\alpha - 1) t dt} = \frac{\alpha \left((\alpha+1)^{\frac{2}{\alpha}} + 1 \right)}{2(\alpha+2)(\alpha+1)^{\frac{2}{\alpha}}},$$

with $\alpha > 0$.

Proof. Using Lemma 1 and the preinvexity of $|f'|$, we have

$$\left| \frac{f(a) + \alpha f(a + \eta(b, a))}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{\eta^\alpha(b, a)} J_{(a + \eta(b, a))^-}^\alpha f(a) \right|$$

$$\leq \frac{\eta(b, a)}{\alpha + 1} \int_0^1 |1 - (\alpha + 1)t^\alpha| |f'(a + t\eta(b, a))| dt$$

$$\leq \frac{\eta(b, a)}{\alpha + 1} \left[\int_0^1 \frac{1}{\sqrt[\alpha]{\alpha+1}} (1 - (\alpha + 1)t^\alpha) [(1-t)|f'(a)| + t|f'(b)|] dt \right.$$

$$\left. + \int_0^1 \frac{1}{\sqrt[\alpha]{\alpha+1}} ((\alpha + 1)t^\alpha - 1) [(1-t)|f'(a)| + t|f'(b)|] dt \right]$$

$$\leq \frac{\eta(b, a)}{\alpha + 1} \left[\frac{\alpha \left(2(\alpha + 2)(\alpha + 1)^{\frac{1}{\alpha}-1} - 1 \right)}{2(\alpha + 2)(\alpha + 1)^{\frac{2}{\alpha}}} |f'(a)| \right.$$

$$\left. + \frac{\alpha}{2(\alpha + 2)(\alpha + 1)^{\frac{2}{\alpha}}} |f'(b)| \right]$$

$$+ \frac{\alpha \left(2(\alpha + 2)(\alpha + 1)^{\frac{1}{\alpha}-1} - 1 - (\alpha + 1)^{\frac{2}{\alpha}} \right)}{2(\alpha + 2)(\alpha + 1)^{\frac{2}{\alpha}}} |f'(a)|$$

$$\left. + \frac{\alpha \left((\alpha + 1)^{\frac{2}{\alpha}} + 1 \right)}{2(\alpha + 2)(\alpha + 1)^{\frac{2}{\alpha}}} |f'(b)| \right].$$

So, the proof is completed.

Remark 2. In Theorem 4,

1. If we take $\eta(b, a) = b - a$, then we get the inequality given in [[17], Theorem 4.1].

2. If we take $\eta(b, a) = b - a$ and $\alpha = 1$, then we get the inequality given in [[29], Theorem 2.2].

Theorem 5. Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta: K \times K \rightarrow \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b, a)$. Suppose $f: K \rightarrow \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a + \eta(b, a)])$. If $|f'|^q$ is preinvex on $[a, a + \eta(b, a)]$ for $q \geq 1$, then the following Riemann-Liouville fractional integral inequality holds:

$$\left| \frac{f(a) + \alpha f(a + \eta(b, a))}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{\eta^\alpha(b, a)} J_{(a + \eta(b, a))^-}^\alpha f(a) \right|$$

$$\leq \frac{\eta(b, a)}{\alpha + 1} \left(\frac{2\alpha}{(\alpha + 1)^{\frac{1}{\alpha} + 1}} \right)^{1 - \frac{1}{q}}$$

$$\times \left[M_1(\alpha) |f'(a)|^q + M_2(\alpha) |f'(b)|^q \right]^{\frac{1}{q}}$$

$$+ \left[M_3(\alpha) |f'(a)|^q + M_4(\alpha) |f'(b)|^q \right]^{\frac{1}{q}},$$

where $M_1(\alpha)$, $M_2(\alpha)$, $M_3(\alpha)$ and $M_4(\alpha)$ are given as Theorem 4 and $\alpha > 0$.

Proof. Using Lemma 1, power mean inequality and the preinvexity of $|f'|^q$, we have

$$\left| \frac{f(a) + \alpha f(a + \eta(b, a))}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{\eta^\alpha(b, a)} J_{(a + \eta(b, a))^-}^\alpha f(a) \right|$$

$$\leq \frac{\eta(b, a)}{\alpha + 1} \int_0^1 |1 - (\alpha + 1)t^\alpha| |f'(a + t\eta(b, a))| dt$$

$$\leq \frac{\eta(b, a)}{\alpha + 1} \left[\left(\int_0^1 |1 - (\alpha + 1)t^\alpha| dt \right)^{1 - \frac{1}{q}} \right.$$

$$\left. \times \left(\int_0^1 |1 - (\alpha + 1)t^\alpha| |f'(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \right]$$

$$\leq \frac{\eta(b, a)}{\alpha + 1} \left[\left(\int_0^1 \frac{1}{\sqrt[\alpha]{\alpha+1}} (1 - (\alpha + 1)t^\alpha) dt + \int_0^1 \frac{1}{\sqrt[\alpha]{\alpha+1}} ((\alpha + 1)t^\alpha - 1) dt \right)^{1 - \frac{1}{q}} \right.$$

$$\begin{aligned} & \times \left[\int_0^{\frac{1}{\sqrt[\alpha]{\alpha+1}}} (1-(\alpha+1)t^\alpha) \left[(1-t)|f'(a)|^q + t|f'(b)|^q \right] dt \right. \\ & \left. + \int_{\frac{1}{\sqrt[\alpha]{\alpha+1}}}^1 ((\alpha+1)t^\alpha - 1) \left[(1-t)|f'(a)|^q + t|f'(b)|^q \right] dt \right]^{\frac{1}{q}} \\ & \leq \frac{\eta(b,a)}{\alpha+1} \left(\frac{2\alpha}{(\alpha+1)^{\frac{1}{\alpha}+1}} \right)^{1-\frac{1}{q}} \\ & \times \left[\begin{aligned} & M_1(\alpha)|f'(a)|^q + M_2(\alpha)|f'(b)|^q \\ & + M_3(\alpha)|f'(a)|^q + M_4(\alpha)|f'(b)|^q \end{aligned} \right] \end{aligned}$$

This completes the proof.

Remark 3. In Theorem 5,

1. If we take $\eta(b,a) = b - a$, then we get the inequality given in [[17], Theorem 4.2].

2. If we take $\eta(b,a) = b - a$ and $\alpha = 1$, then we get the inequality given in [[20], Theorem 1].

Theorem 6. Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta: K \times K \rightarrow \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b, a)$. Suppose $f: K \rightarrow \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a + \eta(b, a)])$. If $|f'|^q$ is preinvex on $[a, a + \eta(b, a)]$ for $q > 1$, then the following Riemann-Liouville fractional integral inequality holds:

$$\begin{aligned} & \left| \frac{f(a) + \alpha f(a + \eta(b, a))}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{\eta^\alpha(b, a)} J^\alpha_{(a+\eta(b,a))^-} f(a) \right| \\ & \leq \frac{\eta(b, a)}{\alpha + 1} \left(\frac{M_5(\alpha, p)}{+M_6(\alpha, p)} \right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \end{aligned}$$

Where

$$\begin{aligned} M_5(\alpha, p) &= \int_0^{\frac{1}{\sqrt[\alpha]{\alpha+1}}} (1-(\alpha+1)t^\alpha)^p dt, \\ M_6(\alpha, p) &= \int_{\frac{1}{\sqrt[\alpha]{\alpha+1}}}^1 ((\alpha+1)t^\alpha - 1)^p dt, \end{aligned}$$

with $p^{-1} + q^{-1} = 1$ and $\alpha > 0$.

Proof. Using Lemma 1, Hölder's integral inequality and the preinvexity of $|f'|^q$, we have

$$\left| \frac{f(a) + \alpha f(a + \eta(b, a))}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{\eta^\alpha(b, a)} J^\alpha_{(a+\eta(b,a))^-} f(a) \right|$$

$$\begin{aligned} & \leq \frac{\eta(b, a)}{\alpha + 1} \int_0^1 |1 - (\alpha + 1)t^\alpha| |f'(a + t\eta(b, a))| dt \\ & \leq \frac{\eta(b, a)}{\alpha + 1} \left[\left(\int_0^1 |1 - (\alpha + 1)t^\alpha|^p dt \right)^{\frac{1}{p}} \right. \\ & \quad \left. \times \left(\int_0^1 |f'(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \right] \\ & \leq \frac{\eta(b, a)}{\alpha + 1} \left[\left(\int_0^{\frac{1}{\sqrt[\alpha]{\alpha+1}}} (1 - (\alpha + 1)t^\alpha)^p dt \right. \right. \\ & \quad \left. \left. + \int_{\frac{1}{\sqrt[\alpha]{\alpha+1}}}^1 ((\alpha + 1)t^\alpha - 1)^p dt \right)^{\frac{1}{p}} \right. \\ & \quad \left. \times \left(\int_0^1 [(1-t)|f'(a)|^q + t|f'(b)|^q] dt \right)^{\frac{1}{q}} \right] \\ & \leq \frac{\eta(b, a)}{\alpha + 1} \left(\frac{M_5(\alpha, p)}{+M_6(\alpha, p)} \right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \end{aligned}$$

This completes the proof.

Remark 4. In Theorem 6,

1. If we take $\eta(b,a) = b - a$, then we get the inequality given in [[17], Theorem 4.3].

2. If we take $\eta(b,a) = b - a$ and $\alpha = 1$, then we get the inequality given in [[29], Theorem 2.3].

4. Conclusion

We have derived new fractional Hermite-Hadamard type integral inequalities via preinvex functions involving only the right Riemann-Liouville fractional integral. We have obtained new generalizations of the right side of the Hermite-Hadamard type inequalities for functions whose first derivatives absolute values are preinvex via right Riemann-Liouville fractional integrals. It has shown that previously known results can be obtained as special cases from our results. It is expected that idea of this article may attract interested readers.

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