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# Generalized Gould-Hopper Based Fully Degenerate Central Bell Polynomials

**Ugur Duran[1,*], Mehmet Acikgoz[2]**

[1]Department of the Basic Concepts of Engineering, Faculty of Engineering and Natural Sciences,
Iskenderun Technical University, TR-31200 Hatay, Turkey
[2]University of Gaziantep, Faculty of Science and Arts, Department of Mathematics, TR-27310 Gaziantep, Turkey
*Corresponding author: mtdrnugur@gmail.com, ugur.duran@iste.edu.tr

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**Abstract** In this paper, we first provide the generalized degenerate Gould-Hopper polynomials via the degenerate exponential functions and then give various relations and formulas such as addition formula and explicit identity. Moreover, we consider the generalized Gould-Hopper based degenerate central factorial numbers of the second kind and present several identities and relationships. Furthermore, we introduce the generalized Gould-Hopper based fully degenerate central Bell polynomials and investigated multifarious correlations and formulas including summation formulas, derivation rule and correlations with the Stirling numbers of the first kind, the generalized Gould-Hopper based degenerate central factorial numbers of the second kind and the generalized degenerate Gould-Hopper polynomials. We then acquire some relations with the degenerate Bernstein polynomials for the generalized Gould-Hopper based fully degenerate central Bell polynomials. Finally, we consider the Gould-Hopper based fully degenerate Bernoulli, Euler and Genocchi polynomials and by utilizing these polynomials, we develop some representations for the generalized Gould-Hopper based fully degenerate central Bell polynomials.

***Keywords:*** *Degenerate exponential function, Central factorial numbers, Central Bell polynomials, Gould-Hopper polynomials, Stirling numbers of the first kind*

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## 1. Motivation

Special functions and special polynomials possess a lot of significance in numerous fields of physics, mathematics, applied sciences, engineering and other related research areas including, functional analysis, differential equations, quantum mechanics, mathematical analysis, mathematical physics, and so on. Especially, the family of special polynomials is one of the most applicable, widespread and useful family of special functions. Some of the most considerable polynomials in the theory of special polynomials are the generalized Hermite-Kampé de Fériet (or Gould-Hopper) polynomials (see [1]) and the Bell polynomials (see [2]).

The Bell polynomials considered by Bell [2] appear as a standard mathematical tool and arise in combinatorial analysis. In recent years, the usual Bell polynomials and the familiar central Bell polynomials have been extensively investigated by several mathematicians, *cf.* [2-11] and see also the references cited therein.

In the theory of special functions and special polynomials, the degenerate forms for polynomials and functions have been worked and developed by several mathematicians *cf.* [5,6,8,9,10,12-23] and see also the references cited therein. For example, Carlitz [12] considered the degenerate Euler polynomials of higher order and presented diverse properties. Carlitz [13] introduced the degenerate Staudt-Clausen theorem and also illustrated it for the degenerate Bernoulli numbers. Kim *et al.* [17] introduced the degenerate Bernstein polynomials and examined recurrence relations, their generating function, symmetric identities and various connections with the earlier polynomials. Kim *et al.* [8] considered the degenerate central Bell numbers and polynomials and provided several properties, identities, and recurrence relations. Kim *et al.* [9] worked on degenerate Bell numbers and polynomials and gave diverse new formulas for those numbers and polynomials. Kim *et al.* [18] handled multifarious explicit formulas and recurrence relationships for the degenerate Mittag-Leffler polynomials and investigated diverse relationships between Mittag-Leffler polynomials and other known families of polynomials. Kim *et al.* [19] introduced the degenerate gamma function and degenerate Laplace transform and proved some interesting and novel formulas.

Throughout this paper, the familiar symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ are referred to the set of all complex numbers, the set of all real numbers, the set of all integers, the set of all natural numbers and the set of all non-negative integers, respectively.

The rest of this paper is structured as follows: Section 2 provides the definition of the generalized degenerate Gould-Hopper polynomials by means of the degenerate exponential functions and also includes various relations and formulas for these polynomials. Section 3 deals with the generalized Gould-Hopper based degenerate central factorial numbers of the second kind and covers several identities and relationships. Section 4 considers the generalized Gould-Hopper based fully degenerate central Bell polynomials and presents multifarious correlations and formulas associated with the degenerate Bernstein polynomials and the Gould-Hopper based fully degenerate Bernoulli, Euler and Genocchi polynomials for the mentioned Bell polynomials. The last section of this paper examines the results derived in this paper.

## 2. Introduction and Preliminaries

In this section, we consider the generalized degenerate Gould-Hopper polynomials via the degenerate exponential functions. Before defining these polynomials, we provide some information that we need.

The Gould-Hopper polynomials are given by means of the following Taylor series expansion at $t=0$ (see [1,14,24]):

$$\sum_{n=0}^{\infty} H_n^{(j)}(x,y)\frac{t^n}{n!}=e^{xt+yt^j}, \tag{2.1}$$

with $j\geq 2$. For the special case $j=1$, the Gould-Hopper polynomials reduce to the representation of the Newton binomial formula. When $j=2$ in (2.1), we get the usual Hermite polynomials denoted by $H_n(x,y)$ that have been utilized to generalize several special numbers and polynomials, for instance, Bell, Euler and Bernoulli polynomials and numbers (see [25]).

Here are several basic notations and definitions in order to define the generalized degenerate Gould-Hopper polynomials.

For $r\in\mathbb{C}$, the $r$-falling factorial $(x)_{n,r}$ is defined by [26]

$$(x)_{n,r}=\begin{cases} x(x-r)(x-2r)\cdots(x-(n-1)r), & n=1,2,\dots \\ 1 & n=0. \end{cases} \tag{2.2}$$

The $r$-rising factorial $x^{(n,r)}$ is given by (see [26])

$$x^{(n,r)}=\begin{cases} x(x+r)(x+2r)\cdots(x+(n-1)r), & n=1,2,\dots \\ 1 & n=0. \end{cases} \tag{2.3}$$

In the case $r$ = 1, the $r$-falling factorial reduces to the familiar falling factorial (see [26])

$$(x)_{n,1}=(x)_n=x(x-1)\cdots(x-n+1)$$

and $r$-rising factorial becomes the usual rising factorial [25,26,27]

The Stirling numbers of the first kind $S_1(n,m)$ are defined by means of the falling factorial as follows

$$(x)_n=\sum_{m=0}^{n} S_1(n,m)x^m, \tag{2.4}$$

*cf.* [2-11,26,27] and see also references cited therein.

The $r$-falling factorial and the $r$-rising factorial satisfy the following relation

$$x^{(n,r)}=(-1)^n(-x)_{n,r}. \tag{2.5}$$

The $\Delta_r$ difference operator is defined by (see [25])

$$\Delta_r f(x)=\frac{1}{r}(f(x+r)-f(x)), \quad r\neq 0. \tag{2.6}$$

**Proposition 1.** *(cf. [25]) The following difference rule holds true:*

$$\Delta_r{}^k(x)_{(n,r)}=\frac{n!}{(n-k)!}(x)_{n-k,r}, \quad 0\leq k\leq n. \tag{2.7}$$

The following Lemma will be useful in the derivation of several results.

**Lemma 1.** *(cf. [14]) The following elementary series manipulations hold:*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/j\rfloor} A(k,n-jk). \tag{2.8}$$

The degenerate exponential function $e_\lambda^x(t)$ for a real number $\lambda$ is given by (cf. [5,6,8,9,10,12-23]):

$$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\%\lambda}} \text{ and } e_\lambda^1(t):=e_\lambda(t). \tag{2.9}$$

It is readily seen that $\lim_{\lambda\to 0} e_\lambda^x(t)=e^{xt}$. From (2.2) and (2.9), we obtain the following relation

$$e_\lambda^x(t)=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \tag{2.10}$$

satisfying

$$\Delta_\lambda e_\lambda^x(t)=te_\lambda^x(t). \tag{2.11}$$

We now give our definition as follows.

**Definition 1.** *Let $j\in\mathbb{Z}$ with $j>0$, and let $\beta,\gamma\in\mathbb{R}\setminus\{0\}$. We define the generalized degenerate Gould-Hopper polynomials $H_{n,\beta,\gamma}^{(j)}(r,\rho)$ by the following generating function to be*

$$G_{\beta,\gamma}(r,\rho,t)=\sum_{n=0}^{\infty} H_{n,\beta,\gamma}^{(j)}(r,\rho)\frac{t^n}{n!}=e_\beta^r(t)e_\gamma^\rho(t^j) \tag{2.12}$$
$$=(1+\beta t)^{\frac{r}{\beta}}\left(1+\gamma t^j\right)^{\frac{\rho}{\gamma}}.$$

We now examine some special cases of the generalized degenerate Gould-Hopper polynomials as follows.

**Remark 1.**

(1) *When $\beta=\gamma$, we get the fully degenerate Gould-Hopper polynomials denoted by $H_{n,\beta}^{(j)}(r,\rho)$* (see [14]).

(2) *Setting* $\beta=\gamma$ *and* $j=2$, *we get the fully degenerate Hermite polynomials denoted by* $H_{n,\beta}(r,\rho)$ (*cf.* [16]).

(3) *When* $\beta\to 0$ *and* $\gamma\to 0$, *we have the Gould-Hopper polynomials denoted by* $H_n^{(j)}(r,\rho)$ (*cf.* [1,14,28]).

(4) *When* $\beta\to 0$, $\gamma\to 0$ *and* $j=2$, *we reach the classical Hermite polynomials denoted by* $H_n(r,\rho)$ *(see* [1,16,24,28,29,30,31,35]).

We now give four Theorems follow from Definition 1 and the transformation formula (2.8) without proofs.

**Theorem 1.** *The generalized degenerate Gould-Hopper polynomials* $H_{n,\beta,\gamma}^{(j)}(r,\rho)$ *satisfy the following explicit formula*

$$H_{n,\beta,\gamma}^{(j)}(r,\rho)=n!\sum_{k=0}^{\lfloor n/j\rfloor}\frac{(r)_{n-jk,\beta}(\rho)_{k,\gamma}}{(n-jk)!k!}, \quad (2.13)$$

where $\lfloor\cdot\rfloor$ is the Gauss notation, and represents the maximum integer which does not exceed the number in the square brackets.

Here is the inversion formula for the generalized degenerate Gould-Hopper polynomials $H_{n,\beta,\gamma}^{(j)}(r,\rho)$.

**Theorem 2.** *The following inversion formula holds true.*

$$(r)_{n,\beta}=n!\sum_{k=0}^{\lfloor n/j\rfloor}\frac{(\rho)_{k,\gamma}}{(n-jk)!k!}H_{n-jk,\beta,\gamma}^{(j)}(r,\rho). \quad (2.14)$$

**Theorem 3.** *The following addition formula is valid.*

$$\begin{aligned}&H_{n,\beta,\gamma}^{(j)}(r_1+r_2,\rho_1+\rho_2)\\&=\sum_{k=0}^{n}\binom{n}{k}H_{n-k,\beta,\gamma}^{(j)}(r_1,\rho_1)H_{k,\beta,\gamma}^{(j)}(r_2,\rho_2).\end{aligned} \quad (2.15)$$

**Theorem 4.** *For* $\theta\in\mathbb{C}\setminus\{0\}$, *the following equation holds true*

$$H_{n,\beta,\gamma}^{(j)}\left(\theta r,\theta^j\rho\right)=\theta^n H_{n,\beta/\theta,\gamma/\theta^j}^{(j)}(r,\rho). \quad (2.16)$$

## 3. The Generalized Gould-Hopper Based Degenerate Central Factorial Numbers

In this section, we perform to analyze and investigate degenerate forms of some special polynomials and numbers. We focus on the generalized Gould-Hopper based degenerate central factorial numbers of the second kind. We then derive several properties and formulas for these numbers.

For non-negative integer $n$, the central factorial numbers of the second kind $T(n,m)$ are defined by the following exponential generating function

$$\sum_{n=0}^{\infty}T(n,m)\frac{t^n}{n!}=\frac{\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^m}{m!} \quad \text{(cf. [5,7,8,10])}(3.1)$$

or by recurrence relation for a fixed non-negative integer $n$,

$$x^n=\sum_{m=0}^{n}T(n,m)x^{[m]}, \quad (3.2)$$

where the notation $x^{[m]}$ called as the central factorial equals to $x\left(x+\frac{m}{2}-1\right)\left(x+\frac{m}{2}-2\right)\cdots\left(x-\frac{m}{2}+1\right)$ with initial condition $x^{[0]}=1$, cf. [5,7,8,10] and see also references cited therein.

For non-negative integer $n$, the degenerate central factorial numbers of the second kind $T_{2,\lambda}(n,m)$ are defined by the following exponential generating function

$$\sum_{n=0}^{\infty}T_{2,\lambda}(n,m)\frac{t^n}{n!}=\frac{\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)^m}{m!} \quad \text{(cf. [8]). } (3.3)$$

When $\lambda$ approaches to $0$, the degenerate central factorial numbers of the second kind (3.3) reduces to the central factorial numbers of the second kind (3.1), namely $\lim_{\lambda\to 0}T_{2,\lambda}(n,m)=T(n,m)$.

We are now ready to define the generalized Gould-Hopper based degenerate central factorial numbers of the second kind.

**Definition 2.** *Let* $\lambda,\beta,\gamma\in\mathbb{R}\setminus\{0\}$. *The generalized Gould-Hopper based degenerate central factorial numbers of the second kind* $T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,m:r,\rho)$ *are introduced by means of the following generating function*

$$\begin{aligned}&\sum_{n=0}^{\infty}T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,m:r,\rho)\frac{t^n}{n!}\\&=\frac{\left(e_\lambda^{\omega}(t)-e_\lambda^{-\omega}(t)\right)^m}{m!}e_\beta^r(t)e_\gamma^\rho\left(t^j\right).\end{aligned} \quad (3.4)$$

We here analyze some circumstances of the generalized Gould-Hopper based degenerate central factorial numbers of the second kind $T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,m:r,\rho)$ as follows.

**Remark 2.**

(1) *When* $r=\rho=0$, *we get the unified degenerate central factorial numbers of the second kind*

$$\sum_{n=0}^{\infty}T_{2,\lambda,\omega}(n,m)\frac{t^n}{n!}=\frac{\left(e_\lambda^{\omega}(t)-e_\lambda^{-\omega}(t)\right)^m}{m!}, \quad \text{cf. ([5]).}(3.5)$$

(2) *When* $\rho=0$, *we get an extension for the* $r$*-central factorial numbers, termed the unified degenerate* $r$*-central factorial numbers of the second kind*:

$$\sum_{n=0}^{\infty} T_{2,\lambda,\omega;\beta}(n,m:r)\frac{t^n}{n!}=\frac{\left(e_\lambda^{\omega}(t)-e_\lambda^{-\omega}(t)\right)^m}{m!}e_\beta^r(t).$$
(3.6)

(3) *When $r=0$, we get a new polynomial which is an extension of the central factorial numbers*:

$$\sum_{n=0}^{\infty} T_{2,\lambda,\gamma}^{[j,\omega]}(n,m:\rho)\frac{t^n}{n!}=\frac{\left(e_\lambda^{\omega}(t)-e_\lambda^{-\omega}(t)\right)^m}{m!}e_\gamma^{\rho}\left(t^j\right). \quad (3.7)$$

(4) *When $\omega=\frac{1}{2}$ and $r=\rho=0$, we get the degenerate central factorial numbers of the second kind $T_{2,\lambda}(n,m)$ in (3.3), cf.* [8].

(5) *When $\lambda\to 0$, generalized Gould-Hopper based degenerate central factorial numbers of the second kind $T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,m:r,\rho)$ reduce to the $\omega$-analog of the degenerate Gould-Hopper based central factorial numbers of the second kind denoted by $T_{2,\omega;\beta,\gamma}^{(j)}(n,m:r,\rho)$, which is also novel generalization of the factorial numbers of the second kind $T_2(n,m)$ in (3.1), given by*

$$\sum_{n=0}^{\infty} T_{2,\beta,\gamma}^{[j,\omega]}(n,m:r,\rho)\frac{t^n}{n!}=\frac{\left(e^{\omega t}-e^{-\omega t}\right)^m}{m!}e_\beta^r(t)e_\gamma^{\rho}\left(t^j\right). \quad (3.8)$$

(6) *When $\omega=\frac{1}{2}$ and $\lambda,\beta,\gamma\to 0$, we attain the familiar central factorial numbers of the second kind $T(n,m)$ in (3.1), cf.* [5,7,8,10].

We now investigate some properties of the generalized Gould-Hopper based degenerate central factorial numbers of the second kind $T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,m:r,\rho)$. Hence, we give the following Theorem 5.

**Theorem 5.** *For $k,m,n\in\mathbb{N}$ and $\lambda,\beta,\gamma\in\mathbb{R}\setminus\{0\}$, we have*

$$T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,k+m:r,\rho)=\frac{k!m!}{(k+m)!}\sum_{u=0}^{n}\binom{n}{u}T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(u,k:r,\rho)T_{2,\lambda,\omega}(n-u,m) \quad (3.9)$$

$$=\frac{k!m!}{(k+m)!}\sum_{u=0}^{n}\binom{n}{u}T_{2,\lambda,\omega}(u,k)\times T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n-u,m:r,\rho) \quad (3.10)$$

$$=\frac{k!m!}{(k+m)!}\sum_{u=0}^{n}\binom{n}{u}T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(u,k:r)\times T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n-u,m:\rho) \quad (3.11)$$

*Proof.* In view of Definition 2 and using (3.5), (3.6) and (3.7), we write

$$\frac{\left(e_\lambda^{\omega}(t)-e_\lambda^{-\omega}(t)\right)^{k+m}}{(k+m)!}e_\beta^r(t)e_\gamma^r\left(t^j\right)$$
$$=\frac{k!m!}{(k+m)!}\frac{\left(e_\lambda^{\omega}(t)-e_\lambda^{-\omega}(t)\right)^k}{k!}$$
$$\times e_\beta^r(t)e_\gamma^r\left(t^j\right)\frac{\left(e_\lambda^{\omega}(t)-e_\lambda^{-\omega}(t)\right)^m}{m!}$$
$$=\sum_{n=0}^{\infty}T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,k:r,\rho)\frac{t^n}{n!}\sum_{n=0}^{\infty}T_{2,\lambda,\omega}(n,m)\frac{t^n}{n!}$$
$$=\frac{k!m!}{(k+m)!}\sum_{n=0}^{\infty}\sum_{u=0}^{n}\binom{n}{u}T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(u,k:r,\rho)T_{2,\lambda,\omega}(n-u,m)\frac{t^n}{n!}.$$

which implies the asserted result (3.9). The equations (3.10) and (3.11) can be derived similarly. So, the proof is completed.

Here are the differentiation rules for the generalized Gould-Hopper based degenerate central factorial numbers of the second kind.

$$T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,m:r,\rho).$$

**Theorem 6.** *The following relation holds true for $\lambda,\beta,\gamma\in\mathbb{R}\setminus\{0\}$.*

$$\frac{\partial}{\partial\omega}T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,m:r,\rho)=\frac{n!}{(n-u)!}\sum_{s=0}^{n-u}\binom{n-u}{s}\sum_{k=0}^{m}\binom{m}{k}\sum_{u=1}^{\infty}\frac{(-1)^{u+k+1}}{m!}\times\frac{m-2k}{u}\lambda^{u-1}(\omega(m-2k))_{s,\lambda}H_{n-u-s,\beta,\gamma}^{(j)}(r,\rho), \quad (3.12)$$

$$\frac{\partial}{\partial r}T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,m:r,\rho)=\frac{n!}{(n-u)!}\sum_{u=1}^{\infty}\frac{(-1)^{u+1}}{u}\beta^{u-1}T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n-u,m:r,\rho) \quad (3.13)$$

*and*

$$\frac{\partial}{\partial\rho}T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,m:r,\rho)=\frac{n!}{(n-ju)!}\sum_{u=1}^{\infty}\frac{(-1)^{u+1}}{u}\gamma^{u-1}T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n-ju,m:r,\rho). \quad (3.14)$$

*Proof.* From (3.4), we get

$$\frac{\partial}{\partial\omega}\sum_{n=0}^{\infty}T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,m:r,\rho)\frac{t^n}{n!}$$
$$=\frac{\partial}{\partial\omega}\frac{\left(e_\lambda^{\omega}(t)-e_\lambda^{-\omega}(t)\right)^m}{m!}e_\beta^r(t)e_\gamma^{\rho}\left(t^j\right)$$
$$=\sum_{k=0}^{m}\binom{m}{k}(-1)^k\frac{e_\beta^r(t)e_\gamma^{\rho}\left(t^j\right)}{m!}$$
$$\times(1+\lambda t)^{\frac{\omega(m-2k)}{\lambda}}\ln\left((1+\lambda t)^{\frac{m-2k}{\lambda}}\right)$$

$$= \sum_{k=0}^{m}\binom{m}{k}\sum_{u=1}^{\infty}\frac{(-1)^{u+k+1}}{m!}\frac{m-2k}{u}(1+\lambda t)^{\frac{\omega(m-2k)}{\lambda}} \times \lambda^{u-1}t^{u}e_{\beta}^{r}(t)e_{\gamma}^{\rho}\left(t^{j}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{s=0}^{n}\binom{n}{s}\sum_{k=0}^{m}\binom{m}{k}\sum_{u=1}^{\infty}\frac{(-1)^{u+k+1}}{m!}\frac{m-2k}{u}\lambda^{u-1}\left(\omega(m-2k)\right)_{s,\lambda}H_{n-s,\beta,\gamma}^{(j)}(r,\rho)\right)\frac{t^{n+u}}{n!}$$

which implies the claimed result (3.12). The proofs of the results in (3.13) and (3.14) can be done by the similar proof method used above.

We here give the following correlation.

**Theorem 7**. *The following correlation*

$$T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,m:r,\rho) = \sum_{l=0}^{n}\binom{n}{l}T_{2,\lambda,\omega}(l,m)H_{n-l,\beta,\gamma}^{(j)}(r,\rho) \tag{3.15}$$

is valid for $\lambda,\beta,\gamma\in\mathbb{R}\setminus\{0\}$.

*Proof*. By Definition 2 and (2.12), we obtain

$$\sum_{n=0}^{\infty}T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,m:r,\rho)\frac{t^{n}}{n!} = \frac{\left(e_{\lambda}^{\omega}(t)-e_{\lambda}^{-\omega}(t)\right)^{m}}{m!}e_{\beta}^{r}(t)e_{\gamma}^{\rho}\left(t^{j}\right) = \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}T_{2,\lambda,\omega}(l,m)H_{n-l,\beta,\gamma}^{(j)}(r,\rho)\frac{t^{n}}{n!}.$$

which provides the desired result (3.15).

We give the following theorem.

**Theorem 8.** *The following relation*

$$T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,m:r,\rho) = \sum_{k=0}^{n}\sum_{u=0}^{\infty}\binom{n}{k}\frac{r!}{(r-u)!}T_{2,\beta,\omega}(n-k,u) \times T_{2,\lambda,\beta,\gamma}^{[j,\omega]}\left(k,m:\frac{r}{\omega}+\omega(u-m),\rho\right) \tag{3.16}$$

*holds true for* $\alpha,\lambda,\beta,\gamma\in\mathbb{R}\setminus\{0\}$.

*Proof*. By Definition 3, we get

$$\sum_{n=0}^{\infty}T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,m:r,\rho)\frac{t^{n}}{n!} = \frac{\left(e_{\lambda}^{\omega}(t)-e_{\lambda}^{-\omega}(t)\right)^{m}}{m!}e_{\beta}^{r}(t)e_{\gamma}^{\rho}\left(t^{j}\right)$$

$$= \sum_{u=0}^{\infty}\binom{r}{u}\left(e_{\beta}^{\omega}(t)-e_{\beta}^{-\omega}(t)\right)^{u}\frac{\left(e_{\lambda}^{\omega}(t)-e_{\lambda}^{-\omega}(t)\right)^{m}}{m!} \times e_{\beta}^{\frac{r}{\omega}-\omega(m-u)}(t)e_{\gamma}^{\rho}\left(t^{j}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{u=0}^{\infty}\binom{n}{k}\frac{r!}{(r-u)!}T_{2,\beta,\omega}(n-k,u) \times T_{2,\lambda,\beta,\gamma}^{[j,\omega]}\left(k,m:\frac{r}{\omega}+\omega(u-m),\rho\right)\frac{t^{n}}{n!},$$

which implies the desired result (3.16).

We here give the following correlation.

**Theorem 9.** *The following correlation*

$$T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,m:r,\rho) = \sum_{l=0}^{n}T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(l,m:r,\rho)S_{1}(n,l)\lambda^{n-l} \tag{3.17}$$

*is valid for* $\alpha,\lambda,\beta,\gamma\in\mathbb{R}\setminus\{0\}$.

*Proof*. By Definition 2 and the identity (2.9), we obtain

$$\sum_{n=0}^{\infty}T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,m:r,\rho)\frac{t^{n}}{n!} = \frac{\left(e_{\lambda}^{\omega}(t)-e_{\lambda}^{-\omega}(t)\right)^{m}}{m!}e_{\beta}^{r}(t)e_{\gamma}^{\rho}\left(t^{j}\right)$$

$$= \frac{1}{m!}\left(e^{\frac{\omega}{\lambda}\log(1+\lambda t)}-e^{-\frac{\omega}{\lambda}\log(1+\lambda t)}\right)^{m}e_{\beta}^{r}(t)e_{\gamma}^{\rho}\left(t^{j}\right)$$

$$= \sum_{l=0}^{\infty}T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(l,m:r,\rho)\lambda^{-l}\frac{\left(\log(1+\lambda t)\right)^{l}}{l!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(l,m:r,\rho)S_{1}(n,l)\lambda^{n-l}\right)\frac{t^{n}}{n!},$$

which provides the desired result (3.17).

## 4. Construction of Generalized Gould-Hopper Based Fully Degenerate Central Bell Polynomials

In this part, we introduce the generalized Gould-Hopper based fully degenerate central Bell polynomials and investigated multifarious correlations and formulas including summation formulas, derivation rule and correlations with the Stirling numbers of the first kind, the generalized Gould-Hopper based degenerate central factorial numbers of the second kind and the generalized degenerate Gould-Hopper polynomials.

The classical Bell polynomials $Bel_{n}(x)$ (also called exponential polynomials) and central Bell polynomials $Bel_{n}^{(c)}(x)$ (also called central exponential polynomials) are defined by means of the following generating functions:

$$\sum_{n=0}^{\infty}Bel_{n}(x)\frac{t^{n}}{n!} = e^{x\left(e^{t}-1\right)} \quad \text{(cf. [2,3,4,6,9,11])} \tag{4.1}$$

and

$$\sum_{n=0}^{\infty} Bel_n^{(c)}(x)\frac{t^n}{n!} = e^{x\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)} \text{ (cf. [5,7,8,10]).} \quad (4.2)$$

The classical Bell numbers $Bel_n$ and usual central Bell numbers $Bel_n^{(c)}$ are acquired by choosing $x=1$ in (4.1) and (4.2), that is $Bel_n(1) := Bel_n$ and $Bel_n^{(c)}(1) := Bel_n^{(c)}$, which are given by the following exponential generating function:

$$\sum_{n=0}^{\infty} Bel_n \frac{t^n}{n!} = e^{\left(e^t-1\right)}$$
$$\text{and } \sum_{n=0}^{\infty} Bel_n^{(c)} \frac{t^n}{n!} = e^{\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)} \quad (4.3)$$

The Bell polynomials extensively studied by Bell [2] appear as a standard mathematical tool and arise in combinatorial analysis. The familiar Bell polynomials and the central Bell polynomials have been intensely studied by many mathematicians, cf. [2-11] and see also the references cited therein. The large investigations of the Bell polynomials and numbers yield a motivation to improve this mathematical tool.

The central Bell polynomials and central factorial numbers of the second kind satisfy the following relation (cf. [5,7,8,10]).

$$Bel_n^{(c)}(x) = \sum_{m=0}^{n} T(n,m)x^m. \quad (4.4)$$

The degenerate classical Bell polynomials and the degenerate central Bell polynomials are given by the following Taylor series expansion at $t=0$ as follows:

$$\sum_{n=0}^{\infty} Bel_{n,\lambda}(x)\frac{t^n}{n!} = e^{x\left(e_\lambda(t)-1\right)} \text{ (cf. [9])} \quad (4.5)$$

and

$$\sum_{n=0}^{\infty} Bel_{n,\lambda}^{(c)}(x)\frac{t^n}{n!} = e^{x\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)} \text{ (cf. [8]),} \quad (4.6)$$

When $x=1$ in (4.5) and (4.6), the mentioned polynomials ($Bel_{n,\lambda}(x)$ and $Bel_{n,\lambda}^{(c)}(x)$) reduce to the corresponding numbers

$$Bel_{n,\lambda}(1) := Bel_{n,\lambda} \text{ and } Bel_{n,\lambda}^{(c)}(1) := Bel_{n,\lambda}^{(c)} \quad (4.7)$$

termed as the degenerate Bell numbers and the degenerate central Bell numbers, respectively.

**Remark 3.** *We note that using (2.9), the degenerate classical Bell polynomials (4.5) and the degenerate central Bell polynomials (4.6) reduce the classical Bell polynomials (4.1) and the central Bell polynomials (4.2) in the following limit cases:*

$$\lim_{\lambda\to 0} Bel_{n,\lambda}(x) = Bel_n(x)$$
$$\text{and } \lim_{\lambda\to 0} Bel_{n,\lambda}^{(c)}(x) = Bel_n^{(c)}(x). \quad (4.8)$$

The degenerate central Bell polynomials and the degenerate central factorial numbers of the second kind satisfy the following relation (cf. [8])

$$Bel_{n,\lambda}^{(c)}(x) = \sum_{m=0}^{n} T_{2,\lambda}(n,m)x^m. \quad (4.9)$$

We are now ready to define the generalized Gould-Hopper based fully degenerate central Bell polynomials and numbers by the following Definition 3.

**Definition 3.** *Let $\alpha,\lambda,\beta,\gamma \in \mathbb{R}\setminus\{0\}$. The generalized Gould-Hopper based fully degenerate central Bell polynomials $Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)$ are defined by the following exponential generating function*

$$\sum_{n=0}^{\infty} B_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)\frac{t^n}{n!}$$
$$= e_\alpha^x\left(e_\lambda^\omega(t)-e_\lambda^{-\omega}(t)\right)e_\beta^r(t)e_\gamma^\rho\left(t^j\right) \quad (4.10)$$

When $x=1$, the generalized Gould-Hopper based fully degenerate central Bell polynomials reduce to the corresponding numbers

$$Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(1:\omega,r,\rho) := Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(\omega,r,\rho)$$

termed as generalized Gould-Hopper based fully degenerate central Bell numbers:

$$\sum_{n=0}^{\infty} Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(\omega,r,\rho)\frac{t^n}{n!}$$
$$= e_\alpha\left(e_\lambda^\omega(t)-e_\lambda^{-\omega}(t)\right)e_\beta^r(t)e_\gamma^\rho\left(t^j\right) \quad (4.11)$$

We now analyze various special circumstances of the generalized Gould-Hopper based fully degenerate central Bell polynomials as follows.

**Remark 4.**

(1) *When $\omega=\frac{1}{2}$, the polynomials $Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)$ and numbers $Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(\omega,r,\rho)$ in (4.10) and (4.11) reduce to the generalized Gould-Hopper based degenerate central Bell polynomials $Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:r,\rho)$ and numbers $Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(\omega,r,\rho)$ in (4.12), which are also new generalizations of the central Bell polynomials $Bel_n^{(c)}(x)$ and numbers $Bel_n^{(c)}$ in (4.2) and (4.3), given by*

$$\sum_{n=0}^{\infty} Bel_{n,\lambda,\alpha}^{(c)}(x)\frac{t^n}{n!} = e_\alpha^x\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)e_\beta^r(t)e_\gamma^\rho\left(t^j\right) \quad (4.12)$$

*and*

$$\sum_{n=0}^{\infty} Bel_{n,\lambda,\alpha}^{(c)} \frac{t^n}{n!} = e_\alpha\left(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t)\right) e_\beta^r(t) e_\gamma^\rho\left(t^j\right).$$

(2) *Upon setting* $\alpha,\beta,\gamma\to 0$ *, the polynomials* $Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)$ *and numbers* $Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(\omega,r,\rho)$ *in (4.10) and (4.11) reduce to the Gould Hopper based generalized degenerate central Bell polynomials* $Bel_{n,\lambda}^{(c,j)}(x:\omega,r,\rho)$ *and numbers* $Bel_{n,\lambda}^{(c,j)}(\omega,r,\rho)$ *in (4.12), which are also novel extensions of the central Bell polynomials* $Bel_n^{(c)}(x)$ *and numbers* $Bel_n^{(c)}$ *in (4.2) and (4.3), shown by*

$$\sum_{n=0}^{\infty} Bel_{n,\lambda}^{(c,j)}(x:\omega,r,\rho)\frac{t^n}{n!} = e^{x\left(e_\lambda^{\omega}(t)-e_\lambda^{-\omega}(t)\right)} e^{rt+\rho t^j}$$
$$\text{and } Bel_{n,\lambda}^{(c,j)}(\omega,r,\rho)\frac{t^n}{n!} = e^{\left(e_\lambda^{\omega}(t)-e_\lambda^{-\omega}(t)\right)} e^{rt+\rho t^j}. \tag{4.13}$$

(3) *When* $\omega=\frac{1}{2}$, $\rho=r=0$ *and* $\alpha\to 0$*, we obtain the degenerate central Bell polynomials and numbers denoted by* $Bel_{n,\lambda}^{(c,)}(x)$ *and* $Bel_{n,\lambda}^{(c)}$ *in (4.6) and (4.7) (cf. [8])*

(4) *Setting* $\omega=\frac{1}{2}$ *and* $\lambda\to 0$*, we attain the degenerate central Bell polynomials and numbers denoted by* $Bel_{n,\lambda}^{(c)}(x)$ *and* $Bel_{n,\lambda}^{(c)}$ *, which is different from the polynomials and numbers in (4.2) and (4.3) given by Kim et al. [8]*:

$$\sum_{n=0}^{\infty} Bel_{n,\alpha}^{(c)}(x)\frac{t^n}{n!} = e_\alpha^x\left(e_\lambda^{1/2}(t)-e_\lambda^{-1/2}(t)\right)$$
$$\text{and } Bel_{n,\alpha}^{(c)}\frac{t^n}{n!} = e_\alpha\left(e_\lambda^{1/2}(t)-e_\lambda^{-1/2}(t)\right). \tag{4.14}$$

(5) *In the special case* $\alpha,\beta,\gamma,\lambda\to 0$ *and* $\omega=\frac{1}{2}$*, the polynomials* $Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)$ *and numbers* $Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(\omega,r,\rho)$ *in (4.10) and (4.11) reduce to the Gould Hopper based central Bell polynomials* $Bel_{n,\lambda}^{(c,j)}(x:r,\rho)$ *and numbers* $Bel_{n,\lambda}^{(c,j)}(r,\rho)$ *in (4.15), which are also new generalizations of the central Bell polynomials* $Bel_n^{(c)}(x)$ *and numbers* $Bel_n^{(c)}$ *in (4.2) and (4.3), given by*

$$\sum_{n=0}^{\infty} Bel_{n,\lambda}^{(c,j)}(x:r,\rho)\frac{t^n}{n!} = e^{x\left(e^{\frac{1}{2}}(t)-e^{-\frac{1}{2}}(t)\right)} e^{rt+\rho t^j}$$
$$\text{and } Bel_{n,\lambda}^{(c,j)}(r,\rho)\frac{t^n}{n!} = e^{\left(e^{\frac{1}{2}}(t)-e^{-\frac{1}{2}}(t)\right)} e^{rt+\rho t^j}. \tag{4.15}$$

(6) When $\alpha,\lambda\to 0$, $\rho=r=0$ and $\omega=\frac{1}{2}$, we arrive at the central Bell polynomials and numbers in (4.2) and (4.3) (cf. [5,7,8,10]).

## 4.1. Simple Identities for $Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)$

We now list a few properties of $Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)$ which follow straightforwardly from Definition 3. So we omit the proofs.

**Theorem 10.** *For* $\alpha,\lambda,\beta,\gamma\in\mathbb{R}\setminus\{0\}$, *we have*

$$Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho) = \sum_{l=0}^{n}\binom{n}{l} Bel_{l,\alpha,\lambda}^{(c)}(x:\omega) H_{n-l,\beta,\gamma}^{(j)}(r,\rho). \tag{4.16}$$

**Theorem 11.** *The following relation*

$$Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho) = \sum_{m=0}^{n} T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,m:r,\rho)(x)_{m,\alpha} \tag{4.17}$$

*holds true for* $\alpha,\lambda,\beta,\gamma\in\mathbb{R}\setminus\{0\}$.

We now state two summation formulas for $Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)$ as follows.

**Theorem 12.** *The following summation formulas*

$$Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x+y:\omega,r,\rho) = \sum_{m=0}^{n}\binom{n}{m} Bel_{n-m,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho) \times Bel_{m,\alpha,\lambda}^{(c)}(y:\omega), \tag{4.18}$$

*and*

$$Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x+y:\omega,r,\rho) = \sum_{m=0}^{n}\binom{n}{m} Bel_{n-m,\alpha,\lambda,\beta}^{(c)}(x:\omega,r)\, Bel_{m,\alpha,\lambda,\gamma}^{(c,j)}(y:\omega,\rho) \tag{4.19}$$

*are valid for* $\alpha,\lambda,\beta,\gamma\in\mathbb{R}\setminus\{0\}$.

We now provide a correlation as follows.

**Theorem 13.** *The following formula*

$$Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho) = \sum_{m=0}^{n}\sum_{l=0}^{n}(x)_{m,\alpha} T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(l,m:r,\rho) S_1(n,l)\lambda^{n-l} \tag{4.20}$$

*holds true for* $\alpha,\lambda,\beta,\gamma\in\mathbb{R}\setminus\{0\}$.

We here provide an explicit formula for $Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)$ as follows.

**Theorem 14.** *The following explicit formula*

$$\begin{aligned}&Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)\\&=\sum_{u=0}^{n}\sum_{j=0}^{u}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{n}{u}\binom{u}{j}\binom{m}{k}(-1)^{m-k}\frac{(n-u)!}{m!}\\&\times(x)_{m,\alpha}(\omega k)_{j,\lambda}(\omega(k-m))_{u-j,\lambda}\\&\times\sum_{k=0}^{\lfloor(n-u)/j\rfloor}\frac{(r)_{n-u-jk,\beta}(\rho)_{k,\gamma}}{(n-u-jk)!k!}.\end{aligned}\tag{4.21}$$

*holds true for* $\alpha,\lambda,\beta,\gamma\in\mathbb{R}\setminus\{0\}$.

## 4.2. A Partial Derivative for $Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)$

Theorem 15 includes the partial derivative of $Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)$ with respect to $x$ as follows.

**Theorem 15.** *The following relation*

$$\begin{aligned}&\frac{\partial}{\partial x}Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)\\&=\sum_{k=0}^{n}\sum_{m=1}^{\infty}\binom{n}{k}(m-1)!(-\alpha)^{m-1}\\&\times Bel_{n-k,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)T_{2,\lambda;\omega}(k,m)\end{aligned}\tag{4.22}$$

*holds true for* $\alpha,\lambda,\beta,\gamma\in\mathbb{R}\setminus\{0\}$.

*Proof.* By Definition 3 and formulas (2.9) and (2.10), we get

$$\begin{aligned}&\frac{\partial}{\partial x}\sum_{n=0}^{\infty}Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)\frac{t^n}{n!}\\&=\frac{\partial}{\partial x}e_\alpha^x\left(e_\lambda^\omega(t)-e_\lambda^{-\omega}(t)\right)e_\beta^r(t)e_\gamma^\rho\left(t^j\right)\\&=\left(1+\alpha\left(e_\lambda^\omega(t)-e_\lambda^{-\omega}(t)\right)\right)^{\frac{x}{\alpha}}\\&\times\ln\left(\left(1+\alpha\left(e_\lambda^\omega(t)-e_\lambda^{-\omega}(t)\right)\right)^{\alpha^{-1}}\right)e_\beta^r(t)e_\gamma^\rho\left(t^j\right)\\&=\sum_{n=0}^{\infty}Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)\frac{t^n}{n!}\\&\times\sum_{m=1}^{\infty}(m-1)!(-\alpha)^{m-1}\frac{\left(e_\lambda^\omega(t)-e_\lambda^{-\omega}(t)\right)^m}{m!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\sum_{m=1}^{\infty}(m-1)!(-\alpha)^{m-1}\\&\times Bel_{n-k,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)T_{2,\lambda;\omega}(k,m)\frac{t^n}{n!}\end{aligned}$$

which means the claimed result (4.22).

## 4.3. Relations for $Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)$

Here, we perform to get several diverse relations for $Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)$ with some other degenerate polynomials including degenerate Bernstein, Bernoulli, Genocchi and Euler polynomials.

We firstly perform to attain some relations with the degenerate Bernstein

$$\sum_{n=k}^{\infty}\mathfrak{B}_{k,n}(x:\lambda)\frac{t^n}{n!}=\frac{(x)_{k,\lambda}}{k!}t^k(1+\lambda t)^{\frac{1-x}{\lambda}}.\tag{4.23}$$

**Remark 5.** *Upon setting* $\lambda\to0$ *in (4.23), the generation function of degenerate Bernstein polynomials reduce to the generating function of familiar Bernstein polynomials as follows*:

$$\sum_{n=k}^{\infty}B_{k,n}(x)\frac{t^n}{n!}=\frac{x^k}{k!}t^ke^{(1-x)t},$$

*which is firstly given by Acikgoz and Araci in [32].*

We consider that

$$\begin{aligned}&\sum_{n=0}^{\infty}Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)\frac{t^n}{n!}\\&=\sum_{m=0}^{\infty}(x)_{m,\alpha}\frac{\left(e_\lambda^\omega(t)-e_\lambda^{-\omega}(t)\right)^m}{m!}e_\beta^r(t)e_\gamma^\rho\left(t^j\right)\\&=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{(x)_{m,\alpha}}{t^k(1-\omega(m-2k))_{k,\lambda}}\\&\times\frac{(-1)^{m-k}}{(m-k)!}\frac{(1+\omega(m-2k))_{k,\lambda}}{k!}\\&\times t^k(1+\lambda t)^{\frac{\omega(2k-m)}{\lambda}}e_\beta^r(t)e_\gamma^\rho\left(t^j\right)\\&=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{(x)_{m,\alpha}}{(1+\omega(m-2k))_{k,\lambda}}\frac{(-1)^{m-k}}{(m-k)!}\\&\times\sum_{n=0}^{\infty}\sum_{u=0}^{n}\binom{n}{u}\mathfrak{B}_{k,u}(1+\omega(m-2k):\lambda)H_{n-u,\beta,\gamma}^{(j)}(r,\rho)\frac{t^{n-k}}{n!}.\end{aligned}$$

Hence, we arrive at the following theorem.

**Theorem 16.** *The following correlation*

$$\begin{aligned}&Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)\\&=n!\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{(x)_{m,\alpha}}{(1+\omega(m-2k))_{k,\lambda}}\frac{(-1)^{m-k}}{(m-k)!(n+k)!}\\&\times\sum_{u=0}^{n+k}\binom{n+k}{u}\mathfrak{B}_{k,u}(1+\omega(m-2k):\lambda)\\&\times H_{n+k-u,\beta,\gamma}^{(j)}(r,\rho).\end{aligned}\tag{4.24}$$

*holds true*.

Let

$$\begin{aligned}I&=(x)_{k,\alpha}\frac{\left((1+\lambda t)^{\frac{\omega}{\lambda}}-(1+\lambda t)^{-\frac{\omega}{\lambda}}\right)^k}{k!}\\&\times\left(1+\alpha\left((1+\lambda t)^{\frac{\omega}{\lambda}}-(1+\lambda t)^{-\frac{\omega}{\lambda}}\right)\right)^{\frac{1-x}{\alpha}}(1+\beta t)^{\frac{r}{\beta}}\left(1+\gamma t^j\right)^{\frac{\rho}{\gamma}}.\end{aligned}$$

Therefore, from (3.4) and (4.23), we obtain

$$I=\sum_{u=0}^{\infty}\mathfrak{B}_{k,u}(x:\alpha)\frac{\left((1+\lambda t)^{\frac{\omega}{\lambda}}-(1+\lambda t)^{-\frac{\omega}{\lambda}}\right)^{u}}{u!}\times(1+\beta t)^{\frac{r}{\beta}}\left(1+\gamma t^{j}\right)^{\frac{\rho}{\gamma}}$$
$$=\sum_{u=0}^{\infty}\mathfrak{B}_{k,u}(x:\alpha)\sum_{n=0}^{\infty}T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,u:r,\rho)\frac{t^{n}}{n!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{u=0}^{n}\mathfrak{B}_{k,u}(x:\alpha)T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,u:r,\rho)\right)\frac{t^{n}}{n!}.$$

and by (3.5) and (4.10), similarly

$$I=\sum_{n=0}^{\infty}Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(1-x:\omega,r,\rho)\frac{t^{n}}{n!}\times\sum_{n=0}^{\infty}T_{2,\lambda;\omega}(n,k)\frac{t^{n}}{n!}(x)_{k,\alpha}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{u=0}^{n}\binom{n}{u}Bel_{u,\alpha,\lambda,\beta,\gamma}^{(c,j)}(1-x:\omega,r,\rho)\,T_{2,\lambda;\omega}(n-u,k)(x)_{k,\alpha}\right)\frac{t^{n}}{n!}.$$

Thus, we arrive at the following theorem.

**Theorem 17.** *The following summation equality*

$$\sum_{u=0}^{n}\mathfrak{B}_{k,u}(x:\alpha)T_{2,\lambda,\beta,\gamma}^{[j,\omega]}(n,u:r,\rho)=\sum_{u=0}^{n}\binom{n}{u}(x)_{k,\alpha}\,Bel_{u,\alpha,\lambda,\beta,\gamma}^{(c,j)}(1-x:\omega,r,\rho)\times T_{2,\lambda;\omega}(n-u,k)\quad(4.25)$$

*is valid.*

We here generalize the classical Bernoulli $B_n(x)$, Euler $E_n(x)$ and Genocchi $G_n(x)$ polynomials above via the generalized degenerate Gould-Hopper polynomials (2.12) as follows.

**Definition 4.** *The generalized degenerate Gould-Hopper based\ fully degenerate Bernoulli $B_{n,\lambda;\beta,\gamma}^{[j]}(r,\rho)$, Euler $E_{n,\lambda;\beta,\gamma}^{[j]}(r,\rho)$ and Genocchi $G_{n,\lambda;\beta,\gamma}^{[j]}(r,\rho)$ polynomials are defined by the following exponential generating functions:*

$$\sum_{n=0}^{\infty}B_{n,\lambda;\beta,\gamma}^{[j]}(r,\rho)\frac{t^{n}}{n!}=\frac{t}{e_{\lambda}(t)-1}e_{\beta}^{r}(t)e_{\gamma}^{\rho}\left(t^{j}\right)\quad(4.26)$$

$$\sum_{n=0}^{\infty}E_{n,\lambda;\beta,\gamma}^{[j]}(r,\rho)\frac{t^{n}}{n!}=\frac{2}{e_{\lambda}(t)+1}e_{\beta}^{r}(t)e_{\gamma}^{\rho}\left(t^{j}\right)\quad(4.27)$$

$$\sum_{n=0}^{\infty}G_{n,\lambda;\beta,\gamma}^{[j]}(r,\rho)\frac{t^{n}}{n!}=\frac{2t}{e_{\lambda}(t)+1}e_{\beta}^{r}(t)e_{\gamma}^{\rho}\left(t^{j}\right)\quad(4.28)$$

*for $\lambda,\beta,\gamma\in\mathbb{R}\setminus\{0\}$.*

**Remark 6.** *As the parameters $\lambda,\beta$ and $\gamma$ goes to $0$, the generalized degenerate Gould-Hopper based fully degenerate Bernoulli, Euler and Genocchi polynomials reduce to the usual Bernoulli, Euler and Genocchi polynomials, cf.* [12,15,21,23,24,26,27,29,30,33,34].

When $r=\rho=0$, the polynomials in (4.26), (4.27) and (4.28) reduce to the corresponding degenerate numbers, namely $B_{n,\lambda;\beta,\gamma}^{[j]}(0,0):=B_{n,\lambda}$, $E_{n,\lambda;\beta,\gamma}^{[j]}(0,0):=E_{n,\lambda}$ and $G_{n,\lambda;\beta,\gamma}^{[j]}(0,0):=G_{n,\lambda}$, see [12,23] and the references cited therein for further details on the mentioned numbers.

Note that the following relation holds true as has been in the usual Genocchi and Euler numbers:

$$\frac{G_{n+1,\lambda;\beta,\gamma}^{[j]}(r,\rho)}{n+1}=E_{n,\lambda;\beta,\gamma}^{[j]}(r,\rho)\text{ for }n\geqq 0.$$

The summation formulas for the polynomials $B_{n,\lambda;\beta,\gamma}^{[j]}(r,\rho)$, $E_{n,\lambda;\beta,\gamma}^{[j]}(r,\rho)$ and $G_{n,\lambda;\beta,\gamma}^{[j]}(r,\rho)$ are stated in the following theorem without proofs.

**Theorem 18.** *The following formulas are valid*:

$$B_{n,\lambda;\beta,\gamma}^{[j]}(r+\varsigma,\rho+\vartheta)=\sum_{k=0}^{n}\binom{n}{k}B_{n-k,\lambda;\beta,\gamma}^{[j]}(\varsigma,\vartheta)H_{k,\beta,\gamma}^{(j)}(r,\rho)$$

$$E_{n,\lambda;\beta,\gamma}^{[j]}(r+\varsigma,\rho+\vartheta)=\sum_{k=0}^{n}\binom{n}{k}E_{n-k,\lambda;\beta,\gamma}^{[j]}(\varsigma,\vartheta)H_{k,\beta,\gamma}^{(j)}(r,\rho)$$

$$G_{n,\lambda;\beta,\gamma}^{[j]}(r+\varsigma,\rho+\vartheta)=\sum_{k=0}^{n}\binom{n}{k}G_{n-k,\lambda;\beta,\gamma}^{[j]}(\varsigma,\vartheta)H_{k,\beta,\gamma}^{(j)}(r,\rho).$$

We now perform to acquire some representations for $Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)$ by means of the Gould-Hopper based fully degenerate Bernoulli, Euler and Genocchi polynomials and fully degenerate central Bell polynomials.

**Theorem 19.** *The following correlation*

$$Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)=\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\frac{(1)_{n+1-k,\lambda}}{n+1-k}Bel_{k-m,\lambda,\alpha}^{(c)}(x:\omega)\times B_{m,\lambda;\beta,\gamma}^{[j]}(r,\rho)\quad(4.29)$$

*holds true.*

*Proof*. By (4.10) and (4.26), we get

$$\sum_{n=0}^{\infty}Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}(x:\omega,r,\rho)\frac{t^{n}}{n!}$$
$$=e_{\alpha}^{x}\left(e_{\lambda}^{\omega}(t)-e_{\lambda}^{-\omega}(t)\right)e_{\beta}^{r}(t)e_{\gamma}^{\rho}\left(t^{j}\right)\frac{t}{e_{\lambda}(t)-1}\frac{e_{\lambda}(t)-1}{t}$$

$$=\sum_{n=0}^{\infty} Bel_{n,\lambda,\alpha}^{(c)}\left(x:\omega\right)\frac{t^n}{n!}\sum_{n=0}^{\infty} B_{n,\lambda;\beta,\gamma}^{[j]}\left(r,\rho\right)\frac{t^n}{n!}$$
$$\times\sum_{n=0}^{\infty}\left(1\right)_{n+1,\lambda}\frac{t^n}{\left(n+1\right)!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\frac{\left(1\right)_{n+1-k,\lambda}}{n+1-k} Bel_{k-m,\lambda,\alpha}^{(c)}\left(x:\omega\right)$$
$$\times B_{m,\lambda;\beta,\gamma}^{[j]}\left(r,\rho\right)\frac{t^n}{n!},$$

which implies the desired result (4.29).

**Theorem 20.** *The following summation formula*

$$Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}\left(x:\omega,r,\rho\right)$$
$$=\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\frac{\left(1\right)_{n-k,\lambda}}{2} Bel_{k-m,\lambda,\alpha}^{(c)}\left(x:\omega\right)$$
$$\times E_{m,\lambda;\beta,\gamma}^{[j]}\left(r,\rho\right)$$
$$+\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k} Bel_{n-k,\lambda,\alpha}^{(c)}\left(x:\omega\right) E_{k,\lambda;\beta,\gamma}^{[j]}\left(r,\rho\right)$$

*is valid*.

*Proof*. From (4.10) and (4.27), the proof can be completed by utilizing a similar proof method in Theorem 19. So we omit the proof.

**Theorem 21.** *The following relation*

$$Bel_{n,\alpha,\lambda,\beta,\gamma}^{(c,j)}\left(x:\omega,r,\rho\right)$$
$$=\frac{1}{n+1}\sum_{k=0}^{n+1}\sum_{m=0}^{k}\binom{n+1}{k}\binom{k}{m}\frac{\left(1\right)_{n+1-k,\lambda}}{2}$$
$$\times Bel_{k-m,\lambda,\alpha}^{(c)}\left(x:\omega\right) G_{m;\beta,\gamma}^{[j]}\left(r,\rho\right)$$
$$+\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{Bel_{n+1-k,\lambda,\alpha}^{(c)}\left(x:\omega\right) G_{k;\beta,\gamma}^{[j]}\left(r,\rho\right)}{2\left(n+1\right)}$$

*holds true*.

*Proof*. Because of (4.10) and (4.28), the proof can be done by using a similar proof method in Theorem 19. So we omit the proof.

## 5. Conclusion

In this paper, we have first defined the generalized degenerate Gould-Hopper polynomials via the degenerate exponential functions and then have given various relations and formulas such as addition formula and explicit identity. Also, we have defined the generalized Gould-Hopper based degenerate central factorial numbers of the second kind and have presented several identities and relationships. We have considered the generalized Gould-Hopper based fully degenerate central Bell polynomials and have derived multifarious correlations and formulas including summation formulas, derivation rule and correlations with the Stirling numbers of the first kind, the generalized Gould-Hopper based degenerate central factorial numbers of the second kind and the generalized degenerate Gould-Hopper polynomials. We then have investigated some relations with the degenerate Bernstein polynomials for the generalized Gould-Hopper based fully degenerate central Bell polynomials. Lastly, by introducing the Gould-Hopper based fully degenerate Bernoulli, Euler and Genocchi polynomials, we have proved many representations for the generalized Gould-Hopper based fully degenerate central Bell polynomials via the introduced polynomials.

## References


[1] Appell, P.; Kampé de Fériet, J., Fonctions hypergéométriques. Polynômes d'Hermite, Gauthier-Villars, Paris, 1926.

[2] Bell, E. T. Exponential polynomials. Ann. Math. 1934, 35, 258-277.

[3] Bouroubi, S.; Abbas, M. New identities for Bell's polynomials: New approaches. Rostock. Math. Kolloq. 2006, 61, 49-55.

[4] Carlitz, L. Some remarks on the Bell numbers. Fibonacci Quart. 1980, 18, 66-73.

[5] Duran, U.; Acikgoz, M. Unified degenerate central Bell polynomials. J. Math. Anal. 2020, 11(2), 18-33.

[6] Duran, U.; Acikgoz, M. On generalized degenerate Gould-Hopper based fully degenerate Bell polynomials. Accepted for publication in J. Math. Comp. Sci. 2020.

[7] Kim, T.; Kim, D.S. A note on central Bell numbers and polynomials. Russ. J. Math. Phys. 27, 1, 2020, 76-81.

[8] Kim. T.; Kim, D.S. Degenerate central Bell numbers and polynomials. RACSAM. 2019.

[9] Kim. T.; Kim, D.S. On degenerate Bell numbers and polynomials. RACSAM. 2017, 111, 435-446.

[10] Kim, D.S.; Dolgy, D.V.; Kim, T.; Kim, D. Extended degenerate *r*-central factorial numbers of the second kind and extended degenerate *r*-central bell polynomials. Symmetry. 2019, 11, 595.

[11] Mihoubi, M. Bell polynomials and binomial type sequences. Discrete Math. 2008, 308, 2450-2459.

[12] Carlitz, L. Degenerate Stirling, Bernoulli and Eulerian numbers. Utilitas Math. 1975, 15, 51-88.

[13] Carlitz, L. A degenerate Staudt-Clausen theorem. Arch Math (Basel). 1956, 7, 28-33.

[14] Duran, U.; Sadjang, P.N. On Gould-Hopper-based fully degenerate poly-Bernoulli polynomials with a *q*-parameter, Mathematics, 2019, 7, 121.

[15] Howard, F.T. Explicit formulas for degenerate Bernoulli numbers. Discrete Math. 1996, 162, 175-185.

[16] Khan, W. A. A note on degenerate Hermite poly-Bernoulli numbers and polynomials. J. Class. Anal. 2016, 8, 65-76.

[17] Kim. T.; Kim, D.S. Degenerate Bernstein polynomials. RACSAM. 2018.

[18] Kim, D.S.; Kim, T.; Mansour, T.; Seo, J.-J. Degenerate Mittag-Leffer polynomials. Appl. Math. Comput. 2016, 274, 258-266.

[19] Kim. T.; Kim, D.S. Degenerate Laplace transform and degenerate gamma function, Russ. J. Math. Phys. 2017, 24, 241-248.

[20] Kim, T.; Kim, D.S.; Jang, G.-W.; Kwon, J. A note on degenerate Bernstein polynomials. J. Inequal. Appl. 2019, 2019:129.

[21] Kim, W.J.; Kim, D.S.; Kim, H.Y.; Kim, T. Some identities of degenerate Euler polynomials associated with degenerate Bernstein polynomials. J. Inequal. Appl. 2019, 2019:160.

[22] Kwon, H.I.; Kim, T.; Seo, J.-J. A note on degenerate Changhee numbers and polynomials. Proc. Jangjeon Math. Soc. 2015, 18, 295-305.

[23] Lim, D. Some identities of degenerate Genocchi polynomials. Bull. Korean Math. Soc. 2016, 53, 569-579.

[24] Duran, U.; Acikgoz, M.; Araci, S. Hermite based poly-Bernoulli polynomials with a *q*-parameter. Adv. Stud. Contemp. Math. 2018, 28, 285-296.

[25] Dattoli, G.; Lorenzutta S.; Cesarano, C. Finite sums and generalized forms of Bernoulli polynomials, Rend. Math. Appl. 1999, 19, 385-391.

[26] Srivastava, H.M.; Choi, J. Zeta and *q*-Zeta functions and associated series and integrals; Elsevier Science Publishers: msterdam, The Netherlands, 2012, 674 p.

[27] Rainville E.D. Special Functions, The Macmillan Company, New York, 1960.

[28] Araci, S.; Khan, W.A.; Acikgoz, M.; Özel, C.; Kumam, P. A new generalization of Apostol type Hermite-Genocchi polynomials and its applications, SpringerPlus. 2016, 5:860.

[29] Kurt, B.; Simsek, Y. On the Hermite based Genocchi polynomials. Adv. Stud. Contemp. Math., 2013, 23, 13-17.

[30] Pathan, M.A. A new class of generalized Hermite-Bernoulli polynomials. Georgian Math. J. 2012, 19, 559-573.

[31] Widder, D.V. The Heat Equation, Academic Press, New York, 1975.

[32] Acikgoz, M.; Araci, S. On the generating function for Bernstein polynomials. AIP Conference Proceedings, 1281:1, 2010, 1141-1143.

[33] Araci, S.; Riyasat, M.; Khan, S.,; Wani, S.A. Some uni.ed formulas involving generalized-Apostol-type-Gould-Hopper polynomials and multiple power sums. J. Math. Comp. Sci. 2019, 19, 2, 97-115.

[34] Ozarslan, M.A. Hermite-based unified Apostol-Bernoulli, Euler and Genocchi polynomials, Adv. Difference Equ. 2013, 2013:116.

[35] Cheikh Y.B., Zaghouani A. Some discrete *d*-orthogonal polynomials sets. J. Comput. Appl. Math. 2003, 156, 253-263.