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Inequalities for Three-times Differentiable Arithmetic-Harmonically Functions

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Abstract In this work, by using an integral identity together with both the Hölder integral inequality and the power-mean integral inequality we establish several new inequalities for three-times differentiable arithmetic-harmonically-convex function. Then, using this inequalities, we obtain some new inequalities connected with means.

Keywords: *convex function, arithmetic-harmonically-convex function, Hermite-Hadamard's inequality*

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1. Preliminaries and Fundamentals

It is well known that theory of convex sets and convex functions play an important role in mathematics and the other pure and applied sciences. Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences.

Definition 1.1 *A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality*

$$f\left(tx+(1-t)y\right) \le tf\left(x\right)+(1-t)f\left(y\right)$$

valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \varnothing$. This definition is well known in the literature.

Theorem 1.2 *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a,b \in I$ with $a < b$. The following inequality*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \qquad (1.1)$$

holds.

The Hermite-Hadamard integral inequality is very important for convex functions. See [1-6], for the results of the generalization, improvement and extention of the famous integral inequality (1.1).

Definition 1.3 [7,8] *A function $f: I \subset \mathbb{R} \to (0,\infty)$ is said to be arithmetic-harmonically (AH) convex function if for all $x, y \in I$ and $t \in [0,1]$ the equality*

$$f\left(tx+(1-t)y\right) \le \frac{f(x)f(y)}{tf(y)+(1-t)f(x)} \qquad (1.2)$$

holds. If the inequality (1.2) is reversed then the function $f(x)$ is said to be arithmetic-harmonically (AH) concave function.

Theorem 1.4 (Hölder Inequality for Integrals) *Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$ then*

$$\int_a^b |f(x)g(x)|dx \le \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}},$$

with equality holding if and only if $A|f(x)|^p = B|g(x)|^q$ almost everywhere, where A and B are constants.

In order to establish some inequalities of Hermite-Hadamard type integral inequalities for-AH convex functions, we will use the following lemma [5].

Lemma 1.5 *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be three-times differentiable mapping on I° and $f''' \in L[a,b]$, where $a,b \in I^\circ$ with $a<b$, we have the identity*

$$\begin{aligned} &bf(b)-af(a)-\frac{b^2f'(b)-a^2f'(a)}{2} \\ &+\frac{b^3f''(b)-a^3f''(a)}{6}-\int_a^b f(x)dx \qquad (1.3)\\ &=\frac{1}{6}\int_a^b x^3 f'''(x)dx. \end{aligned}$$

In this study, using both Hölder integral inequality, power-mean integral inequality and the identity (1.3) in order to provide inequality for functions whose third derivatives in absolute value at certain power are arithmetic-harmonically-convex functions.

For shortness, throught this paper, we will use the following notations for special means of two nonnegative numbers a,b with $b>a$:

1. The arithmetic mean

$$A := A(a,b) = \frac{a+b}{2}, \quad a,b>0,$$

2. The geometric mean

$$G := G(a,b) = \sqrt{ab}, \quad a,b \geq 0$$

3. The logarithmic mean

$$L := L(a,b) = \begin{cases} \dfrac{b-a}{\ln b - \ln a}, & a \neq b \\ a, & a = b \end{cases}; \; a,b>0$$

4. The p-logaritmic mean

$$L_p := L_p(a,b) = \begin{cases} \left(\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)} \right)^{\frac{1}{p}}, & a \neq b, p \in \mathbb{R} \setminus \{-1,0\} \\ a, & a = b \end{cases}$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$. In addition, we will use the following notation for shortness:

$$I_f(a,b) := bf(b) - af(a) - \frac{b^2 f'(b) - a^2 f'(a)}{2} + \frac{b^3 f''(b) - a^3 f''(a)}{6} - \int_a^b f(x)dx.$$

2. Main Results

Theorem 2.1 *Let $f : I \subset \mathbb{R} \to (0,\infty)$ be a three times-differentiable mapping on I°, and $a,b \in I^\circ$ with $a<b$. If $\left|f'''\right|$ is an arithmetic-harmonically convex function on the interval $[a,b]$, then the following inequality holds:*

i) If $\left|f'''(a)\right| - \left|f'''(b)\right| \neq 0$ then,

$$I_f(a,b) \leq \frac{(b-a)\left|f'''(a)\right|\left|f'''(b)\right|}{3!} \left\{ \frac{1}{B_f}\left(\frac{b^3-a^3}{3} - C_f \frac{b^2-a^2}{2} \right) + C_f^2 (b-a) - \frac{\left(b\left|f'''(b)\right| - a\left|f'''(a)\right| \right)^3}{L\left(\left|f'''(a)\right|, \left|f'''(b)\right| \right)} \right\}. \quad (2.1)$$

ii) *If $\left|f'''(a)\right| - \left|f'''(b)\right| = 0$ then,*

$$\left|I_f(a,b)\right| \leq \frac{(b-a)\left|f'''(b)\right|}{3!} L_3^3(a,b).,$$

where

$$B_f = B_f(a,b;n) = \left|f'''(a)\right| - \left|f'''(b)\right|$$

$$C_f = C_f(a,b;n) = \frac{b\left|f'''(b)\right| - a\left|f'''(a)\right|}{B_f}.$$

Proof. i) *Let $\left|f'''(a)\right| - \left|f'''(b)\right| \neq 0$. If $\left|f'''\right|$ is an arithmetic-harmonically convex function on the interval $[a,b]$, using Lemma 1.5 and*

$$\left|f'''(x)\right| = \left|f'''\left(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b \right)\right| \leq \frac{(b-a)\left|f'''(a)\right|\left|f'''(b)\right|}{(b-x)\left|f'''(b)\right| + (x-a)\left|f'''(a)\right|}$$

we get

$$\begin{aligned} \left|I_f(a,b)\right| &\leq \frac{1}{6}\int_a^b x^3 \left|f'''(x)\right| dx \\ &\leq \frac{1}{6}\int_a^b \frac{x^3(b-a)\left|f'''(a)\right|\left|f'''(b)\right|}{(b-x)\left|f'''(b)\right| + (x-a)\left|f'''(a)\right|} dx \\ &= \frac{(b-a)\left|f'''(a)\right|\left|f'''(b)\right|}{6} \times \int_a^b \frac{x^3}{(b-x)\left|f'''(b)\right| + (x-a)\left|f'''(a)\right|} dx. \end{aligned} \quad (2.2)$$

We can write the following inequality according to B_f and C_f:

$$\begin{aligned} \left|I_f(a,b)\right| &\leq \frac{(b-a)\left|f'''(a)\right|\left|f'''(b)\right|}{3!B_f} \int_a^b \frac{x^3}{x+C_f} dx \\ &= \frac{(b-a)\left|f'''(a)\right|\left|f'''(b)\right|}{3!B_f} \int_a^b \left(x^2 - xC_f + C_f^2 - \frac{C_f^3}{x+C_f} \right) dx \\ &= \frac{(b-a)\left|f'''(a)\right|\left|f'''(b)\right|}{3!B_f} \left(\frac{x^3}{3} - \frac{C_f}{2}x^2 + C_f^2 x - C_f^3 \ln\left(x+C_f \right) \right) \Bigg|_a^b \end{aligned}$$

$$=\frac{(b-a)\left|f'''(a)\right|\left|f'''(b)\right|}{3!B_f}\left\{\frac{b^3-a^3}{3}-C_f\frac{b^2-a^2}{2}\right.$$

$$\left.+C_f^2(b-a)-C_f^3\left[\ln\left(b+C_f\right)-\ln\left(a+C_f\right)\right]\right\}$$

$$=\frac{(b-a)\left|f'''(a)\right|\left|f'''(b)\right|}{3!}\left\{\frac{1}{B_f}\left(\begin{array}{l}\frac{b^3-a^3}{3}-C_f\frac{b^2-a^2}{2}\\+C_f^2(b-a)\end{array}\right)\right.$$

$$\left.-\frac{\left(b\left|f'''(b)\right|-a\left|f'''(a)\right|\right)^3}{L\left(\left|f'''(a)\right|,\left|f'''(b)\right|\right)}\right\}.$$

Therefore, we obtain the desired inequality.

ii) Let $\left|f'''(a)\right|-\left|f'''(b)\right|=0$. Then, substituting $\left|f'''(a)\right|=\left|f'''(b)\right|$ in the inequality (2.2), we obtain

$$\left|I_f(a,b)\right|\le\frac{(b-a)\left|f'''(a)\right|\left|f'''(b)\right|}{6}$$

$$\times\int_a^b\frac{x^3}{(b-x)\left|f'''(b)\right|+(x-a)\left|f'''(a)\right|}dx \qquad (2.3)$$

$$=\frac{\left|f'''(b)\right|}{6}\int_a^b x^3dx=\frac{(b-a)\left|f'''(b)\right|}{3!}L_3^3(a,b).$$

This completes the proof of theorem.

Theorem 2.2 *Let $f:I\subset(0,\infty)\to(0,\infty)$ be a there-times differentiable mapping on I°, and $a,b\in I^\circ$ with $a<b$. If $\left|f'''\right|^q$ is an arithmetic-harmonically convex function on the interval $[a,b]$, then the following inequality holds:*

i) If $\left|f'''(a)\right|^q-\left|f'''(b)\right|^q\neq 0$ then,

$$\left|I_f(a,b)\right|\le\frac{b-a}{3!}$$

$$\times\frac{L_{3p}^3(a,b)\left|f'''(a)\right|\left|f'''(b)\right|}{\left[L\left(\left|f'''(a)\right|,\left|f'''(b)\right|\right)L_{q-1}^{q-1}\left(\left|f'''(a)\right|,\left|f'''(b)\right|\right)\right]^{\frac{1}{q}}}, \qquad (2.4)$$

ii) If $\left|f'''(a)\right|^q-\left|f'''(b)\right|^q=0$ then,

$$\left|I_f(a,b)\right|\le\frac{b-a}{3!}\left|f'''(b)\right|L_{3p}^3(a,b),$$

where

$$B_{q,f}=B_{q,f}(a,b;n)=\left|f'''(a)\right|^q-\left|f'''(b)\right|^q$$

$$C_{q,f}=C_{q,f}(a,b;n)=\frac{b\left|f'''(b)\right|^q-a\left|f'''(a)\right|^q}{B_{q,f}},$$

and $\frac{1}{p}+\frac{1}{q}=1.$

Proof. i) Let $\left|f'''(a)\right|-\left|f'''(b)\right|\neq 0$. If $\left|f'''\right|^q$ for $q>1$ is an arithmetic-harmonically convex function on the interval $[a,b]$, then by applying the well known Hölder integral inequality to the right-hand side of the Lemma 1.5 and using the following identity

$$\left|f'''(x)\right|^q=\left|f'''\left(\frac{b-x}{b-a}a+\frac{x-a}{b-a}b\right)\right|^q$$

$$\le\frac{(b-a)\left|f'''(a)\right|^q\left|f'''(b)\right|^q}{(b-x)\left|f'''(b)\right|^q+(x-a)\left|f'''(a)\right|^q}$$

we have

$$\left|I_f(a,b)\right|\le\frac{1}{6}\left(\int_a^b x^{3p}dx\right)^{\frac{1}{p}}\left(\int_a^b\left|f'''(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$\le\frac{1}{6}\left(\int_a^b x^{3p}dx\right)^{\frac{1}{p}}\left(\int_a^b\frac{(b-a)\left|f'''(a)\right|^q\left|f'''(b)\right|^q}{\left[\begin{array}{l}(b-x)\left|f'''(b)\right|^q\\+(x-a)\left|f'''(a)\right|^q\end{array}\right]}dx\right)^{\frac{1}{q}}. \qquad (2.5)$$

From here, we can write the following inequality

$$\left|I_f(a,b)\right|$$

$$\le\frac{b-a}{6}L_{3p}^3(a,b)\left|f'''(a)\right|\left|f'''(b)\right|\left(\frac{1}{B_f}\int_a^b\frac{1}{x+C_f}dx\right)^{\frac{1}{q}}$$

$$=\frac{b-a}{6}L_{3p}^3(a,b)\left|f'''(a)\right|\left|f'''(b)\right|\left(\frac{1}{B_f}\ln\left(x+C_f\right)\Big|_a^b\right)^{\frac{1}{q}}$$

$$=-\frac{b-a}{6}L_{3p}^3(a,b)\left|f'''(a)\right|\left|f'''(b)\right|\left(\frac{1}{B_f}\ln\left(\frac{b+C_f}{a+C_f}\right)\right)^{\frac{1}{q}}$$

$$=\frac{b-a}{6}L_{3p}^3(a,b)\left|f'''(a)\right|\left|f'''(b)\right|$$

$$\times\left(\frac{\ln\left(\left|f'''(a)\right|^q\right)-\ln\left(\left|f'''(b)\right|^q\right)}{\left|f'''(a)\right|^q-\left|f'''(b)\right|^q}\right)^{\frac{1}{q}}$$

$$=\frac{b-a}{6}\frac{L_{3p}^3(a,b)\left|f'''(a)\right|\left|f'''(b)\right|}{L^{\frac{1}{q}}\left(\left|f'''(a)\right|^q,\left|f'''(b)\right|^q\right)}$$

$$= \frac{b-a}{3!} \frac{L_{3p}^{3}(a,b)\left|f'''(a)\right|\left|f'''(b)\right|}{\left[L\left(\left|f'''(a)\right|,\left|f'''(b)\right|\right)L_{q-1}^{q-1}\left(\left|f'''(a)\right|,\left|f'''(b)\right|\right)\right]^{\frac{1}{q}}},$$

where

$$\int_a^b x^{3p}dx = (b-a)L_{3p}^{3p}(a,b)$$

$$\frac{\ln\left|f'''(a)\right|^q - \ln\left|f'''(b)\right|^q}{\left|f'''(a)\right|^q - \left|f'''(b)\right|^q}$$

$$= \frac{\ln\left|f'''(a)\right| - \ln\left|f'''(b)\right|}{\left|f'''(a)\right| - \left|f'''(b)\right|} \frac{q\left[\left|f'''(a)\right| - \left|f'''(b)\right|\right]}{\left|f'''(a)\right|^q - \left|f'''(b)\right|^q}$$

$$= \left[L\left(\left|f'''(a)\right|,\left|f'''(b)\right|\right)L_{q-1}^{q-1}\left(\left|f'''(a)\right|,\left|f'''(b)\right|\right)\right]^{-1}.$$

ii) Let $\left|f'''(a)\right| - \left|f'''(b)\right| = 0$. In this case, substituting $\left|f'''(a)\right| = \left|f'''(b)\right|$ in the inequality (2.5) we get the following inequality:

$$\left|I_f(a,b)\right| \le \frac{b-a}{3!}\left|f'''(b)\right| L_{3p}^{3}(a,b). \qquad (2.6)$$

This completes the proof of the Theorem.

Theorem 2.3 *Let $f : I \subset (0,\infty) \to (0,\infty)$ be a there-times differentiable mapping on I°, and $a,b \in I^\circ$ with $a<b$. If $\left|f'''\right|^q, q \ge 1$ is an arithmetic-harmonically convex function on the interval $[a,b]$, then the following inequality holds:*

i) If $\left|f'''(a)\right|^q - \left|f'''(b)\right|^q \ne 0$ then,

$$\left|I_f(a,b)\right| \le \frac{b-a}{6} L_3^{3\left(1-\frac{1}{q}\right)}(a,b)\left|f'''(a)\right|\left|f'''(b)\right|$$
$$\times\left\{\frac{1}{B_{q,f}}\left[\frac{b^3-a^3}{3} - C_{q,f}\frac{b^2-a^2}{2} + C_{q,f}^2(b-a)\right.\right. \qquad (2.7)$$
$$\left.\left. - \frac{C_{q,f}^3}{L\left(\left|f'''(a)\right|^q,\left|f'''(b)\right|^q\right)}\right]\right\}^{\frac{1}{q}},$$

ii) *If $\left|f'''(a)\right|^q - \left|f'''(b)\right|^q = 0$ then,*

$$\left|I_f(a,b)\right| \le \frac{(b-a)L_3^3(a,b)\left|f'''(b)\right|}{3!}$$

where

$$B_{q,f} = B_{q,f}(a,b;n) = \left|f'''(a)\right|^q - \left|f'''(b)\right|^q$$

$$C_{q,f} = C_{q,f}(a,b;n) = \frac{b\left|f'''(b)\right|^q - a\left|f'''(a)\right|^q}{B_{q,f}}.$$

Proof. i) Let $\left|f'''(a)\right|^q - \left|f'''(b)\right|^q \ne 0$. If $\left|f'''\right|^q$ for $q \ge 1$ is an arithmetic-harmonically convex function on the interval $[a,b]$, then by applying well known power-mean integral inequality to the right-hand side of the Lemma 1.5 and using

$$\left|f'''(x)\right|^q \le \frac{(b-a)\left|f'''(a)\right|^q\left|f'''(b)\right|^q}{(b-x)\left|f'''(b)\right|^q + (x-a)\left|f'''(a)\right|^q}$$

we have

$$\left|I_f(a,b)\right| \le \frac{1}{6}\left(\int_a^b x^3dx\right)^{1-\frac{1}{q}}\left(\int_a^b x^3\left|f'''(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$\le \frac{1}{6}\left(\int_a^b x^3dx\right)^{1-\frac{1}{q}}\left(\int_a^b \frac{x^3(b-a)\left|f'''(a)\right|^q\left|f'''(b)\right|^q}{\left[\begin{array}{c}(b-x)\left|f'''(b)\right|^q \\ +(x-a)\left|f'''(a)\right|^q\end{array}\right]}dx\right)^{\frac{1}{q}}$$

$$= \frac{b-a}{6}L_3^{3\left(1-\frac{1}{q}\right)}(a,b)\left|f'''(a)\right|\left|f'''(b)\right|$$
$$\times\left(\int_a^b \frac{x^3}{(b-x)\left|f'''(b)\right|^q + (x-a)\left|f'''(a)\right|^q}dx\right)^{\frac{1}{q}} \qquad (2.8)$$

$$= \frac{b-a}{6}L_3^{3\left(1-\frac{1}{q}\right)}(a,b)\left|f'''(a)\right|\left|f'''(b)\right|$$
$$\times\left[\frac{1}{B_{q,f}}\int_a^b\left(x^2 - xC_{q,f} + C_{q,f}^2 - \frac{C_{q,f}^3}{x+C_{q,f}}\right)dx\right]^{\frac{1}{q}}$$

$$= \frac{b-a}{6}L_3^{3\left(1-\frac{1}{q}\right)}(a,b)\left|f'''(a)\right|\left|f'''(b)\right|$$
$$\times\left[\frac{1}{B_{q,f}}\left(\begin{array}{c}\frac{x^3}{3} - \frac{C_{q,f}}{2}x^2 + C_{q,f}^2x \\ -C_{q,f}^3\ln\left(x+C_{q,f}\right)\end{array}\right)\Bigg|_a^b\right]^{\frac{1}{q}}$$

$$= \frac{b-a}{6} L_3^{3\left(1-\frac{1}{q}\right)}(a,b) \left|f'''(a)\right| \left|f'''(b)\right|$$

$$\left\{ \frac{1}{B_{q,f}} \left[\frac{b^3-a^3}{3} - C_{q,f} \frac{b^2-a^2}{2} + C_{q,f}^2 (b-a) \right] \right.$$

$$\left. - \frac{C_{q,f}^3}{L\left(\left|f'''(a)\right|^q, \left|f'''(b)\right|^q\right)} \right\}^{\frac{1}{q}},$$

where $\int_a^b x^3 dx = (b-a)\left[L_3^3(a,b)\right]$.

ii) Let $\left|f'''(a)\right|^q - \left|f'''(b)\right|^q = 0$. By using the inequality (2.8), we have

$$\left|I_f(a,b)\right| \le \frac{(b-a)L_3^3(a,b)\left|f'''(b)\right|}{3!}. \quad (2.9)$$

This completes the proof of the Theorem.

Corollary 2.4 *If we take $q=1$ in the inequality (2.7), we get the following inequality:*

$$\left| \frac{bf(b)-af(a)}{b-a} - \frac{b^2 f'(b) - a^2 f'(a)}{2(b-a)} + \frac{b^3 f''(b) - a^3 f''(a)}{6(b-a)} - \frac{1}{b-a}\int_a^b f(x)dx \right|$$

$$\le \frac{\left|f'''(a)\right|\left|f'''(b)\right|}{3!} \left\{ \frac{b^3-a^3}{3B_f} - C_f \frac{b^2-a^2}{2B_f} + \frac{C_f^2}{B_f}(b-a) - \frac{C_f^3}{L\left(\left|f'''(a)\right|, \left|f'''(b)\right|\right)} \right\} \quad (2.10)$$

3. Applications for Special Means

If $p \in (-1,0)$ then the function $f(x) = x^p, x>0$ is an arithmetic harmonically-convex function [7]. Using this function we obtain following propositions related to means:

Proposition 3.1 *Let $0<a<b$ and $p \in (-1,0)$. Then we have the following inequalities:*

$$L_{p+3}^{p+3}(a,b)$$

$$\le G^{2p}(a,b) \left\{ -\frac{1}{p} \frac{2A\left(a^2,b^2\right) + G^2(a,b)}{3L_{p-1}^{p-1}(a,b)} - \frac{p+1}{p^2} \frac{A(a,b)L_p^p(a,b)}{L_{p-1}^{2(p-1)}(a,b)} \right.$$

$$\left. - \frac{(p+1)^2}{p^3} \frac{L_p^{2p}(a,b)}{L_{p-1}^{3(p-1)}(a,b)} + \left(\frac{p+1}{p}\right)^3 \frac{L_p^{3p}(a,b)}{L(a,b)L_{p-1}^{4(p-1)}(a,b)} \right\}.$$

Proof. $\left|f'''(x)\right| = x^p$ is convex function for $p \in (-1,0)$. Therefore, the assertion follows from the inequality (2.1) in the Theorem 2.1, for

$$f:(0,\infty) \to \mathbb{R},\ f(x) = \frac{x^{p+3}}{(p+1)(p+2)(p+3)}.$$

Proposition 3.2 *Let $a,b \in (0,\infty)$ with $a<b$, $q>1$ and $m \in (-1,0)$. Then, we have the following inequality:*

$$L_{\frac{m}{q}+3}^{\frac{m}{q}+3}(a,b) \le \frac{L_{3p}^3(a,b) G^{\frac{2m}{q}}(a,b)}{\left[L(a,b)L_{m-1}^{m-1}(a,b)\right]^{\frac{1}{q}}}.$$

Proof. The assertion follows from the inequality (2.4) in the Theorem 2.2. Let

$$f(x) = \frac{q^3}{(m+q)(m+2q)(m+3q)} x^{\frac{m}{q}+3},\quad x \in (0,\infty).$$

Then $\left|f'''(x)\right|^q = x^m$ is an arithmetic harmonically-convex on $(0,\infty)$ and the result follows directly from Theorem 2.2.

Proposition 3.3 *Let $a,b \in (0,\infty)$ with $a<b$, $q>1$ and $m \in (-1,0)$. Then, we have the following inequality:*

$$L_{\frac{m}{q}+3}^{\frac{m}{q}+3}(a,b) \le L_3^{3\left(1-\frac{1}{q}\right)}(a,b) G^{\frac{2m}{q}}(a,b) \quad (3.1)$$

$$\times \left\{ -\frac{1}{m} \frac{2A\left(a^2,b^2\right) + G^2(a,b)}{3L_{p-1}^{p-1}(a,b)} - \frac{m+1}{m^2} \frac{A(a,b)L_m^m(a,b)}{L_{m-1}^{2(m-1)}(a,b)} - \frac{(m+1)^2}{m^3} \frac{L_m^{2m}(a,b)}{L_{m-1}^{3(m-1)}(a,b)} + \left(\frac{m+1}{m}\right)^3 \frac{L_m^{3m}(a,b)}{L(a,b)L_{m-1}^{4(m-1)}(a,b)} \right\}^{\frac{1}{q}}.$$

Proof. The assertion follows from the inequality (2.6) in the Theorem 2.3. Let

$$f(x) = \frac{q^3}{(m+q)(m+2q)(m+3q)} x^{\frac{m}{q}+3},\quad x \in (0,\infty).$$

Then $\left|f'''(x)\right|^q = x^m$ is an arithmetic harmonically-convex on $(0,\infty)$ and the result follows directly from Theorem 2.3.

Corollary 3.4 *If we take* $q=1$ *in the inequality (3.1), we get the following inequality:*

$$L_{m+3}^{m+3}(a,b) \le \frac{G^{2m}(a,b)}{\left[mL_{m-1}^{m-1}(a,b)\right]^3}\left\{\frac{\left[(m+1)L_m^m(a,b)\right]^3}{L(a,b)L_{m-1}^{m-1}(a,b)}\right.$$
$$+(m+1)L_m^m(a,b)+m(m+1)L_m^m(a,b)L_{m-1}^{m-1}(a,b)$$
$$\left.+\left(\frac{2A\left(a^2,b^2\right)+G^2(a,b)}{3}\right)\left[mL_{m-1}^{m-1}(a,b)\right]^2\right\}.$$

4. Conclusion

We established several new inequalities for three-times differentiable arithmetic-harmonically-convex function and obtained some new inequalities connected with means. Similar method can be applied the different type of convexity.

References

[1] Dragomir, S.S. and Pearce, C.E.M., *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[2] Kadakal, H., *Hermite-Hadamard type inequalities for two times differentiable arithmetic-harmonically convex functions*, Cumhuriyet Science Journal, (Accepted for publication), 2019.

[3] Kadakal M., İşcan, İ., Kadakal H. and Bekar K., *On improvements of some integral inequalities*, Researchgate, Preprint, 2019.

[4] Kadakal, H., Kadakal, M. and İşcan, İ., *Some New Integral Inequalities for n-Times Differentiable s-Convex and s-Concave Functions in the Second Sense*, Mathematics and Statistics 5(2), 2017, 94-98.

[5] Maden, S., Kadakal, H., Kadakal, M. and İşcan, İ., *Some new integral inequalities for n-times differentiable convex and concave functions. J. Nonlinear Sci. Appl., 10 2017, 6141–6148.*

[6] Sarikaya, M.Z. and Aktan, N., *On the generalization of some integral inequalities and their applications*, Mathematical and Computer Modelling, 54, (2011), 2175-2182.

[7] Dragomir, S.S., *Inequalities of Hermite-Hadamard type for AH-convex functions*, Stud. Univ. Babes-Bolyai Math. 61(4) (2016), 489-502.

[8] Zhang, T.-Y. and Qi, F., *Integral Inequalities of Hermite–Hadamard Type for m-AH Convex Functions*, Turkish Journal of Analysis and Number Theory, 2(3), 2014, 60-64.