On Some Properties of Coprime Labelled Graphs

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Abstract A *Co − prime − Labeled − Graph* is a labelled graph denoted by $ER − \varphi(G(p,q))$ in which the vertex set V of an $ER − \varphi(G(p,q))$ has p vertices labeled $\{1, 2, \ldots, p\}$ and q edges such that there exist an edge between two distinct vertices labeled $\{i$ and $j\}$, if $\{i$ and $j\}$ are coprime to each other. In this paper, some properties of the ER-$\varphi(G(p,q))$ are studied. An algorithm to compute GCD and LCM of any two numbers between 1 and p by means of an ER-$\varphi(G(p,q))$ graph is also described.

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1. Introduction

In number theory, the **totient** function or *Euler's phi function* $\varphi(n)$ of a positive integer n plays a vital role in group theory [1]. It determines the size of the multiplicative group modulo n. Apart from this, it also finds application in one of the very prevalent cryptographic techniques called RSA algorithm. It is well known that, $\varphi(n)$ is defined as the number of integers less than or equal to n, that are coprime to n. Paul Erdös extensively worked on co-prime graphs [2]. He has established very good fundamental results on number of cycles in coprime graphs. Further in [3], the author has studied exclusively on graph labeling and Euler's Phi function for planar graphs. Motivated by this, we study the co-prime graphs from spectral theory perspective. In this paper we apply graph theoretic approach to study the $\varphi(n)$ based on coprime graph labelling. More graph labelling techniques can be found in [4]. The coprime labeled graph is denoted by ER-$\varphi(G(p,q))$. The vertex set V of ER-$\varphi(G(p,q))$ has p vertices, labeled with $\{1, 2, \ldots, p\}$ and q edges. If two distinct vertices labeled i and j are coprime, then there exists an edge between them.

This paper is organized into four sections. In section 2, main results are discussed. In section 3, algorithms to compute the greatest common divisor (GCD) and the least common multiplier (LCM) of any two given numbers between 1 and p using ER-$\varphi(G(p,q))$ are described.

Final section is for conclusions. For terminology and notations, one can look at [1,5]. The following are the most used notations:

(1) $\pi(x)$ is the set of prime numbers up to a real number x.

(2) (a,b) is the GCD of two numbers a and b and $[a,b]$ is the LCM of them.

(3) A wheel graph W_p is a graph with p vertices, formed by connecting a single vertex to all vertices of a C_{p-1} cycle.

(4) Chromatic number $\chi(G)$ of a graph G is the minimum number of colors to properly color the graph.

(5) Chromatic polynomial $f(G,t)$ of a graph G is the number of different ways of properly coloring a labeled graph G by t colors.

(6) Dominating set of a graph $G(V,E)$ is a subset S of V such that every vertex of $V − S$ is adjacent to some vertex of S.

(7) Domination Number $\gamma(G)$ is the cardinality of the smallest dominating set.

(8) $L(G)$ is the line graph of G.

(9) Permanent $Per(A)$ of a matrix A is the sum $\sum_{\alpha} \Pi_{i=1}^{p} a_{i,\alpha(i)}$, where α runs over all permutations of the set $\{1, 2, \cdots, p\}$.

(10) Energy $E(G)$ of a graph $G(p,q)$ is the sum of the absolute eigenvalues of the adjacency matrix A of G,

that is, $E(G) = \sum_{i=1}^{p} |\lambda_i|$, where λ_i are the eigenvalues of A [6].

(11) Two graphs are said to be equi-energetic if and only if their energies are equal [7].

(12) ER^{π}-$\varphi(G(p,q))$ is a variant of $ER - \varphi(G(p,q))$ in which the vertices of the graph are labeled with the first $p-1$ prime numbers $\{2,3,5,\cdots\}$ and 1.

2. Results

In this section, we state some observations and results of an ER-$\varphi(G(p,q))$ graph.

2.1. Observations

(1) $ER - \varphi(G(p,q))$ is a simple connected graph without pendant vertex.

(2) $\Delta(ER$-$\varphi(G(p,q))) = p-1$.

(3) The trivial graph $G(1,0)$ is an $ER-\varphi$ graph.

(4) K_2 and K_3 are the only complete graphs which are $ER-\varphi$ graphs.

(5) For $p \geq 4$, the ER-$\varphi(G(p,q))$ is not a regular graph.

(6) ER-$\varphi(G(p,q))$ has a star graph $K_{1,p-1}$ as its subgraph.

(7) For the ER-$\varphi(G(p,q))$,

(i) The girth is 3.

(ii) The circumference is p.

(iii) The diameter is 2.

Theorem 2.1. *The number of edges in ER-$\varphi(G(p,q))$ is*

$$q = \sum_{i=2}^{p} \varphi(i) \ for \ p \geq 2.$$

Proof. By definition of an $ER - \varphi$ graph, there exists an edge between any two distinct vertices i and j of the ER-$\varphi(G(p,q))$ if and only if $(i,j) = 1$ and $i \neq j$. Thus, the vertex labelled with p has $\varphi(p)$ number of edges with vertices in $\{1,2,\cdots,p-1\}$. Similarly the vertex labelled with $p-1$ has $\varphi(p-1)$ edges with vertices in $\{1,2,\cdots,p\}$, excluding the edge with vertex p if any, since this edge is already counted within the edge set of vertex labeled by p. Continuing this way for any arbitrary vertex labelled by r, there are $\varphi(r)$ edges with vertices in $\{1,2,\cdots,r-1\}$ excluding the edges with vertices from $\{r+1,r+2,\cdots,p\}$ if any, as these edges are already accounted with the edge set of vertices of $\{r+1,r+2,\cdots,p\}$. Similarly the vertex labelled by 2 is having one edge with vertex labelled by 1 with excluding those edges which are associated with $\{3,4,\cdots,p\}$ and finally vertex labeled by 1 has zero edges with excluding those edges which are associated with $\{2,3,4,\cdots,p\}$.

Thus the total number of edges in ER-$\varphi(G(p,q))$ is

$$q = \sum_{i=2}^{p} \varphi(i).$$

Corollary 2.2. The number of edges in ER-$\varphi(G(p,q))$ is less than or equal to $\binom{p}{2}$.

Proof. From Theorem 2.1,

$$q = \sum_{i=2}^{p} \varphi(i) \ for \ p \geq 2.$$

and using the Euler's totient property, $\varphi(i) \leq i-1$, we get

$$q \leq \sum_{i=2}^{p} (i-1) \ for \ p \geq 2$$

$$q \leq \binom{p+1}{2} - p.$$

Thus $q \leq \binom{p}{2}$.

Example 1. The number of edges in ER-$\varphi(G(3,q))$ is $q = \varphi(2) + \varphi(3) = 1 + 2 = 3$. The number of edges in ER-$\varphi(G(p+1,q))$ is $q + \varphi(p+1)$, where q is the number of edges in ER-$\varphi(G(p,q))$.

Theorem 2.3. If $p \geq 5$ and p is odd, then ER-$\varphi(G(3,q))$ contains a spanning sub-graph as a wheel graph W_p.

Proof. The proof can be given in two steps: 1) The ER-$\varphi(G(p,q))$ has C_{p-1} cycles, and 2) Every vertex of C_{p-1} has an edge with a distinguished vertex. Without loss of generality, let the vertex labelled by 1 be the distinguished vertex and the vertices $\{2,3,\cdots,p\}$ be the possible vertices of C_{p-1}. Since every vertex labelled by $i > 2$ is coprime with the vertices labelled $i+1(mod\ p)$ and $i-1(mod\ p)$. By the virtue of this, it has an edge with the vertices labeled by $i+1(mod\ p)$ and $i-1(mod\ p)$ and since p is odd, the vertex labelled by 2 has an edge with vertex labelled by p and 3. Thus these edges together with the vertices form a cycle C_{p-1}. By definition, every natural number is co-prime to 1. Thus all the vertices labelled by $\{2,3,\cdots,p\}$ of C_{p-1} has an edge with the vertex labelled by 1. Hence ER-$\varphi(G(p,q))$ contains a wheel spanning subgraph W_p.

Corollary 2.4. If $p \geq 4$ and p is even, then ER-$\varphi(G(p,q))$ contains a subgraph $W_p - x$.

Proof. Using Theorem 2.2, as p is even, there can't be an edge between the vertices labelled by 2 and p. Thus no C_{p-1} cycle can be formed with the vertices $\{2,3,\cdots,p\}$. Thus an ER-$\varphi(G(p,q))$ contains a subgraph $W_p - x$, whenever p is even.

Theorem 2.5. *ER-* $\varphi(G(p,q)) - \{p\}$ *is also an ER-* φ *graph and is isomorphic to ER-* $\varphi(G(p-1, q - \varphi(p)))$.

Proof. ER- $\varphi(G(p,q)) - \{p\}$ is a graph obtained by removing the vertex labelled by p and all the edges associated with it. Thus the resultant graph has $p-1$ vertices $\{1, 2, 3, \cdots, p-1\}$ and there exist an edge between any pair of vertices if and only if their labels are co-primes. Hence *ER-* $\varphi(G(p,q)) - \{p\}$ is ER- φ graph having number of vertices and edges as $p-1$ and $q - \varphi(p)$, respectively, and therefore, is isomorphic to $ER - \varphi(G(p-1, q - \varphi(p)))$ graph.

Theorem 2.6. *If* $\pi(p)$ *is the set of primes up to* p, *then maximal clique in ER-* $\varphi(G(p,q))$ *is* $K_{|\pi(p)|+1}$.

Proof. Since any two prime numbers are coprime, there exist edges between all the vertices labelled with prime numbers, and as these numbers are coprime with 1, they together form a complete graph with $|\pi(p)|+1$ vertices. Thus the maximal clique in *ER-* $\varphi(G(p,q))$ is $K_{|\pi(p)|+1}$.

Corollary 2.7. *The chromatic number* χ *of ER-* $\varphi(G(p,q))$ *is* $|\pi(p)|+1$.

Proof. By using Theorem 2.5, maximal clique in *ER-* $\varphi(G(p,q))$ is $K_{|\pi(p)|+1}$. Hence it requires $|\pi(p)|+1$ colors to color the $|\pi(p)|$ prime labelled vertices and one more color to color the vertex labelled by 1. Since the remaining $p - |\pi(p)| - 1$ vertices can be colored by $|\pi(p)|+1$ colors. Thus $\chi(ER$ - $\varphi(G(p,q))$) is $|\pi(p)|+1$.

Theorem 2.8. *Let* $\{p_1, p_2, p_3, ...p_{|\pi(p)|}\}$ *be the set of prime numbers from 1 to* p *with* $p_1 < p_2 < p_3 < \cdots < p_{|\pi(p)|}$ *and* $t \geq |\pi(p)|+1$. *Then chromatic polynomial of ER-* $\varphi(G(p,q))$ *is given by*

$$f(ER - \varphi(G(p,q),t)) = t \prod_{i=1}^{|\pi(p)|} (t-i)^{\left\lfloor \frac{p}{|p_i|} \right\rfloor - \sum_{j=1, i+1>0}^{i-1} \left\lfloor \frac{p}{p_j p_i} \right\rfloor}.$$

Proof. Let there are t colors available to color graph *ER-* $\varphi(G(p,q))$. By corollary 2.7, $f(ER - \varphi(G(p,q),t)) = 0$, for $t < |\pi(p)|+1$. Wlog, let $t >= |\pi(p)|+1$ and we color the vertices labled with 1, $p_1, p_2, p_3, ..., p_i, ...p_{|\pi(p)|}$ in $t, t-1, t-2, ..., t-i, ..., t - p_{|\pi(p)|}$ ways respectively.

Let $P = \{P_0, P_1, P_2, P_3, ..., P_i, P_{i+1}, ..., P_{P_{|\pi(p)|}}\}$ be the set of sets $P_0, P_1, P_2, P_3, ..., P_i, P_{i+1}, ..., P_{P_{|\pi(p)|}}$ with the following properties.

Let P_0 ={vertex labeled with 1}. Let P_1 be the set of vertices which are labeled with the numbers which are divisible by prime p_1. Then $| P_1 | = \left\lfloor \frac{p}{p_1} \right\rfloor$.

Let P_2 be the set of vertices which are labeled with the numbers which are divisible by prime p_2 and not

divisible by prime p_1. Then $| P_2 | = \left\lfloor \frac{p}{p_2} \right\rfloor - \left\lfloor \frac{p}{p_1 p_2} \right\rfloor$..

Wlog, for all i in $[2, p_{|\pi(p)|}]$, let P_i be the set of vertices which are labeled with the numbers which are divisible by prime p_i and not divisible by primes p_1, p_2, p_3, ... p_{i-1}. Then

$$P_i \mid = \left\lfloor \frac{p}{p_i} \right\rfloor - \sum_{j=1}^{i-1} \left\lfloor \frac{p}{p_j p_i} \right\rfloor$$

$$P_0 \bigcap P_1 \bigcap P_2 \bigcap P_3 \bigcap \cdots \bigcap P_i \bigcap P_{i+1} \bigcap \cdots \cdots \bigcap P_{P_{|\pi(p)|}} = \phi.$$

$$\sum_{i=0}^{|\pi(p)|} |P_i| = p.$$

Further vertex of set P_0 can be colored in t ways and for i in $[1, p_\pi]$, all the vertices of the set P_i are not connected to each other and hence all the vertices of this set can be colred with the same color and thus every vertex of this set can be colored in $t-i$ ways. Thus in total all the vertices of P_i can be clored in $(t-i)^{|P_i|}$ or

$$(t-i)^{\left\lfloor \frac{p}{p_i} \right\rfloor - \sum_{j=1, i+1>0}^{i-1} \left\lfloor \frac{p}{p_j p_i} \right\rfloor}$$ ways.

$$\therefore f(ER - \varphi(G(p,q),t)) = t \prod_{i=1}^{|\pi(p)|} (t-i)^{\left\lfloor \frac{p}{p_i} \right\rfloor - \sum_{j=1, i+1>0}^{i-1} \left\lfloor \frac{p}{p_j p_i} \right\rfloor}.$$

Theorem 2.9. If pi is a prime number which is a label of a vertex v in ER- $\varphi(G(p,q))$ then the degree of v is ,

$p - \left\lfloor \frac{p}{p_i} \right\rfloor$.

Proof. Let a vertex v in ER- $\varphi(G(p,q))$ be labelled by a prime number p_i, where $1 \leq p_i \leq p$. The maximum degree of any vertex in $ER - \varphi(G(p,q))$ is $p-1$. Then number of non-coprimes from 1 to p with p_i is $\left\lfloor \frac{p}{p_i} \right\rfloor$. Since p_i is a prime number and no number except 1 and itself can divide p_i. Thus number of coprimes from 1 to p with p_i is $p - \left\lfloor \frac{p}{p_i} \right\rfloor$. By the definition of $ER - \varphi(G(p,q))$, the degree of this vertex v is $p - \left\lfloor \frac{p}{p_i} \right\rfloor$.

Theorem 2.10. The sum of degrees of vertices labelled with non prime vertices in $ER - \varphi(G(p,q))$ is

$$2\sum_{i=2}^{p} \varphi(i) - p | \pi(p) | + \sum_{j \in \pi(p)} \left\lfloor \frac{p}{\% p_j} \right\rfloor, \qquad (1)$$

where $2 \leq i \leq p$ *and each* p_i *is a prime number.*

Proof. Let the vertices v_{x_j} and v_{y_j} are labelled with non-prime and prime numbers respectively. Also let S be the set of numbers such that $S = \{1, 2, ..., p\} \setminus \pi(p)$. It is

well-known that the sum of degrees of all the vertices of any graph is twice the number of edges in it. By using this and Theorems 2.1 and 2.8,

$$\sum_{i=1}^{p} d(v_i) = \sum_{k \in S} d(v_{x_k}) + \sum_{k \in \pi(p)} d(v_{y_j}),$$

i.e.,
$$\sum_{k \in S} d(v_{x_k}) = 2\sum_{i=2}^{p} \varphi(i) - \sum_{j \in \pi(p)} d(v_{y_j})$$

$$= 2\sum_{i=2}^{p} \varphi(i) - \sum_{j \in \pi(p)} \left(p - \left\lfloor \frac{p}{p_j} \right\rfloor \right)$$

i.e., $\sum_{k \in S} d(v_{x_k}) = 2\sum_{i=2}^{p} \varphi(i) - p \mid \pi(p) \mid + \sum_{j \in \pi(p)} \left\lfloor \frac{p}{p_j} \right\rfloor.$

This completes the proof.

Theorem 2.11. If $p \geq 3$, then $ER\text{-}varphi(G(p,q))$ always contains a Hamiltonian cycle.

Proof. Case 1: Let p be odd. By using mathematical Induction: it is true for $p = 3$, the $ER - \varphi(G(3,3))$ is K_3, since K_3 is a Hamiltonian graph. Now assume that it is true for $p = 2n - 1$. We want to prove it for $p = 2n + 1$. By definition and Theorem 2.2, $ER - \varphi(G(2n+1,q))$ has a spanning subgraph as wheel graph W_{2n+1}. But a wheel graph has a Hamiltonian cycle. Thus $ER - \varphi(G(2n+1,q))$ has a Hamiltonian cycle.

Case 2: Let $p = 2n$, $n \geq 2$ be even and $\{v_1, v_2, \ldots, v_{p=2n}\}$ is the vertex set labeled with $\{1, 2, \ldots, 2n\}$. By Corollary 2.3, $ER - \varphi(G(p,q))$ contains a spanning subgraph $W_p - x$ then there exist a path $P_{2n-1} = v_1 v_2 \cdots v_{p=2n}$ and edge or path ($v_1 v_{p=2n}$) between the vertices v_1 (labeled 1) and $v_{p=2n}$ (labeled $p = 2n$). By joining these two paths, a cycle $C_{p=2n}$ is formed, which proves the theorem for even p also. Hence the result is true all values of p. Note that (L($ER\text{-}\varphi(G(p,q))$) is also a Hamiltonian graph).

Theorem 2.12. *The domination number and the total domination number of ER-* $\varphi(G(p,q))$ *is* $\gamma(ER\text{-}\varphi(G(p,q))) = \gamma_T(ER\text{-}\varphi(G(p,q))) = 1.$

Proof. In $ER\text{-}\varphi(G(p,q))$, the vertex labeled by 1 is adjacent to all the other vertices. Thus the set $\{1\}$ is the minimum dominating set of \$ER\$- $\varphi(G(p,q))$ having the cardinality 1. Thus $\gamma(ER\text{-}\varphi(G(p,q))) = 1$. Similarly $\gamma_T(ER\text{-}\varphi(G(p,q))) = 1.$

Theorem 2.13. *If* $p \geq 7$, $ER - \varphi(G(p,q))$ *is non-planar.*

Proof. By Kuratowaski's theorem on planarity, a graph is non-planar if and only if it contains either K_5 or $K_{3,3}$ as its subgraph, or obtains any of these subgraphs by means of homeomorphic operations. By Theorem 2.5, every $ER - \varphi(G(p,q))$ has $K_{\pi(p)+1}$ as its maximal clique. For $p \geq 7$, the number of primes is ≥ 4. Thus $ER - \varphi(G(p,q))$ has Kuratowaski's 2nd graph K_5 as its subgraph. Thus $ER - \varphi(G(p,q))$ is non-planar.

Remark 2.14. *With* $p < 7$, *no Kuratowaski's graph exists as a subgraph in* $ER - \varphi(G(p,q))$ *and also these graphs can be embedded in a plane surface. Hence they are planar graphs.*

Theorem 2.15. *If* $p = 2n + 1$, $n \geq 2$, *then* $ER\text{-}\varphi(G(p,q))$ *has wheel subgraphs* $W_5, W_7, \ldots, W_p$,. *Therefore, the number of wheel subgraphs is* $n - 1$.

Proof. By Theorem 2.4, it is true that, $ER - \varphi(G(p,q))$ has $ER - \varphi(G(p,q)) - \{p\}$ as its subgraph and is also an $ER - \varphi$ graph. Since p is odd, by Theorem 2.2, this graph has $W_{p=2n+1}$ as its spanning subgraph. But the fact that $p - 1$ is even implies $ER\text{-}\varphi(G(p,q)) - \{p\}$ has no wheel subgraphs W_{p-1}. But $p - 2$ is odd and $ER - \varphi(G(p,q)) - \{p, p-1\}$ is also an $ER - \varphi$ graph and has W_{2n-1} as its subgraph. Thus alternating $ER - \varphi$ graphs have wheel subgraphs when the number of vertices is odd. Finally this process is continued till the number of vertices of $ER - \varphi$ reaches 5, which has W_5 as its subgraph. Thus $ER - \varphi(G(p,q))$ has wheel subgraphs $W_5, W_7, \cdots, W_{p=2n+1}$.

Theorem 2.16. *The graph* $ER^\pi\text{-}\varphi(G(p,q))$ *is* K_p.

Proof. By the definition, $p - 1$ vertices of this graph are labelled by prime numbers and one vertex labeled as 1. All these labels are coprime to each other and hence there exist an edge between every pair of vertices. Thus $ER^\pi\text{-}\varphi(G(p,q))$ is K_p.

Theorem 2.17. *The number of edges in* $ER - \varphi(G(p+1,q))$ *is* $q + p$ *if* $p + 1$ *is prime, and* $q + \varphi(p+1)$ *if* $p + 1$ *is composite, where* q *is the number of edges in an ER-* $\varphi(G(p,q))$ *graph.*

Proof. By construction method. Consider an $ER^\pi\text{-}\varphi(G(p,q))$ graph. Let v be the vertex labeled by $p + 1$ of the trivial graph $G(1,0)$. Now the graph $ER\text{-}\varphi(G(p+1,q'))$ is constructed from $ER\text{-}\varphi(G(p,q))$ and $G(1,0)$ by the following procedure. If $p + 1$ is prime, then $p + 1$ is coprime to all the numbers $1, 2, 3, \cdots, p$.

$ER\text{-}\quad \varphi(G(p+1,q')) = ER - \varphi(G(p,q)) \otimes G(1,0).$ Then $q' = q + p$. If $p + 1$ is composite, then $p + 1$ is coprime to $\varphi(p+1)$ numbers among $1, 2, 3, \ldots, p$. Then $ER\text{-}\varphi(G(p+1,q'))$ is constructed by adding $\varphi(p+1)$ edges between the vertices of $ER\text{-}\varphi(G(p,q))$ and the vertex labeled $p + 1$ of $G(1,0)$, such that the labels of the vertices of $ER\text{-}\varphi(G(p,q))$ are coprime with the vertex labeled $p + 1$ of $G(1,0)$. Thus $q' = q + \varphi(p+1)$.

3. Eigenvalues of an *ER-* $\varphi(G(p, q))$ Graph

The adjacency matrix $A(\Gamma)$ of $\Gamma = ER\text{-}\varphi(G(p,q))$ is defined by its entries $a_{ij} = 1$ if $v_i v_j \in E(\Gamma)$ and 0

otherwise. Since $A(\Gamma)$ is symmetric, its eigenvalues are real. Without loss of generality, we can write them as $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_p(\Gamma)$ and call them the eigenvalues of G.

Lemma 3.1. [8] *Let* $\Gamma = (V, E)$ *be a graph with sub-vertex set* $V' = \{v_1, v_2, \ldots, v_k\}$ *having same set of neighbors* $\{v_{k+1}, v_{k+2}, \ldots, v_s\}$ *, where* $V = \{v_1, \ldots, v_k, \ldots, v_s, \ldots, v_n\}$. *Then this graph* Γ *has at least* $(k-1)$ *equal eigenvalues* 0*. Also the corresponding* $k-1$ *eigenvectors are*

$$\underbrace{(1, -1, 0, \ldots, 0)^T}_{2}, \underbrace{(1, 0, -1, 0, \ldots, 0)^T}_{3}, \ldots,$$

$$\text{and } \underbrace{(1, 0, \ldots, -1, 0, \ldots, 0)^T}_{k}.$$

Theorem 3.2. *Let* $ER - \varphi(G(p,q))$ *be a graph of order* p*. Also let* i *be a prime number with positive integer* a_i $(a_i \geq 2)$ *such that* $i^{a_i - 1} \leq p < i^{a_i}$*,* $i = 2, 3, \ldots$*. Then one of the eigenvalues of the adjacency matrix of an* ER-$\varphi(G(p,q))$ *graph is* 0 *of multiplicity* $\displaystyle\sum_{i \in \pi(p)} a_i - 2 \,|\, \pi(p)\,|$.

Proof. Since i is an prime number, vertices with label $i, i^2, i^3, \ldots, i^{a_i - 1}$ have same set of neighbors. By Lemma 3.1, we conclude that ER-$\varphi(G(p,q))$ graph has at least $a_i - 2$ equal eigenvalues 0, $i \in \pi(p)$. This completes the proof.

Theorem 3.3. *If* p *is a prime number, then one of the eigenvalues of the adjacency matrix of an* ER-$\varphi(G(p,q))$*. graph is* -1.

Proof. Since p is a prime, there exist at-least two vertices labeled 1 and p having degree $p-1$. Then the adjacency matrix ($p \times p$ matrix) of the graph is given by

$$A(\Gamma) = \begin{bmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ 1 & 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 0 \end{bmatrix}.$$

Now,

$$|A(\Gamma) - \lambda I_p| = \begin{vmatrix} -\lambda & 1 & 1 & \cdots & 1 \\ 1 & -\lambda & 1 & \cdots & 1 \\ 1 & 1 & -\lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & -\lambda \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda - 1 & 1 & 1 & \cdots & 1 + \lambda \\ 1 & -\lambda & 1 & \cdots & 1 \\ 1 & 1 & -\lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & -\lambda \end{vmatrix},$$

by applying the transform $R_1 \to R_1 - R_p$

$$= \begin{vmatrix} -1 & 0 & 0 & \cdots & 1 \\ 1 & -\lambda & 1 & \cdots & 1 \\ 1 & 1 & -\lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & -\lambda \end{vmatrix}$$

by the determinant property.

To compute the eigenvalues of $A(\Gamma)$, write

$$|A(\Gamma) - \lambda I_p| = 0.$$

From above, we get $\lambda + 1 = 0$, that is, $\lambda = -1$. Hence -1 is one of the eigenvalues of $ER - \varphi(G(p,q))$.

Remark 3.4. *From the proof of Theorem 3.3, we conclude that if there are* k *vertices of* $ER - \varphi(G(p,q))$ *having degree* $p-1$*, then the multiplicity of eigenvalue -1 of* $ER \varphi(G(p,q))$ *is* $k-1$.

In mathematics, Bertrand's postulate (actually a theorem) was proven by several researchers (see, [9]). Using Bertrand's postulate, we prove the following result:

Theorem 3.5. *One of the eigenvalues of the adjacency matrix of* ER-$\varphi(G(p,q))$ *graph is* -1.

Proof. If p is prime, then the proof is same as Theorem 3.3. Otherwise, p is composite. Then the proof is based on prime gaps. A prime gap is the difference between two successive prime numbers. To prove this theorem, the Bertrand's postulates [10,11,12] are applied. It states that there is always a prime number between x and $2x$. Any vertex in ER-$\varphi(G(p,q))$ has degree $p-1$ if it is labelled by 1 or a prime number p_x such that $(p_x, k) = 1$ for every k such that $p_x + 1 \leq k \leq p \Rightarrow p_x$ is not a divisor of the numbers from $p_x + 1$ to p. By Bertrand's postulates, there is always a prime number between $\dfrac{p}{2}$ and p as p is composite. Without loss of generality, let p_x be the nearest prime, which is less than p and $\dfrac{p}{2} \leq p_x \leq p$. Now $2p_x$ is the least number divisible by p_x. Since $p \leq 2p_x$, then p_x is not a divisor of numbers between $p_x + 1$ to p. Hence the degree of the vertex that is labelled by p_x is $p-1$. Thus there exist at least two vertices labeled by 1 and p_x having degree $p-1$. By applying Theorem 3.3 and Bertrand's postulates result is proven.

Let G_1 and G_2 be two graphs with n_1 and n_2 vertices, and m_1 and m_2 edges, respectively. The union of two graphs G_1 and G_2, written $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The join $G_1 \vee G_2$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$, and edge sets $E(G_1)$ and $E(G_2)$ is the graph union $G_1 \cup G_2$ together with all the edges joining $V(G_1)$ and $V(G_2)$. Thus, for example, $\overline{K}_p \vee \overline{K}_q = K_{p,q}$ \,, the complete bipartite graph.

Theorem 3.6. *One of the eigenvalues of the adjacency matrix of* $\overline{ER - \varphi(G(p,q))}$ *graph is zero and its multiplicity is at least* 2 *for* $p \geq 2$.

Proof. Let $\Gamma = ER$ - $\varphi(G(p,q))$. By Theorems 3.3 and 3.5, there exist at least two vertices having degree $p-1$ in an Γ, where p is the number of vertices and q is the number of edges in Γ. Then we have

$$\Gamma = K_2 \vee H,$$

where H is a graph of order $p-2$ with $q - 2p + 3$ edges. Thus we have

$$\overline{\Gamma} = K_1 \cup K_1 \cup \overline{H}.$$

Let $\Phi(\Gamma, x)$ be the characteristic polynomial of graph Γ. From above one can see easily that

$$\Phi(\overline{\Gamma}, x) = x^2 \Phi(\overline{H}, x).$$

This completes the theorem.

The energy of the graph G is defined as

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

This quantity has a long known chemical application; for details see the surveys [13,14,15]. Recently much work on graph energy appeared also in the mathematical literature (see, for instance, [16,17,18,19]).

Theorem 3.7. *If* $p + 1$ *is prime, then* $\overline{ER - \varphi(G(p,q))}$ $ER - \varphi(G(p+1, q+p))$ *are equi-energetic graphs.*

Proof. Denote by $\Gamma_1 = ER - \varphi(G(p,q))$ and

$$\Gamma_2 = ER - \varphi(G(p+1, q+p)).$$

Let $E_1(\overline{\Gamma}_1)$, $E_2(\overline{\Gamma}_2)$ and $A_1(\overline{\Gamma}_1)$, $A_2(\overline{\Gamma}_2)$ be the energies and adjacency matrices of two graphs $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$, respectively. Since $p + 1$ is a prime and is co-prime with all numbers $1, 2, 3, \ldots, p$, this implies that in Γ_2, the vertex labeled by $p + 1$ has edges with all the vertices of Γ_1. Thus we have

$$\overline{\Gamma}_2 = K_1 \cup \overline{\Gamma}_1.$$

Let the eigenvalues of $A_1(\overline{\Gamma}_1)$ be $\lambda_1, \lambda_2, \ldots, \lambda_p$. From above one can see easily that the eigenvalues of $A_2(\overline{\Gamma}_2)$ are $\lambda_1, \lambda_2, \ldots, \lambda_p$ and 0. Therefore

$$E_2(\overline{\Gamma}_2) = \sum_{i=1}^{p} |\lambda_i| + 0 = E_1(\overline{\Gamma}_1) = E_1(\overline{\Gamma}_1).$$ Hence the result follows.

Theorem 3.8. *Permanent of adjacency matrix of* $\overline{ER - \varphi(G(p,q))}$ *is zero.*

Let A be the adjacency matrix of $\overline{ER - \varphi(G(p,q))}$. Since the vertex labeled by 1 is adjacent to allthe vertices of the graph ER- $\varphi(G(p,q))$ and $\overline{ER - \varphi(G(p,q))}$, it is not adjacent to none of the vertices of $\overline{ER - \varphi(G(p,q))}$.

By Theorems 3.3 and 3.5, this implies that A has at least two rows with allzeros. Without loss of generality, let the 1st and j th rows bezero rows. Using the definition of the permanent of matrix we get $Per(A) = \sum \Pi a_{i, \alpha_i} = \sum a_{1, i_1}, a_{2, i_2} \ldots a_{p, i_p}$, where $\{i_1, i_2, \ldots, i_p\}$ is a permutation over $\{1, 2, \ldots, p\}$. In each of the products of sums, there exists at least one term which has elements of the 1st and j^{th} rows. Thus all the products of sums vanishes to zero. Hence the permanent of adjacency matrix is zero.

Theorem 3.9. *The number of edges of a line graph of* $ER - \varphi(G(p,q))$ *is*

$$\frac{1}{2} \left[(p-1)^2 + \sum_{i \in \pi(p)} \left(p - \left\lfloor \frac{p}{\% p_i} \right\rfloor \right)^2 + \sum_{j=2, j \notin \pi(p)}^{p} d(v_j)^2 \right] - \sum_{i=2}^{p} \varphi(i),$$

where v_j is labelled with non-primes.

Proof. The number of edges q_L of a \$line\$ \$graph\$ of $ER - \varphi(G(p,q))$ is given by $q_L = -q + \frac{1}{2} \sum_{i=1}^{p} d_i^2$, where d_i is the degree of the vertex of G. By Theorem 2.1, for an $ER - \varphi(G(p,q))$, $q = \sum_{i=2}^{p} \varphi(i)$. The vertices of the graph $ER - \varphi(G(p,q))$ can be partitioned into 3 sets: vertices labeled with prime numbers, vertices labeled with composite numbers and vertex labeled with 1 and their cardinalities are $\pi(p), p - \pi(p) - 1$ and 1, respectively. By using Theorems 2.1 and 2.3, the theorem is proved.

Let $D(\Gamma)$ be the diagonal matrix of vertex degrees of graph Γ. Then the Laplacian matrix of Γ is $L(\Gamma) = D(\Gamma) - A(\Gamma)$, where $A(\Gamma)$ is the adjacency matrix of Γ. Let $\mu_1(\Gamma) \geq \mu_2(\Gamma) \geq \cdots \geq \mu_{n-1}(\Gamma) \geq \mu_n(\Gamma) = 0$ denote the eigenvalues of $L(\Gamma)$. They are usually called the Laplacian eigenvalues of Γ. Among all eigenvalues of the Laplacian of a graph, the most studied is the second smallest, called the algebraic connectivity of a graph. It is well known that a graph is connected if and only if $a(\Gamma) = \mu_{n-1}(\Gamma) > 0$. Besides the algebraic connectivity, $\mu_1(\Gamma)$ is the invariant that interested the graph theorists. The following two results are obtained in [20] and [21].

Lemma 3.10. [20] *Let G be a graph with Laplacian spectrum* $\{0 = \mu_n, \mu_{n-1}, \ldots, \mu_2, \mu_1\}$. *Then the Laplacian spectrum of $\overline{G}$ is* $\{0, n - \mu_1, n - \mu_2, \ldots, n - \mu_{n-2}, n - \mu_{n-1}\}$, *where $\overline{G}$ is the complement of the graph G.*

Lemma 3.11. [21] *Let $G = (V, E)$ be a graph with vertex subset $V' = \{v_1, v_2, \ldots, v_k\}$ having the same set of neighbors $\{v_{k+1}, v_{k+2}, \ldots, v_s\}$, where $V = \{v_1, \ldots, v_k, \ldots, v_s, \ldots, v_n\}$. Then this graph G has at least $(k-1)$ equal eigenvalues*

and they are all equal to the cardinality of the neighbor set. Also the corresponding $(k-1)$ eigenvectors are

$$(\underbrace{1,-1,0,\ldots,0}_{2})^{T},(\underbrace{1,0,-1,0,\ldots,0}_{3})^{T},\ldots,$$

$$\text{and } (\underbrace{1,0,\ldots,-1,0,\ldots,0}_{k})^{T}.$$

Theorem 3.12. *Let ER-$\varphi(G(p,q))$ be a graph of order p. Also let i be a prime number with positive integer a_i $(a_i \geq 2)$ such that $i^{a_i-1} \leq p < i^{a_i}$, $i = 2,3,\ldots$. Then the eigenvalues of the Laplacian matrix of an ER-$\varphi(G(p,q))$ graph is $p - \left\lfloor \dfrac{p}{\%i} \right\rfloor$ of multiplicity $a_i - 2$, $i \in \pi(p)$. Moreover, the largest two Laplacian eigenvalues are n.*

Proof. Since i is an prime number, vertices with label $i, i^2, i^3, \ldots, i^{a_i-1}$ have same set of neighbors. By Lemma 3.11 and Theorem 2.9, we conclude that ER-$\varphi(G(p,q))$ graph has at least $a_i - 2$ equal eigenvalues $p - \left\lfloor \dfrac{p}{\%i} \right\rfloor$, $i \in \pi(p)$

We have that two adjacent vertices are adjacent to all the remaining vertices of the graph ER-$\varphi(G(p,q))$, by the proofs of Theorems 3.3 and 3.5. This implies that graph ER-$\varphi(G(p,q))$ has at least two vertices of degree $p-1$, that is, graph $\overline{ER-\varphi(G(p,q))}$ has two isolated vertices. By Lemma 3.10, we conclude that the largest two Laplacian eigenvalues are n. This completes the proof.

4. An Algorithm to Compute GCD and LCD of Two Numbers.

Though the $ER-\varphi(G(p,q))$ is constructed using GCD concept, from its adjacency matrix, the GCD and LCM of any two numbers from 1 to p can be computed with less complexity and this can easily be extended to higher order of the graphs. This is one of the techniques for finding GCD and LCM. The algorithm to compute the GCD of two numbers is as follows:

4.1. GCD Computation of Two Numbers a and b, where $1 \leq a, b < p$

(1) Determine the adjacency matrix A of $ER-\varphi(G(p,q))$.

(2) If A(a, b) = 1 or a = b, then (a,b) = 1.

(3) If $a\vert b$ then (a, b) = a or (if $b\vert a$, then (a, b) = b); else find the common non neighbors of i and j, excluding 1.

(4) MinCommonNeighbour = minimum(common non-neighbors of i and j.)

(5) (a; b) = minimum(a, b; minCommonNeighbour)

4.2 LCD computation of two numbers a and b, where $1 \leq$ a, b $<$p

(1) Determine the adjacency matrix A of $ER-\varphi(G(p,q))$.

(2) If $a = b$, $[a,b] = a$;

(3) If $A(a,b) = 1$ or $a = b$, then $(a,b) = 1$.

(4) Compute (a, b) using algorithm 3.1.

(5) $[a,b] = \dfrac{a*b}{(a,b)}$.

5. Conclusion

In this paper, we studied the spectral properties of co-prime labeled graphs ($ER-\varphi$) based on Euler's φ function. Several properties including connectivity, coloring and domination of these graphs are stated and proved. As part of further research, the study of generic spectral properties of these graphs is in progress.

Acknowledgements

References

[1]   D. M. Burton, Elementary Number Theory, 6th Edition, McGraw Hill, India (2007).

[2]   Paul Erdös, Gabor. N.Sarkozy, On Cycles in the Coprime graph of integers, Electronic Journal of Combinatorics 4 (2) (1997) #R8.

[3]   J. Baskar Babujee, Euler's Phi Function and Graph Labeling, Int. J. Math. Sciences 5 (20) (2010) 977-984.

[4]   J. A. Gallian, A Dynamic Survey of Graph Labeling, The Electronic Journal of Combinatorics 16 (2009) #DS6.

[5]   F. Harary, Graph Theory, Addison-Wesley, Reading Mass, (1969).

[6]   I. Gutman, The energy of a graph: old and new results, Algebraic Combinatorics and Applications, Springer Berlin, (2001), 196-211.

[7]   K. C. Das and P. Kumar, Some new bounds on the spectral radius of graphs, Discrete Math. 281 (2004) 149-161.

[8]   H. S. Ramane, H . B. Walikar, S. B. Rao, B. D. Acharya, P. R. Hampiholi, S. R. Jog, I. Gutman, Equienergetic graphs, Kragujevac, J. Math. 26 (2004), preceding paper.

[9]   S. Ramanujan, A proof of Bertrand's postulate, Journal of the Indian Mathematical Society 11 (1919) 181-182.

[10]   G.H. Hardy, An Introduction to the Theory of Numbers, 5th Edition, Oxford Science Publications, (1980).

[11]   M. N. Huxley, On the Difference Between Consecutive Primes, Inventiones math. 15 (1972) 164-170.

[12]   Thomas. R. Nicely, New Maximal Prime Gaps and First Occurences, Mathematics of Computation, AMS 68 (227) (1999) 1311-1315.

[13]   I. Gutman, Total π-electron energy of benzenoid hydrocarbons, Topics Curr. Chem. 162 (1992) 29-63.

[14]   I. Gutman, The energy of a graph: old and new results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), Algebraic Combinatorics and Applications, Springer-Verlag, Berlin, 2001, pp. 196-211.

[15]   I. Gutman, Topology and stability of conjugated hydrocarbons. The dependence of total π-electron energy on molecular topology, J. Serb. Chem. Soc. 70 (2005) 441-456.

[16]   R. Balakrishnan, The energy of a graph, Linear Algebra Appl. 387 (2004) 287-295.

[17] D. Stevanović, Energy and NEPS of graphs, Linear Multilinear Algebra 53 (2005) 67-74.

[18] D. Stevanović, I. Stanković, Remarks on hyperenergetic circulant graphs, Linear Algebra Appl. 400 (2005) 345-348.

[19] W. Yan, L. Ye, On the minimal energy of trees with a given diameter, Appl. Math. Lett. 18 (2005) 1046-1052.

[20] R. Merris, Laplacian matrices of graphs: A survey, Linear Algebra Appl. 197,198 (1994) 143-176.

[21] K. C. Das, Sharp Lower Bounds on the Laplacian eigenvalues of Trees, Linear Algebra Appl. 384 (2004) 155-169.