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Strongly Multiplicatively *P*-function and Some New Inequalities

Mahir Kadakal*

Department of Mathematics, Faculty of Sciences and Arts, Giresun University, Giresun-Turkey
*Corresponding author: mahirkadakal@gmail.com

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Abstract In this study, we present a new definition of convexity. This definition is the class of strongly multiplicatively P-functions. Some new Hermite-Hadamard type inequalities are derived for strongly multiplicatively P-functions. Some applications to special means of real numbers are given. Ideas of this paper may stimulate further research.

Keywords: *Convex function, multiplicatively P-function, strongly multiplicatively P-function, Hölder and Power-Mean Integral inequalities, Hermite-Hadamard inequality*

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1. Introduction

In this section, we firstly give several definitions and some known results.

Definition 1: A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then the function f is said to be concave on interval $I \neq \emptyset$.

This definition is well known in the literature. Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences.

One of the most important integral inequalities for convex functions is the Hermite-Hadamard inequality. The classical Hermite–Hadamard inequality provides estimates of the mean value of a continuous convex function $f: [a,b] \to \mathbb{R}$. The following double inequality is well known as the Hadamard inequality in the literature.

Definition 2: Let $f: [a,b] \to \mathbb{R}$ be a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is known as the Hermite-Hadamard inequality.

Some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors [1-6] and the Authors obtained a new refinement of the Hermite-Hadamard inequality for convex functions.

Definition 3: A nonnegative function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be P-function if the inequality

$$f(tx + (1-t)y) \leq f(x) + f(y)$$

holds for all $x, y \in I$ and $t \in (0,1)$.

We will denote by $P(I)$ the set of P-functions on the interval I. Note that $P(I)$ contain all nonnegative convex and quasi-convex functions.

In [7], Dragomir et al. proved the following inequality of Hadamard type for class of P-functions.

Theorem 1: Let $f \in P(I)$, $a, b \in I$ with $a < b$ and $f \in L[a,b]$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b f(x)dx \leq 2\left[f(a)+f(b)\right].$$

Definition 4: [8] Let $I \subset \mathbb{R}$ be an interval and c be a positive number. A function $f: I \to \mathbb{R}$ is called strongly convex with modulus c if

$$\begin{aligned} &f(tx+(1-t)y) \\ &\leq tf(x)+(1-t)f(y)-ct(1-t)(x-y)^2 \end{aligned}$$

for all $x, y \in I$ and $t \in [0,1]$.

In [9], Kadakal gave the definition of multiplicatively P-function (or log-P-function) and related Hermite-Hadamard inequality. It should be noted that the concept of log-P-convex, which we consider in our study and given below, was first defined by Noor et al in 2013 [10]. Then, the algebraic properties of this definition with the name of multiplicatively P-function are examined in detail by us.

Definition 5: [9,10] Let $I \neq \emptyset$ be an interval in $\mathbb{R}$. The function $f: I \to (0,\infty)$ is said to be multiplicatively P-function (or log-P-function), if the inequality

$$f(tx+(1-t)y) \leq f(x)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Denote by $MP(I)$ the class of all multiplicatively P-functions on I. Clearly, $f: I \to 0, \infty)$ is multiplicatively P-function if and only if $\log f$ is P-function. The range of the multiplicatively P-functions is greater than or equal to 1.

Theorem 2: Let the function $f: I \to (1, \infty)$, be a multiplicatively P-function and $a, b \in I$ with $a < b$. *If* $f \in L[a,b]$, then the following inequalities hold:

i) $$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx \leq [f(a)f(b)]^2$$

ii) $$f\left(\frac{a+b}{2}\right) \leq f(a)f(b)\frac{1}{b-a}\int_a^b f(x)dx \leq [f(a)f(b)]^2.$$

Dragomir and Agarwal in [11] used the following lemma to prove Theorems.

Lemma 1: The following equation holds true:

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t) f'(ta+(1-t)b)dt.$$

In [12], U. S. Kırmacı used the following lemma to prove Theorems.

Lemma 2: Let $f: I^* \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^*, $a, b \in I^*$ $(I^*$ is the interior of $I)$ with $a < b$. If $f' \in L[a,b]$, then we have the following equation holds true:

$$\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) = (b-a)\begin{bmatrix} \int_0^{\frac{1}{2}} f'(ta+(1-t)b)\,tdt \\ +\int_{\frac{1}{2}}^1 (t-1) f'(ta+(1-t)b)dt \end{bmatrix}.$$

The main purpose of this paper is to establish new estimations and refinements of the Hermite–Hadamard inequality for functions whose derivatives in absolute value are strongly multiplicatively P-function.

2. Strongly Multiplicatively *P*-functions and Their Some Properties

In this section, we begin by setting some algebraic properties for strongly multiplicatively P-functions.

Definition 6: Let $I \neq \emptyset$ be an interval in $\mathbb{R}$. The function $f: I \to (0, \infty)$ is said to be strongly multiplicatively P-function with modulus $c > 0$, if the inequality

$$f(tx+(1-t)y) \leq f(x)f(y) - ct(1-t)(x-y)^2$$

holds for all $x, y \in I$ and $t \in [0,1]$.

We will denote by $SMP(I)$ the class of all strongly multiplicatively P-functions on interval I.

Remark 1: The range of the strongly multiplicatively P-functions is greater than or equal to 1.

Proof: Using the definition of the strongly multiplicatively P-function, for $t = 1$;

$$f(x) \leq f(x)f(y) \Longrightarrow f(x)[1-f(y)] \leq 0.$$

Here, $f(x) \geq 0$, so we obtain $f(y) \geq 1$. Similarly, for $t = 0$,

$$f(y) \leq f(x)f(y) \Longrightarrow f(y)[1-f(x)] \leq 0.$$

Since $f(y) \geq 0$, we get $f(x) \geq 1$.

3. Hermite-Hadamard Type Inequalities for Strongly Multiplicatively *P*-functions

The goal of this paper is to develop concept of the strongly multiplicatively P-functions and to establish some inequalities of Hermite-Hadamard type for these classes of functions.

Theorem 3: Let the function $f: I \to (1, \infty)$, be a strongly multiplicatively P-function and $a, b \in I$ with $a < b$. *If* $f \in L[a,b]$, then the following inequalities hold:

i) $$f\left(\frac{a+b}{2}\right) + \frac{c}{12}(a-b)^2 \leq \frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx \leq [f(a)f(b)]^2 - \frac{c}{3}(b-a)\int_a^b f(x)dx - \frac{c^2(a-b)^4}{30}$$

ii) $$f\left(\frac{a+b}{2}\right) \leq f(a)f(b)\frac{1}{b-a}\int_a^b f(x)dx \leq [f(a)f(b)]^2 - \frac{c}{12}(a-b)^2.$$

Proof: *i)* Since the function f is a strongly multiplicatively P-function, we write the following inequality:

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{[ta+(1-t)b]+[tb+(1-t)a]}{2}\right) \leq f(ta+(1-t)b)\, f(tb+(1-t)a) - \frac{c}{4}(2t-1)^2(a-b)^2.$$

By integrating this inequality on $[0,1]$ and changing the variable as $x = ta + (1-t)b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx - \frac{c}{4}(a-b)^2\int_0^1 (2t-1)^2 dt = \frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx - \frac{c}{12}(a-b)^2.$$

Moreover, a simple calculation give us that

$$\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx \leq [f(a)f(b)]^2 - \frac{c}{3}(b-a)\int_a^b f(x)dx - \frac{c^2(a-b)^4}{30}.$$

So, we get the desired result.

ii) Similarly, as f is a strongly multiplicatively P-function, we write the following:

$$f\left(\frac{a+b}{2}\right)$$
$$\leq f\left(ta+(1-t)b\right) f\left(tb+(1-t)a\right)-ct(1-t)(a-b)^2$$
$$\leq f(a) f(b) f\left(tb+(1-t)a\right)-\frac{c}{4}(2t-1)^2(a-b)^2.$$

Here, by integrating this inequality on $[0,1]$ and changing the variable as $x = tb + (1-t)a$, then, we have

$$f\left(\frac{a+b}{2}\right)\leq f(a) f(b)\frac{1}{b-a}\int_a^b f(x)dx-\frac{c}{12}(a-b)^2.$$

Since,

$$\frac{1}{b-a}\int_a^b f(x)dx\leq f(a) f(b),$$

we obtain

$$f\left(\frac{a+b}{2}\right)\leq f(a) f(b)\frac{1}{b-a}\int_a^b f(x)dx$$
$$\leq\left[f(a) f(b)\right]^2-\frac{c}{12}(a-b)^2.$$

This completes the proof of theorem.

Remark 2: Above Theorem (i) and (ii) can be written together as follows:

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x) f(a+b-x)dx$$
$$\leq f(a) f(b)\frac{1}{b-a}\int_a^b f(x)dx$$
$$\leq\left[f(a) f(b)\right]^2-\frac{c}{12}(a-b)^2.$$

Proof: By integrating the following inequality on $[0,1]$, the desired result can be obtained:

$$f\left(\frac{a+b}{2}\right)\leq f\left(\frac{A_t+A_{1-t}}{2}\right)\leq f(A_t) f(A_{1-t})$$
$$\leq f(a) f(b) f(A_t)-\frac{c}{12}(a-b)^2,$$

where $A_t = ta + (1-t)b$.

Theorem 4: Let $f: I\to\mathbb{R}$ be a differentiable function on I° such that the function $|f'|$ is strongly multiplicatively P-function. Suppose that $a,b\in I$ with $a<b$ and $f'\in L[a,b]$. Then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\left[\left|f'(a)\right|\left|f'(b)\right|-\frac{c(a-b)^2}{8}\right]. \quad (3.1)$$

Proof: Using Lemma 1, since $|f'|$ is strongly multiplicatively P-function, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$=\left|\frac{b-a}{2}\int_0^1(1-2t) f'\left(ta+(1-t)b\right)dt\right|$$
$$\leq\frac{b-a}{2}\int_0^1|1-2t|\left|f'\left(ta+(1-t)b\right)\right|dt$$
$$\leq\frac{b-a}{2}\int_0^1|1-2t|\left|f'\left(ta+(1-t)b\right)\right|dt$$
$$\leq\frac{b-a}{2}\left[\begin{array}{l}\left|f'(a)\right|\left|f'(b)\right|\int_0^1|1-2t|dt\\+c(a-b)^2\int_0^1 t(1-t)|1-2t|dt\end{array}\right]$$
$$=\frac{b-a}{4}\left[\left|f'(a)\right|\left|f'(b)\right|-\frac{c(a-b)^2}{8}\right],$$

where

$$\int_0^1|1-2t|dt=\frac{1}{2},$$

$$\int_0^1 t(1-t)|1-2t|dt=\frac{1}{16}.$$

This completes the proof of theorem.

Theorem 5: Let $f: I\to\mathbb{R}$ be a differentiable function on I°. Assume $q\in\mathbb{R}$, $q>1$, is such that the function $|f'|^q$ is strongly multiplicatively P-function. Suppose that $a,b\in I$ with $a<b$ and $f'\in L[a,b]$. Then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left|f'(a)\right|^q\left|f'(b)\right|^q-\frac{c(a-b)^2}{6}\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1.$

Proof: Let $a,b\in I$. By assumption, Hölder's integral inequality, Lemma 1 and the inequality

$$\left|f'\left(ta+(1-t)b\right)\right|^q$$
$$\leq\left|f'(a)\right|^q\left|f'(b)\right|^q-ct(1-t)(a-b)^2,$$

we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{2}\int_0^1 |1-2t|\left|f'\left(ta+(1-t)b\right)\right|dt$$

$$\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$= \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left[\begin{matrix}\left|f'(a)\right|^q\left|f'(b)\right|^q\\ -ct(1-t)(a-b)^2\end{matrix}\right]dt\right)^{\frac{1}{q}}$$

$$= \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left|f'(a)\right|^q\left|f'(b)\right|^q-\frac{c(a-b)^2}{6}\right)^{\frac{1}{q}},$$

where

$$\int_0^1 |1-2t|^p\,dt=\frac{1}{p+1}.$$

This completes the proof of theorem.

A more general inequality using Lemma 1 is as follows.

Theorem 6: Let $f: I \to \mathbb{R}$ be a differentiable function on I°. Assume $q \in \mathbb{R}$, $q \geq 1$, is such that the function $|f'|^q$ is strongly multiplicatively P-function. Suppose that $a,b \in I$ with $a < b$ and $f' \in L[a,b]$. Then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{4}\left(\left|f'(a)\right|^q\left|f'(b)\right|^q-\frac{c(a-b)^2}{8}\right)^{\frac{1}{q}}. \tag{3.2}$$

Proof: Let $a,b \in I^\circ$. Since the function $|f'|^q$ is a strongly multiplicatively P-function, from Lemma 1 and the power-mean integral inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$\leq \frac{b-a}{2}\int_0^1 |1-2t|\left|f'\left(ta+(1-t)b\right)\right|dt$$

$$\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left[\begin{matrix}\left|f'(a)\right|^q\left|f'(b)\right|^q\\ -ct(1-t)(a-b)^2\end{matrix}\right]dt\right)^{\frac{1}{q}}$$

$$= \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\begin{matrix}\left|f'(a)\right|^q\left|f'(b)\right|^q\int_0^1|1-2t|\,dt\\ -c(a-b)^2\int_0^1 t(1-t)|1-2t|\,dt\end{matrix}\right)^{\frac{1}{q}}$$

$$= \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'(a)\right|^q\left|f'(b)\right|^q}{2}-\frac{c(a-b)^2}{16}\right)^{\frac{1}{q}}$$

$$= \frac{b-a}{4}\left(\left|f'(a)\right|^q\left|f'(b)\right|^q-\frac{c(a-b)^2}{8}\right)^{\frac{1}{q}}.$$

This completes the proof.

Corollary 1: If we take $q = 1$ in inequality (3.2), we obtain the following inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{4}\left(\left|f'(a)\right|\left|f'(b)\right|-\frac{c(a-b)^2}{8}\right).$$

This inequality coincides with the inequality (3.1).

Theorem 7: Let $f: I \to \mathbb{R}$ be a differentiable function on I° such that the function $|f'|$ is strongly multiplicatively P-function. Suppose that $a,b \in I$ t $a < b$ and $f' \in L[a,b]$. Then the following inequality holds:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left[\left|f'(a)\right|\left|f'(b)\right|-\frac{5c(a-b)^2}{24}\right]. \tag{3.3}$$

Proof: Using Lemma 2, since $|f'|$ is strongly multiplicatively P-function, we obtain

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right|$$

$$\leq (b-a)\left[\begin{matrix}\int_0^{\frac{1}{2}} t\left|f'\left(ta+(1-t)b\right)\right|dt\\ +\int_{\frac{1}{2}}^1 |t-1|\left|f'\left(ta+(1-t)b\right)\right|dt\end{matrix}\right]$$

$$\leq (b-a)\left[\left(\int_0^{\frac{1}{2}} t\left[\begin{matrix}\left|f'(a)\right|\left|f'(b)\right|\\ -ct(1-t)(a-b)^2\end{matrix}\right]dt\right)+\left(\int_{\frac{1}{2}}^1 (1-t)\left[\begin{matrix}\left|f'(a)\right|\left|f'(b)\right|\\ -ct(1-t)(a-b)^2\end{matrix}\right]dt\right)\right]$$

$$= \frac{b-a}{4}\left[\left|f'(a)\right|\left|f'(b)\right|-\frac{5c(a-b)^2}{24}\right]$$

where

$$\int_0^{\frac{1}{2}} t^2(1-t)\,dt=\int_{\frac{1}{2}}^1 t(1-t)^2\,dt=\frac{5}{192}.$$

This completes the proof of theorem.

Theorem 8: Let $f: I \to \mathbb{R}$ be a differentiable function on I°. Assume $q \in \mathbb{R}$, $q > 1$, is such that the function $|f'|^q$ is strongly multiplicatively P-*f*unction. Suppose that $a,b \in I$

with $a < b$ *and* $f' \in L[a,b]$. Then the following inequality holds:

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|$$

$$\leq 2(b-a)\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q|f'(b)|^q}{2} - \frac{c(a-b)^2}{12}\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof: Since the function $|f'|^q$ is a strongly multiplicatively P-function, from Lemma 2 and the Hölder's integral inequality, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|$$

$$\leq (b-a)\left[\int_0^{\frac{1}{2}} t\left|f'(ta+(1-t)b)\right|dt + \int_{\frac{1}{2}}^1 |t-1|\left|f'(ta+(1-t)b)\right|dt\right]$$

$$\leq (b-a)\left[\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 |t-1|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right]$$

$$\leq (b-a)\left[\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left[|f'(a)|^q|f'(b)|^q - ct(1-t)(a-b)^2\right]dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 |t-1|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left[|f'(a)|^q|f'(b)|^q - ct(1-t)(a-b)^2\right]dt\right)^{\frac{1}{q}}\right]$$

$$= (b-a)\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}\left\{\left(\frac{|f'(a)|^q|f'(b)|^q}{2} - \frac{c(a-b)^2}{12}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q|f'(b)|^q}{2} - \frac{c(a-b)^2}{12}\right)^{\frac{1}{q}}\right\}$$

$$= 2(b-a)\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q|f'(b)|^q}{2} - \frac{c(a-b)^2}{12}\right)^{\frac{1}{q}}$$

where

$$\int_0^{\frac{1}{2}} t^p dt = \int_{\frac{1}{2}}^1 |t-1|^p dt = \frac{1}{(p+1)2^{p+1}}$$

$$\int_0^{\frac{1}{2}} t(1-t)dtt = \int_{\frac{1}{2}}^1 t(1-t)dt = \frac{1}{12}.$$

This completes the proof of theorem.

Theorem 9: Let $f: I \to \mathbb{R}$ be a differentiable function on I°. Assume $q \in \mathbb{R}$, $q > 1$, is such that the function $|f'|^q$ is multiplicatively P-function. Suppose that $a, b \in I$ with $a < b$ and $f' \in L[a,b]$. Then the following inequality holds:

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left(|f'(a)|^q|f'(b)|^q - \frac{c(a-b)^2}{24}\right)^{\frac{1}{q}}, \quad (3.4)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof: Since the function $|f'|^q$ is a multiplicatively P-function, from Lemma 2 and the power-mean integral inequality, we obtain

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|$$

$$\leq (b-a)\left[\int_0^{\frac{1}{2}} t\left|f'(ta+(1-t)b)\right|dt + \int_{\frac{1}{2}}^1 |t-1|\left|f'(ta+(1-t)b)\right|dt\right]$$

$$\leq (b-a)\left[\left(\int_0^{\frac{1}{2}} tdt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 |t-1|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 |1-t|\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right]$$

$$\leq (b-a)\left[\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t\left[|f'(a)|^q|f'(b)|^q - ct(1-t)(a-b)^2\right]dt\right)^{\frac{1}{q}} + \left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 |1-t|\left[|f'(a)|^q|f'(b)|^q - ct(1-t)(a-b)^2\right]dt\right)^{\frac{1}{q}}\right]$$

$$= (b-a)\left[\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left(\frac{|f'(a)|^q|f'(b)|^q}{8} - \frac{5c(a-b)^2}{192}\right)^{\frac{1}{q}} + \left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left(\frac{|f'(a)|^q|f'(b)|^q}{8} - \frac{5c(a-b)^2}{192}\right)^{\frac{1}{q}}\right]$$

$$= \frac{b-a}{4}\left(|f'(a)|^q|f'(b)|^q - \frac{c(a-b)^2}{24}\right)^{\frac{1}{q}}.$$

where

$$\int_0^{\frac{1}{2}} t\,dt = \int_{\frac{1}{2}}^{1} |t-1|\,dt = \frac{1}{8}$$

$$\int_0^{\frac{1}{2}} t^2(1-t)\,dt = \int_{\frac{1}{2}}^{1} t(1-t)^2\,dt = \frac{5}{192}.$$

Corollary 2: If we take $q = 1$ in inequality (3.4), we obtain the following inequality:

$$\left| \frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \right| \le \frac{b-a}{4}\left(|f'(a)||f'(b)| - \frac{c(a-b)^2}{24} \right).$$

This inequality coincides with the inequality (3.3).

4. Conclusion

We derived some new Hermite-Hadamard type inequalities for strongly multiplicatively P-functions. Similar method can be applied to the different type of convex functions.

References

[1] Dragomir, S.S. and Pearce, C.E.M. Selected Topics on Hermite-Hadamard Inequalities and Its Applications, RGMIA Monograph, 2002.

[2] Hadamard, J. Etude sur les proprietes des fonctions entieres en particulier d'une fonction consideree par Riemann, J. Math. Pures Appl. 58, 171-215, 1893.

[3] Kadakal, H., New Inequalities for Strongly *r*-Convex Functions, Journal of Function Spaces, Volume 2019, Article ID 1219237, 10 pages.

[4] Kadakal, M., Kadakal, H. and İşcan, İ., Some New Integral Inequalities for *n*-Times Differentiable Strongly Convex Functions, Karaelmas Science and Engineering Journal, 8(1): 147-150, 2018.

[5] Pečarić, J.E. Proschan, F. and Tong, Y.L. Convex Functions, Partial Orderings, and Statistical Applications, Academic Press Inc., 1992, p. 137.

[6] Zabandan, G. A new refinement of the Hermite-Hadamard inequality for convex functions, J. Inequal. Pure Appl. Math. 10(2), Article ID 45, 2009.

[7] Dragomir, S.S. Pečarić, J. and Persson, L.E. Some inequalities of Hadamard Type, Soochow Journal of Mathematics, Vol.21, No:3, pp. 335-341, July 1995.

[8] Polyak, B.T., Existence theorems and convergence of minimizing sequences in extremum problems with restrictions, Soviet Math. Dokl. 7, 72-75, 1966.

[9] Kadakal, H., Multiplicatively P-functions and some new inequalities, New Trends in Mathematical Sciences, NTMSCI 6, No. 4, 111-118, 2018.

[10] Noor, M.A., Qi, F. and Awan, M.U., Some Hermite-Hadamard type inequalities for log-h-convex functions, Analysis 33, 1-9, 2013.

[11] Dragomir, S. S. and Agarwal, R. P. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett. 11, 1998, 91-95.

[12] Kirmaci U. S. Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Applied Mathematics and Computation 147, 2004. 137-146.