

A Certain Type of Regular Diophantine Triples and Their Non-Extendability

Özen ÖZER*

Department of Mathematics, Faculty of Science and Arts, Kırklareli University, Kırklareli, 39100, Turkey
*Corresponding author: ozenozer39@gmail.com

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Abstract In the present paper, we consider some $D(s)$ Diophantine triples for a prime integer s with its negative case/value although there exist infinitely many Diophantine triples. We give several properties of such Diophantine triples and prove that they are non – extendability to $D(s)$ Diophantine quadruple using algebraic and elementary number theory structures.

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1. Introduction and Preliminaries

Number Theory includes lots of questions on whole types of numbers. These questions on numbers also related with sets, equations and so on... The Diophantine n- tuple was started to work by Diophantus of Alexandria who was one of the crucial mathematicians in the algebra. Diophantine problems can not be solved easily. They contain different kind of methods and some of them related with the integer solutions of Pell equations.

There are lots of papers on the Pell equations and Diophantine n- tuple in the literature. In [1] Beardon & his coauthor and Dujella et al. in [2] gave some special and significant results on some of Diophantine triples such as strong Diophantine triples, regular Diophantine triples etc... Gopalan [3,4] worked on both dio special diophantine triples and pellian equation.

The author Özer [5,6,7,8,9] has prepared her papers on various types of Diophantine triple with different algebraic methods. Some of the basic informations (Definitions, Examples, Theorems, Lemmas etc...) such as quadratic residues, quadratic reciprocity law, legendre & Jacobi symbols are given for algebraic and elementary number theory in [10], [11] and [12]. For further results and informations, we refer to [13-21] books cited herein for readers.

In the current paper we choose $s = \mp 29$ and consider some of the $D(\mp 29)$ Diophantine triples. We prove that they aren't extendable to $D(\mp 29)$ Diophantine quadruple. Then, we demonstrate some properties useful to determine of the such sets. For the proof of theorems, we use the

factorization method in set of integers and integer solutions of Pell (Pellian) equations or Pell like equations as well as quadratic residue, quadratic reciprocity theorem, legendre symbol, etc...

Now, we give some basic notations useful for proving our theorems as follow:

Definition 1.1. [1,13] A Diophantine n-tuple with the property $D(s)$ (it sometimes representatives as P_s with n-tuple) for an integer s is a n-tuple of different positive integers $\{\mu_1, \dots, \mu_n\}$ such that $\mu_i \mu_j + s$ is always a square of an integer for every distinct i, j .

As a special case, If $n = 3$ then it is called $D(s)$ - Diophantine triple.

Definition 1.2. [2] If $D(s)$ - Diophantine triple $\{\alpha, \beta, \gamma\}$ satisfies the condition

$$(\gamma - \beta - \alpha)^2 = 4(\alpha \cdot \beta + s) \quad (1.1)$$

Then, it is called $D(s)$ - Regular Diophantine Triple.

Definition 1.3. [10,12,21] (Quadratic Residue) Let p be an odd prime, $\beta \not\equiv 0 \pmod{p}$. We call that β is a quadratic residue modulo p if a nonzero number β is a square modulo p and is abbreviated QR. A number β which isn't congruent to a square \pmod{p} is called a quadratic nonresidue \pmod{p} and is shortened NR. If a number β that is congruent to 0 modulo p is neither a residue nor a nonresidue.

Lemma 1.1. [10] Let $\theta \not\equiv 0 \pmod{q}$ where q is an odd prime. Then θ is a quadratic residue \pmod{q} if and only

$$\theta^{\frac{p-1}{2}} \equiv 1 \pmod{q}.$$

Definition 1.4. [4] Legendre Symbol) Legendre symbol is introduced the following notation for prime p :

$$\left(\frac{\gamma}{p}\right) = \begin{cases} 1 & \text{if } \gamma \text{ is a quadratic residue mod } p \\ -1 & \text{if } \gamma \text{ is a quadratic non-residue mod } p \end{cases} \quad (1.2)$$

Note from above that $\left(\frac{\gamma}{p}\right) \equiv \gamma^{(p-1)/2} \pmod{p}$.

Lemma 1.2. [12,17] Let $\left(\cdot\right)$ be a Legendre Symbol and p be a prime number. Then, followings are satisfied.

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4} \\ -1 & \text{if } p \equiv 3 \pmod{4} \end{cases} \quad (1.3)$$

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{8} \text{ or } p \equiv 7 \pmod{8} \\ -1 & \text{if } p \equiv 3 \pmod{8} \text{ or } p \equiv 5 \pmod{8} \end{cases} \quad (1.4)$$

Note. The legendre symbol $\left(\frac{a}{b}\right)$ can be naturally generalized to the case when a and b are odd and coprime numbers.

Definition 1.5. [11,20] (Jacobi Symbol) Suppose $b = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$. Then the Jacobi symbol, also represented as $\left(\frac{a}{b}\right)$, is defined as follows:

$$\left(\frac{a}{b}\right) = \begin{cases} 0 & \text{if } (a,b) \neq 1 \\ \prod_{i=1}^k \left(\frac{a}{p_i}\right)^{\alpha_i} & \text{otherwise} \end{cases} \quad (1.5)$$

The Jacobi symbol satisfies some multiplicative properties as follows:

$$\left(\frac{c_1 c_2}{k}\right) = \left(\frac{c_1}{k}\right) \left(\frac{c_2}{k}\right) \quad (1.6)$$

and $\left(\frac{c}{k_1 k_2}\right) = \left(\frac{c}{k_1}\right) \left(\frac{c}{k_2}\right)$.

As usual with empty products, we set $\left(\frac{a}{1}\right) = 1$.

Theorem 1.1. [14,15,16] Quadratic Reciprocity Theorem) If p, q are distinct odd primes, then following equation is hold.

$$\left(\frac{p}{q}\right) \left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{2}} = \begin{cases} 1 & \text{if } p \text{ or } q \equiv 1 \pmod{4} \\ -1 & \text{otherwise} \end{cases} \quad (1.7)$$

Theorem 1.2. [12,21] (Quadratic Reciprocity Law) If u, v are odd numbers such that $(u, v) = 1$, then followings can be obtained by Theorem 1.1.

$$\left(\frac{2}{v}\right) = (-1)^{\frac{v^2-1}{8}} \quad (1.8)$$

$$\left(\frac{u}{v}\right) \left(\frac{v}{u}\right) = (-1)^{\frac{(u-1)(v-1)}{2}} \quad (1.9)$$

2. Theorems and Results

Theorem 2.1. A set $P_{+29} = \{1, 167, 196\}$ is a non-extendible regular Diophantine triple.

Proof. Let us consider Definition 1.2 and regularity condition (1.1). Then, we can see that $P_{+29} = \{1, 167, 196\}$ is a regular triple. Assuming that $\{1, 167, 196, \sigma\}$ is a $D(+29)$ Diophantine quadruple. Then, there exist x_1, x_2, x_3 integers such that;

$$\sigma + 29 = x_1^2 \quad (2.1)$$

$$167\sigma + 29 = x_2^2 \quad (2.2)$$

$$196\sigma + 29 = x_3^2. \quad (2.3)$$

Dropping σ between (2.1) and (2.3), we have

$$196x_1^2 - x_3^2 = 5655 \quad (2.4)$$

the both side of (2.4) can be factorized in the set of integers. So, we have following table:

Table 1. Solutions of $196x_1^2 - x_3^2 = 5655$ in integer

Solutions	1. Class of Solutions	2. Class of Solutions	3. Class of Solutions	4. Class of Solutions
x_1	∓ 202	∓ 16	∓ 14	∓ 8
x_3	∓ 2827	∓ 211	∓ 181	∓ 83

Elimination of σ between (2.1) and (2.2), we get

$$167x_1^2 - x_2^2 = 4814. \quad (2.5)$$

Considering above solutions, we have $x_1^2 = 40804$, $x_1^2 = 256$, $x_1^2 = 196$ and $x_1^2 = 64$ respectively. If we substitute these values x_1^2 into the (2.5) we obtain $x_2^2 = 6809454$, $x_2^2 = 37938$, $x_2^2 = 27918$ and $x_2^2 = 5874$. This shows that any values of x_2 is not integer solution of (2.5).

So, there is not any such $\sigma \in \mathbb{Z}^+$ and the $P_{+29} = \{1, 167, 196\}$ can not be extended to $D(+29)$ Diophantine quadruple.

Theorem 2.2. $P_{+29} = \{2, 70, 98\}$ and $P_{+29} = \{2, 98, 130\}$ are regular Diophantine triple but they can not extendible to P_{+29} Diophantine quadruple.

Proof. From the (1.1) condition in Definition 1.2, it is easily seen that both $P_{+29} = \{2, 70, 98\}$ and $P_{+29} = \{2, 98, 130\}$ are regular P_{+29} triples. Let's suppose that there exists a positive integer ω such that $\{2, 70, 98, \omega\}$ is a P_{+29} Diophantine quadruple. Then the following equations have integral solutions for $y_1, y_2, y_3 \in \mathbb{Z}$.

$$2\omega + 29 = y_1^2 \quad (2.6)$$

$$70\omega + 29 = y_2^2 \quad (2.7)$$

$$98\omega + 29 = y_3^2. \quad (2.8)$$

Let consider (2.6) and (2.8) and eliminate ω . Then, we obtain

$$49y_1^2 - y_3^2 = 1392 \quad (2.9)$$

Factorization of the left side of (2.9), we get

$$(7y_1 - y_3)(7y_1 + y_3) = 1392 \quad (2.10)$$

If we search the solutions of the (3.13), we have

Table 2. Solutions of $49y_1^2 - y_3^2 = 1392$ in the set of integers

Solutions	1. Class of Solutions	2. Class of Solutions
(y_1, y_3)	$(\mp 17, \mp 113)$	$(\mp 13, \mp 83)$

We obtain the equation

$$35y_1^2 - y_2^2 = 986 \tag{2.11}$$

from (2.6) and (2.7). If we use Table 2, we get $y_1^2 = 289$ or $y_1^2 = 169$. By putting these values into the (2.11), we have $y_3^2 = 9129$ or $y_3^2 = 4929$ respectively. This is a contradiction since they are not integer solutions of (2.11).

Let us look at the $P_{+29} = \{2, 98, 130\}$ and assume that it can be extendible to $D(+29)$ Diophantine quadruple. In a similar way, we get

$$2\alpha + 29 = y_4^2 \tag{2.12}$$

$$98\alpha + 29 = y_5^2 \tag{2.13}$$

$$130\alpha + 29 = y_6^2 \tag{2.14}$$

for $y_5, y_6, y_4 \in \mathbb{Z}$ where α positive integer. From (2.12) and (2.14), we obtain

$$65y_4^2 - y_6^2 = 1856. \tag{2.15}$$

If we put the solutions of y_4 (we get Table 2 for solutions (y_4, y_5) if we consider (2.12) and (2.13)) from the Table 2 into the (2.15), then we have $y_6^2 = 166429$, $y_6^2 = 9129$ which $y_6 \notin \mathbb{Z}$ satisfying the equation (2.15).

Therefore, it is a contradiction and $P_{+29} = \{2, 98, 130\}$ is nonextendible.

Theorem 2.3. $D(29) = \{4, 23, 49\}$ and $D(29) = \{4, 49, 83\}$ are not only regular Diophantine triple but also non-extendible.

Proof. It is seen that $D(29) = \{4, 23, 49\}$ is a regular Diophantine triple by applying (1.1) condition from Definition 1.2. Suggesting that $\{4, 23, 49, \kappa\}$ is a $D(29)$ Diophantine quadruple where κ is positive integer. So, there are $z_1, z_2, z_3 \in \mathbb{Z}$ such that

$$4\kappa + 29 = z_1^2 \tag{2.16}$$

$$23\kappa + 29 = z_2^2 \tag{2.17}$$

$$49\kappa + 29 = z_3^2. \tag{2.18}$$

By simplifying κ between (2.16) and (2.18), we obtain

$$49z_1^2 - 4z_3^2 = 1305 \tag{2.19}$$

and likewise a (2.16) and (2.17) we get

$$23z_1^2 - 4z_2^2 = 551. \tag{2.20}$$

By factorizing (2.19), we obtain following table for solutions.

Table 3. Solutions of $49z_1^2 - 4z_3^2 = 1305$ in the set of integers

Solutions	1. Class of Solutions	2. Class of Solutions
(z_1, z_3)	$(\mp 19, \mp 64)$	$(\mp 11, \mp 34)$

By substituting $z_1^2 = 361$ or $z_1^2 = 121$ into the (2.20), we get $z_2^2 = 1938$ or $z_2^2 = 558$ respectively. It shows that this is a contradiction since $z_2 \notin \mathbb{Z}$. As a result, there isn't any such $\kappa \in \mathbb{Z}^+$ and also the $\{4, 23, 49\}$ can not be extended to P_{29} Diophantine quadruple.

Given that $D(29) = \{4, 49, 83, \lambda\}$ be a Diophantine quadruple for $\lambda \in \mathbb{Z}^+$. Using Definition 1.1, we obtain

$$4\lambda + 29 = t_1^2 \tag{2.21}$$

$$49\lambda + 29 = t_2^2 \tag{2.22}$$

$$83\lambda + 29 = t_3^2 \tag{2.23}$$

for $t_1, t_2, t_3 \in \mathbb{Z}$. By eliminating λ from (2.21) and (2.22), we get an equation similar to (2.19). So, we can use results of Table 3 for (t_1, t_2) instead of (z_1, z_3) . From (2.21) and (2.23), we also have

$$83t_1^2 - 4t_3^2 = 2291. \tag{2.24}$$

Putting $t_1^2 = 361$ or $t_1^2 = 121$ into the (2.24), we get $t_3^2 = 6918$ or $t_3^2 = 1938$ respectively. This is a contradiction because t_3 is not an integer solution of (2.24). So, $D(29) = \{4, 49, 83\}$ is nonextendible.

Theorem 2.4. $D(29) = \{5, 7, 28\}$, $D(29) = \{7, 28, 65\}$ diophantine triples are regular and nonextendible to $D(29)$ Diophantine quadruple.

Proof. Let us start with the regularity of $D(29) = \{5, 7, 28\}$ and $D(29) = \{7, 28, 65\}$. Both of them satisfy the (1.1) condition in the Definition 1.2. So, they are regular.

By a contraction method, assume that $\{5, 7, 28, \xi\}$ is Diophantine quadruple for $\xi \in \mathbb{Z}^+$. From the Definition of 1.1, we get following equations

$$5\xi + 29 = u_1^2 \tag{2.25}$$

$$7\xi + 29 = u_2^2 \tag{2.26}$$

$$28\xi + 29 = u_3^2 \tag{2.27}$$

for $u_1, u_2, u_3 \in \mathbb{Z}$. Simplification of (2.26) and (2.27), following equation is got;

$$4u_2^2 - u_3^2 = 87. \tag{2.28}$$

And a similar way, we obtain

$$7u_1^2 - 5u_2^2 = 58 \tag{2.29}$$

from (2.25) and (2.26). Since (2.28) can be factorized, we get following tables for (u_2, u_3) solutions in the set of integers.

Table 4. Solutions of $4u_2^2 - u_3^2 = 87$ in the set of integers

Solutions	1. Class of Solutions	2. Class of Solutions
(u_2, u_3)	$(\mp 22, \mp 43)$	$(\mp 8, \mp 13)$

Using the u_2 values from Table 4 and substitute into the (2.29), we have $u_1^2 = 354$ or $u_1^2 = 54$ show that u_1 is not integer solution of (2.29). This is a contradiction. So, $D(29) = \{5, 7, 28\}$ is a regular $D(+29)$ nonextendible Diophantine triple.

In the same vein, if we suppose that $D(29) = \{7, 28, 65, \varrho\}$ is Diophantine quadruple for positive integer ϱ , then we obtain

$$7\varrho + 29 = v_1^2 \tag{2.30}$$

$$28\varrho + 29 = v_2^2 \tag{2.31}$$

$$65\varrho + 29 = v_3^2 \tag{2.32}$$

for $v_1, v_2, v_3 \in \mathbb{Z}$. From (2.30) and (2.31), we obtain similar equation of (2.28) and we get same solutions of Table 4 for (v_1, v_2) . We have

$$65v_1^2 - 7v_3^2 = 1682 \tag{2.33}$$

by dropping ϱ from (2.30) and (2.32). If we put $v_1^2 = 484$ or $v_1^2 = 64$ into the (2.33), $v_3^2 = 4254$ or $v_3^2 = 354$ are obtained respectively. It is easy to see that

v_3 is not integer solution for (2.33) which is a contradiction. Thus, $D(29) = \{7, 28, 65\}$ can not be extendible.

Remark 2.1. There are lots of various regular P_{+29} Diophantine triples and one may determine others using our method.

Theorem 2.5. There is no $D(+29)$ involves \wp elements satisfy either of the states of affairs as follows:

- (a) $\wp \in \mathbb{Z}^+$, \wp is divided by 3 or multiplies of 3.
- (b) $\wp \in \mathbb{Z}^+$, \wp is divided by 8 or multiplies of 8.
- (c) $\wp \in \mathbb{Z}^+$, \wp is divided by 11 or multiplies of 11.
- (d) $\wp \in \mathbb{Z}^+$, \wp is divided by 17 or multiplies of 17.
- (e) $\wp \in \mathbb{Z}^+$, \wp is divided by 19 or multiplies of 19.

Proof. (a) Let us assume that $\wp = 3\alpha$ and m are elements of set P_{+29} for $\alpha \in \mathbb{Z}^+$. We have

$$3\alpha m + 29 = \mathcal{A}^2 \tag{2.34}$$

for an integer \mathcal{A} . If we apply (mod 3) on (2.34), then we obtain

$$\mathcal{A}^2 \equiv 2 \pmod{3} \tag{2.35}$$

and from (1.3) of Lemma 1.2,

$$\left(\frac{2}{3}\right) = -1 \tag{2.36}$$

is got. It is a contradiction. Thus, there isn't any $D(+29)$ contains the types of elements in (a).

(b) Given that n and 8β , ($n, \beta \in \mathbb{Z}^+$) are elements of $D(+29)$. By the Definition 1.1, we have

$$8\beta n + 29 = \mathcal{B}^2 \tag{2.37}$$

for integer \mathcal{B} . If we apply (mod 8) on (2.37), then following quadratic equivalent is had.

$$\mathcal{B}^2 \equiv 5 \pmod{8}. \tag{2.38}$$

It is clear that $\mathcal{B}^2 \equiv 0, 1, 4 \pmod{8}$ is satisfied from Lemma 1.1, Definition 1.3 and residue classes of modulo 8 is satisfied. It shows that (2.38) can not solved implies that it is a contradiction. Therefore, 5 is non-quadratic residue of (mod 8). So, there is no $D(+29)$ involves the types of elements in (b).

(c) Supposing that $r \in \mathbb{Z}^+$ and, $\wp \in \mathbb{Z}^+$ is divided by 11 or multiplies of 11 in $D(+29)$. Then,

$$11r\gamma + 29 = \mathcal{C}^2 \tag{2.39}$$

satisfies for integer \mathcal{C} . Using modulo 11, we have

$$\mathcal{C}^2 \equiv 7 \pmod{11}. \tag{2.40}$$

From (1.7) in Theorem 1.1, we obtain

$$\left(\frac{7}{11}\right)\left(\frac{11}{7}\right) = (-1)^{\frac{7-1}{2} \cdot \frac{11-1}{2}} = -1. \tag{2.41}$$

Applying properties of Legendre symbols on $\left(\frac{11}{7}\right)$ and using (1.6) from Definition 1.5, we have

$$\left(\frac{11}{7}\right) \equiv \left(\frac{4}{7}\right) = +1. \tag{2.42}$$

Then, we obtain $\left(\frac{7}{11}\right) = -1$. It is a contradiction.

Consequently, \wp can not be an element of $D(+29)$ Diophantine set.

(d) Assuming that k is an element of set $D(+29)$. If $\wp \in \mathbb{Z}^+$, \wp is divided by 17 or multiplies of 17 is an element of set $D(+29)$, so we obtain

$$17\theta k + 29 = \mathcal{D}^2 \tag{2.43}$$

for the integer \mathcal{D} . In a similar manner, we get

$$\mathcal{D}^2 \equiv 12 \pmod{17}. \tag{2.44}$$

Using Definition 1.5, we have

$$\left(\frac{12}{17}\right) = \left(\frac{3}{17}\right)\left(\frac{4}{17}\right) \tag{2.45}$$

From Theorem 1.1 (Quadratic Reciprocity Theorem), following equation is got.

$$\left(\frac{3}{17}\right)\left(\frac{17}{3}\right) = (-1)^{\frac{3-1}{2} \cdot \frac{17-1}{2}} = +1. \tag{2.46}$$

We write $\left(\frac{17}{3}\right) \equiv \left(\frac{2}{3}\right)$ from the properties of Legendre symbol. Using (1.4) (Lemma 1.2) we have

$$\left(\frac{2}{13}\right) = -1. \tag{2.47}$$

Besides, $\left(\frac{4}{17}\right) = +1$ holds since it satisfies (1.6)

condition and $\left(\frac{2}{17}\right) = (-1)^{\frac{289-1}{8}} = +1$.

It is shown that there isn't any \mathcal{D} integer satisfies (2.44). This is a contradiction. So, there is no set $D(+29)$ contains elements such that $\wp \in \mathbb{Z}^+$, \wp divided by 17 or multiplies of 17.

(e) Supposing that $\wp \in \mathbb{Z}^+$, \wp is divided by 19 or multiplies of 19, is an element in $D(+29)$. We get

$$19\vartheta s + 29 = \mathcal{F}^2 \tag{2.48}$$

for any s element of the set $D(+29)$ and it satisfies for integer \mathcal{F} .

Similarly, we obtain

$$\mathcal{F}^2 \equiv 10 \pmod{19}. \tag{2.49}$$

Using Definition 1.5 and condition (1.6), we have

$$\left(\frac{10}{19}\right) = \left(\frac{2}{19}\right)\left(\frac{5}{19}\right). \tag{2.50}$$

Considering (1.7) (Quadratic Reciprocity Theorem), we have.

$$\left(\frac{5}{19}\right)\left(\frac{19}{5}\right) = (-1)^{\frac{5-1}{2} \cdot \frac{19-1}{2}} = +1. \tag{2.51}$$

It is easily seen that $\left(\frac{19}{5}\right) \equiv \left(\frac{4}{5}\right) = \left(\frac{2}{5}\right)\left(\frac{2}{5}\right) = +1$ from (1.4). So, we have

$$\left(\frac{5}{19}\right) = +1. \tag{2.52}$$

Additionally, $\left(\frac{2}{19}\right) = -1$ satisfies due to (1.4). So, we obtain $\left(\frac{10}{19}\right) = -1$.

Hence, there is no \mathcal{F} integer satisfies (2.49). It is a contradiction. As a result, \wp is not an element of $D(+29)$.

Theorem 2.6. A set $P_{-29} = \{6,9,25\}$ is regular but can not be extended to the P_{-29} Diophantine quadruple.

Proof. It is clear that $P_{-29} = \{6,9,25\}$ is regular since it satisfies (1.1) condition in Definition 1.2. Let us suppose that $P_{-29} = \{6,9,25\}$ can be extendible to Diophantine quadruple. Then, there is a positive integer φ such that $P_{-29} = \{6,9,25,\varphi\}$. So, there are ρ_1, ρ_2, ρ_3 integers and following equations are hold.

$$6\varphi - 29 = \rho_1^2 \tag{2.53}$$

$$9\varphi - 29 = \rho_2^2 \tag{2.54}$$

$$25\varphi - 29 = \rho_3^2. \tag{2.55}$$

Dropping φ from (2.54) and (2.55), we have

$$9\rho_3^2 - 25\rho_2^2 = 464. \tag{2.56}$$

If we use factorization method into the (2.56), we obtain solutions as following table:

Table 5. Solutions of $9\rho_3^2 - 25\rho_2^2 = 464$ in the set of integers

Solutions	1. Class of Solutions	2. Class of Solutions
(ρ_3, ρ_2)	$(\mp 39, \mp 23)$	$(\mp 11, \mp 5)$

Reducing φ from (2.53) and (2.55), then we have

$$2\rho_2^2 - 3\rho_1^2 = 29. \tag{2.57}$$

By using Table 5, we calculate $\rho_2^2 = 529, \rho_2 = 23$. If we substitute these values into the (2.57), we have $\rho_1^2 = 343, \rho_1 = 7$ respectively. It shows that ρ_1 is not integer solution of (2.57) and it is contradiction. So, there is no positive integer φ and the set $P_{-29} = \{6,9,25\}$ can be nonextended to P_{-29} Diophantine quadruple.

Theorem 2.7. A $P_{-29} = \{9,25,62\}$ is both regular and nonextendible Diophantine triple.

Proof. Regularity of $P_{-29} = \{9,25,62\}$ Diophantine triple is satisfied by (1.1) from Definition 1.2. In a similar way, let us supposing that $P_{-29} = \{9,25,62,\sigma\}$ be a Diophantine quadruple for positive integer σ . We have $\Delta_1, \Delta_2, \Delta_3 \in \mathbb{Z}$ such that

$$9\sigma - 29 = \Delta_1^2 \tag{2.58}$$

$$25\sigma - 29 = \Delta_2^2 \tag{2.59}$$

$$62\sigma - 29 = \Delta_3^2. \tag{2.60}$$

If we simplify σ from (2.58) and (2.59), we have an equation corresponds to (2.56) for (Δ_1, Δ_2) . We get same solutions of Table 5 for (Δ_1, Δ_2) . We obtain $\Delta_1^2 = 529, \Delta_1 = 23$. If we eliminate σ from (2.58) and (2.60), then following equation is had.

$$9\Delta_3^2 - 62\Delta_1^2 = 1537. \tag{2.61}$$

Substituting $\Delta_1^2 = 529, \Delta_1 = 23$ into the (2.61), we have $\Delta_3^2 = 3815, \Delta_3 = 61$, respectively. It shows that Δ_3 is not integer solution for (2.61) and it is a contradiction.

Hence, there is no positive integer σ and $P_{-29} = \{9,25,62\}$ can not be extended to Diophantine quadruple.

Remark 2.2. There are a great deal of various regular $D(-29)$ Diophantine triples and one may detect others with factorization method used in this paper.

Theorem 2.8. There isn't any $D(-29)$ Diophantine set contains \mathfrak{F} elements hold any of the following circs;

- (i) $\epsilon \in \mathbb{Z}^+, \mathfrak{F}$ is divided by 4 or multiplies of 4.
- (ii) $\epsilon \in \mathbb{Z}^+, \mathfrak{F}$ is divided by 7 or multiplies of 7.
- (iii) $\epsilon \in \mathbb{Z}^+, \mathfrak{F}$ is divided by 17 or multiplies of 17.
- (iv) $\epsilon \in \mathbb{Z}^+, \mathfrak{F}$ is divided by 23 or multiplies of 23.

Proof. (i) Supposing that both u and also $\mathfrak{F} \in \mathbb{Z}^+, \mathfrak{F}$ is divided by 4 or multiplies of 4, be elements in $D(-29)$. By the Definition 1.1, we have

$$4au - 29 = \mathfrak{K}^2 \tag{2.62}$$

for an integer \mathfrak{K} . Applying modulo 4 on the (2.62), we obtain

$$\mathfrak{K}^2 \equiv 3 \pmod{4}. \tag{2.63}$$

If \mathfrak{K} is odd integer, then $\mathfrak{K}^2 \equiv 1 \pmod{4}$ holds. Otherwise, $\mathfrak{K}^2 \equiv 0 \pmod{4}$ satisfies. This implies that (2.63) doesn't have solution and it is a contradiction.

Therefore, If $\mathfrak{F} \in \mathbb{Z}^+$ such that \mathfrak{F} is divided by 4 or multiplies of 4, then $\mathfrak{F} \notin D(-29)$.

(ii) In a same vein, assuming that v is an element of $D(-29)$ and $\mathfrak{F} \in D(-29)$ such that $\epsilon \in \mathbb{Z}^+, \mathfrak{F}$ is divided by 7 or multiplies of 7. Then we have

$$7bv - 29 = \mathfrak{B}^2 \tag{2.64}$$

holds for integer \mathfrak{B} . Considering (2.64) with modulo 7, following is got.

$$\mathfrak{B}^2 \equiv 6 \pmod{7}. \tag{2.65}$$

To see whether or not (2.65) a solution, we can search result of Legendre symbol,

$$\left(\frac{6}{7}\right) = \left(\frac{2}{7}\right)\left(\frac{3}{7}\right). \tag{2.66}$$

From (1.4), we get $\left(\frac{2}{7}\right) = +1$ and from (1.7) we obtain

$$\left(\frac{3}{7}\right)\left(\frac{7}{3}\right) = (-1)^{\frac{3-1}{2} \cdot \frac{7-1}{2}} = -1. \tag{2.67}$$

By use of properties of Legendre Symbol, it is found

$\left(\frac{7}{3}\right) \equiv \left(\frac{1}{3}\right) = +1$. This implies that $\left(\frac{3}{7}\right) = -1$. So, we

calculate $\left(\frac{6}{7}\right) = -1$ and it is a contradiction.

Thus, $\mathfrak{F} \notin D(-29)$ if \mathfrak{F} satisfies the condition (ii).

(iii) Analogously, given that w and $\epsilon \in \mathbb{Z}^+, \mathfrak{F}$ is divided by 17 or multiplies of 17, be elements of $D(-29)$. We can write,

$$17cw - 29 = \mathfrak{C}^2 \tag{2.68}$$

for integer \mathfrak{C} . Applying modulo 17 on (2.68), we get an equivalent as follow;

$$\mathfrak{C}^2 \equiv 5 \pmod{17}. \tag{2.69}$$

Considering Quadratic Reciprocity Theorem (1.7) [from Theorem 1.1], we have

$$\left(\frac{5}{17}\right)\left(\frac{17}{5}\right) = (-1)^{\frac{5-1}{2} \cdot \frac{17-1}{2}} = +1. \tag{2.70}$$

From (1.4) and properties of Legendre Symbol, $\left(\frac{17}{5}\right) \equiv \left(\frac{2}{5}\right) = -1$ holds. It means that $\left(\frac{5}{17}\right) = -1$, if we substituting $\left(\frac{17}{5}\right) = -1$ into the (2.70).

So, $D(-29)$ doesn't include element holds $\epsilon \mathbb{Z}^+$, \mathfrak{F} is divided by 17 or multiplies of 17.

(iv) Given that r and \mathfrak{F} satisfies circ (iv) be elements of $D(-29)$. By the Definition 1.1, we have

$$23br - 29 = \mathfrak{D}^2 \tag{2.71}$$

for integer \mathfrak{D} . Using modulo 23, following equivalent is obtained.

$$\mathfrak{D}^2 \equiv 17 \pmod{23}. \tag{2.72}$$

From (1.7) of Theorem 1.1, then

$$\left(\frac{17}{23}\right)\left(\frac{23}{17}\right) = (-1)^{\frac{17-1}{2} \cdot \frac{23-1}{2}} = +1 \tag{2.73}$$

is got. By use of Legendre Symbol's properties, (1.4) and (1.7), we have $\left(\frac{23}{17}\right) \equiv \left(\frac{6}{17}\right) = \left(\frac{2}{17}\right)\left(\frac{3}{17}\right) = -1$.

$\left(\text{Since } \left(\frac{2}{17}\right) = +1 \text{ and } \left(\frac{3}{17}\right) = -1 \text{ hold}\right)$.

Hence, $\epsilon \mathbb{Z}^+$, \mathfrak{F} is divided by 23 or multiplies of 23, can not be element of $D(-29)$.

3. Conclusion

In the paper, we considered several $D(\mp 29)$ Diophantine triples and proved that they can not be extended to $D(\mp 29)$ Diophantine quadruple. Also, we demonstrated some properties of them by using factorization method in set of integers, integer solutions of Pell (Pellian) equations, quadratic residue, quadratic reciprocity theorem, legendre symbol so on. The results can be used to evaluate or estimate other results on them. Moreover, the conclusions would play significant role in the further study of Diophantiner $D(n)$ sets.

References

- [1] Beardon A.F. and Deshpande M.N., Diophantine Triples, The Mathematical Gazette 86, 258-260, 2002.
- [2] Dujella, A., Jurasic, A., Some Diophantine Triples and Quadruples for Quadratic Polynomials, J. Comp. Number Theory, Vol.3, No.2, 123-141, 2011.
- [3] Gopalan M. A., Vidhyalakshmi S., Mallika S., Some special non-extendable Diophantine triples, Sch. J. Eng. Tech. 2, 159-160, 2014.
- [4] Gopalan M.A., Vidhyalaksfmi S., Özer Ö., "A Collection of Pellian Equation (Solutions and Properties)", Akinik Publications, New Delhi, INDIA, 2018.
- [5] Özer Ö., A Note On The Particular Sets With Size Three, Boundary Field Problems and Computer Simulation Journal, 55, 56-59, 2016.
- [6] Özer Ö., On The Some Particular Sets, Kırklareli University Journal of Engineering and Science, 2, 99-108, 2016.
- [7] Özer Ö., Some Properties of The Certain P₁ Sets, International Journal of Algebra and Statistics, Vol. 6; 1-2, 117-130, 2017.
- [8] Özer Ö., On The Some Nonextendable Regular P₂ Sets, Malaysian Journal of Mathematical Sciences 12(2): 255-266, 2018.
- [9] Özer Ö., Şahin Ç.Z., On Some Particular Regular Diophantine 3-Tuples, Mathematics in Natural Sciences, (Accepted).
- [10] Goldmakher L., Number Theory Lecture Notes, Legendre, Jacobi and Kronecker Symbols Section, 2018.
- [11] Kurur P. P (Instructor), Saptharishi R. (Scribe), Computational Number Theory, Lecture Notes, Quadratic Reciprocity (contd.) Section, 2017.
- [12] Larson, D. and Cantu J., Parts I and II of the Law of Quadratic Reciprocity, Texas A&M University, Lecture Notes, 2015.
- [13] Bashmakova I.G. (ed.), Diophantus of Alexandria, Arithmetics and The Book of Polygonal Numbers, Nauka, Moscow, 1974.
- [14] Biggs N.L., Discrete Mathematics, Oxford University Press, 2003.
- [15] Burton D.M., Elementary Number Theory, Tata McGraw-Hill Education, 2006.
- [16] Cohen H., Number Theory, Graduate Texts in Mathematics, vol. 239, Springer-Verlag, New York, 2007.
- [17] Dickson LE., History of Theory of Numbers and Diophantine Analysis, Vol.2, Dove Publications, New York, 2005.
- [18] Fermat, P. Observations sur Diophante, Oeuvres de Fermat, Vol.1 (P. Tonnerly, C. Henry eds.), p.303, 1891.
- [19] Ireland K. and Rosen M., A Classical Introduction to Modern Number Theory, 2nd ed., Graduate Texts in Mathematics, Vol. 84, Springer-Verlag, New York, 1990.
- [20] Mollin R.A., Fundamental Number Theory with Applications, CRC Press, 2008.
- [21] Silverman, J. H., A Friendly Introduction to Number Theory. 4th Ed. Upper Saddle River: Pearson, 141-157, 2013.
- [22] Kedlaya K.S., Solving constrained Pell equations, Math. Comp. 67, 833-842, 1998.

