

# Some New Inequalities of Ostrowski Type for Double Integrals via Fractional Integral Operators

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**Abstract** In this paper, we first obtain the useful identity for double integrals via fractional integral operators. Then with the help of this identity we outline some significant Ostrowski type integral inequalities for functions in two variables. In accordance with this purpose we benefit from the properties of bounded function and concave mappings on co-ordinates. The established results are extensions of some existing Ostrowski type inequalities in the previous published studies.

**Keywords:** Ostrowski inequality, fractional integral operators, convex function

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## 1. Introduction

The study of various types of integral inequalities has been the focus of great attention for well over a century by a number of mathematicians, interested both in pure and applied mathematics. One of the many fundamental mathematical discoveries of A. M. Ostrowski [1] is the following classical integral inequality associated with the differentiable mappings:

**Theorem 1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  whose derivative  $f' : (a, b) \rightarrow \mathbb{R}$  is bounded on  $(a, b)$ , i.e.  $\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty$ . Then,

we have the inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty,$$

for all  $x \in [a, b]$ .

The constant  $\frac{1}{4}$  is the best possible.

Ostrowski inequality has applications in quadrature, probability and optimization theory, stochastic, statistics, information and integral operator theory. During the past few years, a number of scientists have focused on Ostrowski type inequalities, see for example [2-9]. Until now, a large number of research papers and books have been written on Ostrowski inequalities and their numerous applications.

The remainder of this work is organized as follows: In this section, the related definitions and theorems are summarised. In Section 2, new Ostrowski type integral inequalities are proved via fractional integral operators. At the end some conclusions and further directions of research are discussed in Section 3.

Let us consider a bi-dimensional interval  $\Delta =: [a, b] \times [c, d]$  in  $\mathbb{R}^2$  with  $a < b$  and  $c < d$ . A function  $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  is said to be convex on  $\Delta$  if for all  $(x, y), (z, w) \in \Delta$  and  $t \in [0, 1]$ , it satisfies the following inequality:

$$\begin{aligned} & f(tx + (1-t)z, ty + (1-t)w) \\ & \leq t f(x, y) + (1-t) f(z, w). \end{aligned}$$

A modification for convex function on  $\Delta$  was defined by Dragomir [dragomir], as follows:

A function  $f : \Delta \rightarrow \mathbb{R}$  is said to be convex on the co-ordinates on  $\Delta$  if the partial mappings  $f_y : [a, b] \rightarrow \mathbb{R}$ ,  $f_y(u) = f(u, y)$  and  $f_x : [c, d] \rightarrow \mathbb{R}$ ,  $f_x(v) = f(x, v)$  are convex where defined for all  $x \in [a, b]$  and  $y \in [c, d]$ .

A formal definition for co-ordinated convex function may be stated as follows:

**Definition 1.** A function  $f : \Delta \rightarrow \mathbb{R}$  is called co-ordinated convex on  $\Delta$ , for all  $(x, u), (y, v) \in \Delta$  and  $t, s \in [0, 1]$ , if it satisfies the following inequality:

$$\begin{aligned} & f(tx + (1-t)y, su + (1-s)v) \\ & \leq ts f(x, u) + t(1-s)f(x, v) + s(1-t)f(y, u) \\ & \quad + (1-t)(1-s)f(y, v), \end{aligned} \quad (1.1)$$

The mapping  $f$  is a co-ordinated concave on  $\Delta$  if the inequality (1.1) holds in reversed direction for all  $t, s \in [0, 1]$  and  $(x, u), (y, v) \in \Delta$ .

Note that every convex function  $f : \Delta \rightarrow \mathbb{R}$  is co-ordinated convex but the converse is not generally true (see, [10]).

In [10], Dragomir proved the following inequality which is Hermite-Hadamard type inequality for co-ordinated convex functions on the rectangle from the plane  $\mathbb{R}^2$ .

**Theorem 2.** Suppose that  $f : \Delta \rightarrow \mathbb{R}$  is co-ordinated convex, then we have the following inequalities:

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\ & \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\ & \leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx \right. \\ & \quad \left. + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \\ & \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned} \quad (1.2)$$

The above inequalities are sharp. The inequalities in (1.2) hold in reverse direction if the mapping  $f$  is a co-ordinated concave mapping.

In [11], Raina defined the following results connected with the general class of fractional integral operators.

$$\begin{aligned} \mathcal{F}_{\rho, \lambda}^{\sigma}(x) &= \mathcal{F}_{\rho, \lambda}^{\sigma(0), \sigma(1), \dots}(x) \\ &= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \quad (\rho, \lambda > 0; |x| < \mathcal{R}), \end{aligned} \quad (1.3)$$

where the coefficients  $\sigma(k)$  ( $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ) is a bounded sequence of positive real numbers and  $\mathcal{R}$  is the set of real numbers. With the help of (1.3), in [11] and [12], Raina and Agarwal et al. defined the following left-sided and right-sided fractional integral operators, respectively, as follows:

$$\begin{aligned} & \mathcal{J}_{\rho, \lambda, a+; \omega}^{\alpha} f(x) \\ &= \int_a^x (x-t)^{\lambda-1} \mathcal{F}_{\rho, \lambda}^{\sigma} \left[ \omega(x-t)^{\rho} \right] f(t) dt, \quad x > a, \end{aligned} \quad (1.4)$$

$$\begin{aligned} & \mathcal{J}_{\rho, \lambda, b-; \omega}^{\alpha} f(x) \\ &= \int_x^b (t-x)^{\lambda-1} \mathcal{F}_{\rho, \lambda}^{\sigma} \left[ \omega(t-x)^{\rho} \right] f(t) dt, \quad x < b, \end{aligned} \quad (1.5)$$

where  $\lambda, \rho > 0, \omega \in \mathbb{R}$ , and  $f(t)$  is such that the integrals on the right side exists.

It is easy to verify that  $\mathcal{J}_{\rho, \lambda, a+; \omega}^{\alpha} f(x)$  and  $\mathcal{J}_{\rho, \lambda, b-; \omega}^{\alpha} f(x)$  are bounded integral operators on  $L(a, b)$ , if

$$\mathfrak{M} := \mathcal{F}_{\rho, \lambda+1}^{\sigma} \left[ \omega(b-a)^{\rho} \right] < \infty. \quad (1.6)$$

In fact, for  $f \in L(a, b)$ , we have

$$\left\| \mathcal{J}_{\rho, \lambda, a+; \omega}^{\alpha} f(x) \right\|_1 \leq \mathfrak{M}(b-a)^{\lambda} \|f\|_1 \quad (1.7)$$

and

$$\left\| \mathcal{J}_{\rho, \lambda, b-; \omega}^{\alpha} f(x) \right\|_1 \leq \mathfrak{M}(b-a)^{\lambda} \|f\|_1, \quad (1.8)$$

where

$$\|f\|_p := \left( \int_a^b |f(t)|^p dt \right)^{\frac{1}{p}}.$$

The importance of these operators stems indeed from their generality. Many useful fractional integral operators can be obtained by specializing the coefficient  $\sigma(k)$ . Here, we just point out that the classical Riemann-Liouville fractional integrals  $I_{a+}^{\alpha}$  and  $I_{b-}^{\alpha}$  of order  $\alpha$  defined by (see, [13,14])

$$\begin{aligned} \left( I_{a+}^{\alpha} f \right)(x) &:= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \\ &(x > a; \alpha > 0) \end{aligned} \quad (1.9)$$

and

$$\begin{aligned} \left( I_{b-}^{\alpha} f \right)(x) &:= \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt \\ &(x < b; \alpha > 0) \end{aligned} \quad (1.10)$$

follow easily by setting

$$\lambda = \alpha, \quad \sigma(0) = 1, \quad \text{and } w = 0$$

in (1.4) and (1.5), and the boundedness of (1.9) and (1.10) on  $L(a, b)$  is also inherited from (1.7) and (1.8), (see, [12]). In [15], authors give new definitions related to fractional integral operators for two variables functions:

**Definition 2.** Let  $f \in L_1([a, b] \times [c, d])$ . The fractional integral operators for two variables functions with  $p = (p_1, p_2)$ ,  $\lambda = (\lambda_1, \lambda_2)$ ,  $p, \lambda \in [0, \infty)^2$ ;

$$\begin{aligned} & \left[ (x-a)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(x-a)^{\rho_1} \right] \right. \\ & \left. + (b-x)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(b-x)^{\rho_1} \right] \right] \\ & \times \left[ (y-c)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(y-c)^{\rho_2} \right] \right. \\ & \left. + (d-y)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(d-y)^{\rho_2} \right] \right] f(x, y) + A(x, y) \end{aligned}$$

$$\leq \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{q, \Delta} \left[ \left[ (x-a)^{\lambda_1+1} \mathcal{F}_{\rho_1, \lambda_1+2}^{\sigma_1} \left[ \omega_1 (x-a)^{\rho_1} \right] \right]^{2-\frac{1}{q}} \right. \\ \left. + \left[ (b-x)^{\lambda_1+1} \mathcal{F}_{\rho_1, \lambda_1+2}^{\sigma_1} \left[ \omega_1 (b-x)^{\rho_1} \right] \right]^{1-\frac{1}{q}} \right] \\ \times \left[ \left[ (y-c)^{\lambda_2+1} \mathcal{F}_{\rho_2, \lambda_2+2}^{\sigma_2} \left[ \omega_2 (y-c)^{\rho_2} \right] \right]^{2-\frac{1}{q}} \right. \\ \left. + \left[ (d-y)^{\lambda_2+1} \mathcal{F}_{\rho_2, \lambda_2+2}^{\sigma_2} \left[ \omega_2 (d-y)^{\rho_2} \right] \right]^{1-\frac{1}{q}} \right]$$

and  $a, c \geq 0$  defined by

$$\mathcal{J}_{\rho, \lambda, a+, c+; \omega}^{\sigma} f(x, y) \\ = \iint_{ac}^{xy} \left\{ \begin{array}{l} (x-t)^{\lambda_1-1} (y-s)^{\lambda_2-1} \mathcal{F}_{\rho_1, \lambda_1}^{\sigma_1} \left[ \omega_1 (x-t)^{\rho_1} \right] \\ \mathcal{F}_{\rho_2, \lambda_2}^{\sigma_2} \left[ \omega_2 (y-s)^{\rho_2} \right] f(t, s) \end{array} \right\} ds dt,$$

$x > a, y > c,$

$$\mathcal{J}_{\rho, \lambda, a+, d-; \omega}^{\sigma} f(x, y) \\ = \iint_{ay}^{xd} \left\{ \begin{array}{l} (x-t)^{\lambda_1-1} (s-y)^{\lambda_2-1} \mathcal{F}_{\rho_1, \lambda_1}^{\sigma_1} \left[ \omega_1 (x-t)^{\rho_1} \right] \\ \mathcal{F}_{\rho_2, \lambda_2}^{\sigma_2} \left[ \omega_2 (s-y)^{\rho_2} \right] f(t, s) \end{array} \right\} ds dt$$

$x > a, y < d,$

$$\mathcal{J}_{\rho, \lambda, b-, c+; \omega}^{\sigma} f(x, y) \\ = \iint_{xc}^{by} \left\{ \begin{array}{l} (t-x)^{\lambda_1-1} (y-s)^{\lambda_2-1} \mathcal{F}_{\rho_1, \lambda_1}^{\sigma_1} \left[ \omega_1 (t-x)^{\rho_1} \right] \\ \mathcal{F}_{\rho_2, \lambda_2}^{\sigma_2} \left[ \omega_2 (y-s)^{\rho_2} \right] f(t, s) \end{array} \right\} ds dt$$

$x < b, y > c,$

and

$$\mathcal{J}_{\rho, \lambda, b-, d-; \omega}^{\sigma} f(x, y) \\ = \iint_{xy}^{bd} \left\{ \begin{array}{l} (t-x)^{\lambda_1-1} (s-y)^{\lambda_2-1} \mathcal{F}_{\rho_1, \lambda_1}^{\sigma_1} \left[ \omega_1 (t-x)^{\rho_1} \right] \\ \mathcal{F}_{\rho_2, \lambda_2}^{\sigma_2} \left[ \omega_2 (s-y)^{\rho_2} \right] f(t, s) \end{array} \right\} ds dt$$

$x < b, y < d.$

Similar the above definition, we introduce the following integrals:

$$\mathcal{J}_{\rho_1, \lambda_1, x+; \omega_1}^{\sigma_1} f(b, y) \\ = \int_x^b (b-t)^{\lambda_1-1} \mathcal{F}_{\rho_1, \lambda_1}^{\sigma_1} \left[ \omega_1 (b-t)^{\rho_1} \right] f(t, y) dt, x > a$$

$$\mathcal{J}_{\rho_1, \lambda_1, x-; \omega_1}^{\sigma_1} f(a, y) \\ = \int_a^x (t-a)^{\lambda_1-1} \mathcal{F}_{\rho_1, \lambda_1}^{\sigma_1} \left[ \omega_1 (t-a)^{\rho_1} \right] f(t, y) dt, x < b$$

$$\mathcal{J}_{\rho_2, \lambda_2, y+; \omega_2}^{\sigma_2} f(x, d) \\ = \int_y^d (d-s)^{\lambda_2-1} \mathcal{F}_{\rho_2, \lambda_2}^{\sigma_2} \left[ \omega_2 (d-s)^{\rho_2} \right] f(x, t) dt, y > c$$

and

$$\mathcal{J}_{\rho_2, \lambda_2, y-; \omega_2}^{\sigma_2} f(x, c) \\ = \int_c^y (s-c)^{\lambda_2-1} \mathcal{F}_{\rho_2, \lambda_2}^{\sigma_2} \left[ \omega_2 (s-c)^{\rho_2} \right] f(x, t) dt, y < d.$$

Now the main findings are given below.

## 2. Ostrowski Type Inequalities via Fractional Integral Operators

In this section, we outline some significant lemma, theorems and known properties of some special inequalities used throughout the remaining of the paper. For the theorems, we shall need the fractional integral operator version of the special identity for functions in two independent variables given below (Lemma 11). In accordance with this purpose we prove this special identity and have results which are extension of bivariate Ostrowski type inequalities with the help of fractional integral operator.

Throughout this section,  $A(x, y), B_1(x, p), B_2(x, p), C_1(y, p)$  and  $C_2(y, p)$  denote the following expressions:

$$A(x, y) \\ = \mathcal{J}_{\rho, \lambda, x-, y-; \omega}^{\sigma} f(a, c) + \mathcal{J}_{\rho, \lambda, x-, y+; \omega}^{\sigma} f(a, d) \\ + \mathcal{J}_{\rho, \lambda, x+, y-; \omega}^{\sigma} f(b, c) + \mathcal{J}_{\rho, \lambda, x+, y+; \omega}^{\sigma} f(b, d) \\ - \left[ \begin{array}{l} (x-a)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (x-a)^{\rho_1} \right] \\ + (b-x)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-x)^{\rho_1} \right] \end{array} \right] \\ \times \left[ \begin{array}{l} \mathcal{J}_{\rho_2, \lambda_2, y-; \omega_2}^{\sigma_2} f(x, c) + \mathcal{J}_{\rho_2, \lambda_2, y+; \omega_2}^{\sigma_2} f(x, d) \\ - \left[ \begin{array}{l} (y-c)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (y-c)^{\rho_2} \right] \\ + (d-y)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-y)^{\rho_2} \right] \end{array} \right] \end{array} \right] \\ \times \left[ \begin{array}{l} \mathcal{J}_{\rho_1, \lambda_1, x-; \omega_1}^{\sigma_1} f(a, y) + \mathcal{J}_{\rho_1, \lambda_1, x+; \omega_1}^{\sigma_1} f(b, y) \end{array} \right],$$

$$B_1(x, p) = \left( \int_a^x (t-a)^{p\lambda_1} \left[ \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (t-a)^{\rho_1} \right] \right]^p dt \right)^{\frac{1}{p}},$$

$$B_2(x, p) = \left( \int_x^b (t-a)^{p\lambda_1} \left[ \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-t)^{\rho_1} \right] \right]^p dt \right)^{\frac{1}{p}},$$

$$C_1(y, p) = \left( \int_c^y (s-c)^{p\lambda_2} \left[ \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (s-c)^{\rho_2} \right] \right]^p ds \right)^{\frac{1}{p}}$$

and

$$C_2(y, p) = \left( \int_c^y (d-s)^{p\lambda_2} \left[ \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(d-s)^{\rho_2} \right] \right]^p ds \right)^{\frac{1}{p}}.$$

**Lemma 1.** Let  $f : \Delta := [a, b] \times [c, d]$  be a partial differentiable mapping on  $\Delta^\circ$  with  $a < b$ ,  $c < d$ . If  $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$ , then the following identity for fractional integral operators for function in two independent variables holds

$$\begin{aligned} & \left[ (x-a)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(x-a)^{\rho_1} \right] \right] \\ & \left[ + (b-x)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(b-x)^{\rho_1} \right] \right] \\ & \times \left[ (y-c)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(y-c)^{\rho_2} \right] \right] \\ & \left[ + (d-y)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(d-y)^{\rho_2} \right] \right] f(x, y) \\ & + A(x, y) \\ & = \iint_{ac}^{xy} \left\{ (t-a)^{\lambda_1} (s-c)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(t-a)^{\rho_1} \right] \right. \\ & \left. \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(s-c)^{\rho_2} \right] \frac{\partial^2}{\partial s \partial t} f(t, s) \right\} ds dt \\ & - \iint_{ay}^{xd} \left\{ (t-a)^{\lambda_1} (d-s)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(t-a)^{\rho_1} \right] \right. \\ & \left. \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(d-s)^{\rho_2} \right] \frac{\partial^2}{\partial s \partial t} f(t, s) \right\} ds dt \\ & - \iint_{xc}^{by} \left\{ (b-t)^{\lambda_1} (s-y)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(b-t)^{\rho_1} \right] \right. \\ & \left. \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(s-c)^{\rho_2} \right] \frac{\partial^2}{\partial s \partial t} f(t, s) \right\} ds dt \\ & + \iint_{xy}^{bd} \left\{ (b-t)^{\lambda_1} (d-s)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(b-t)^{\rho_1} \right] \right. \\ & \left. \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(d-s)^{\rho_2} \right] \frac{\partial^2}{\partial s \partial t} f(t, s) \right\} ds dt \end{aligned}$$

for  $(x, y) \in \Delta$ .

*Proof.* Using the integration by parts for double integrals, we have

$$\begin{aligned} I_1 &= \iint_{ac}^{xy} \left\{ (t-a)^{\lambda_1} (s-c)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(t-a)^{\rho_1} \right] \right. \\ & \left. \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(s-c)^{\rho_2} \right] \frac{\partial^2}{\partial s \partial t} f(t, s) \right\} ds dt \\ &= \int_a^x \left[ \int_c^y \left[ \omega_2(s-c)^{\rho_2} \right] \frac{\partial^2}{\partial s \partial t} f(t, s) ds \right] dt \end{aligned}$$

$$\begin{aligned} &= \int_a^x (t-a)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(t-a)^{\rho_1} \right] \\ & \left[ (y-c)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(y-c)^{\rho_2} \right] \frac{\partial}{\partial t} f(t, y) \right. \\ & \left. - \int_c^y (s-c)^{\lambda_2-1} \mathcal{F}_{\rho_2, \lambda_2}^{\sigma_2} \left[ \omega_2(s-c)^{\rho_2} \right] \frac{\partial}{\partial t} f(t, s) ds \right] dt \\ &= (x-a)^{\lambda_1} (y-c)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(x-a)^{\rho_1} \right] \\ & \times \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(y-c)^{\rho_2} \right] f(x, y) \\ & - (y-c)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(y-c)^{\rho_2} \right] \int_a^x (t-a)^{\lambda_1-1} \\ & \mathcal{F}_{\rho_1, \lambda_1}^{\sigma_1} \left[ \omega_1(t-a)^{\rho_1} \right] f(t, y) dt \\ & - (t-a)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(x-a)^{\rho_1} \right] \int_c^y (s-c)^{\lambda_2-1} \\ & \mathcal{F}_{\rho_2, \lambda_2}^{\sigma_2} \left[ \omega_2(s-c)^{\rho_2} \right] f(x, s) ds \\ & + \iint_{ac}^{xy} (t-a)^{\lambda_1-1} (s-c)^{\lambda_2-1} \mathcal{F}_{\rho_1, \lambda_1}^{\sigma_1} \left[ \omega_1(t-a)^{\rho_1} \right] \\ & \mathcal{F}_{\rho_2, \lambda_2}^{\sigma_2} \left[ \omega_2(s-c)^{\rho_2} \right] f(t, s) ds dt \\ &= (x-a)^{\lambda_1} (y-c)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(x-a)^{\rho_1} \right] \\ & \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(y-c)^{\rho_2} \right] f(x, y) \\ & - (y-c)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(y-c)^{\rho_2} \right] \mathcal{J}_{\rho_1, \lambda_1, x-; \omega_1}^{\sigma_1} f(a, y) \\ & - (x-a)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(x-a)^{\rho_1} \right] \mathcal{J}_{\rho_2, \lambda_2, y-; \omega_2}^{\sigma_2} f(x, c) \\ & + \mathcal{J}_{\rho, \lambda, x-, y-; \omega}^{\sigma} f(a, c). \end{aligned}$$

Similarly, following the same steps, we also have

$$\begin{aligned} I_2 &= \iint_{ay}^{xd} \left\{ (t-a)^{\lambda_1} (d-s)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(t-a)^{\rho_1} \right] \right. \\ & \left. \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(d-s)^{\rho_2} \right] \frac{\partial^2}{\partial s \partial t} f(t, s) \right\} ds dt \\ &= -(x-a)^{\lambda_1} (d-y)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(x-a)^{\rho_1} \right] \\ & \times \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(d-y)^{\rho_2} \right] f(x, y) \\ & + (d-y)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(d-y)^{\rho_2} \right] \mathcal{J}_{\rho_1, \lambda_1, x-; \omega_1}^{\sigma_1} f(a, y) \\ & + (x-a)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(x-a)^{\rho_1} \right] \mathcal{J}_{\rho_2, \lambda_2, y+; \omega_2}^{\sigma_2} f(x, d) \\ & - \mathcal{J}_{\rho, \lambda, x-, y+; \omega}^{\sigma} f(a, d), \\ I_3 &= \iint_{xc}^{by} \left\{ (b-t)^{\lambda_1} (s-y)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(b-t)^{\rho_1} \right] \right. \\ & \left. \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(s-c)^{\rho_2} \right] \frac{\partial^2}{\partial s \partial t} f(t, s) \right\} ds dt \end{aligned}$$

$$\begin{aligned}
 &= -(b-x)^{\lambda_1} (y-c)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-x)^{\rho_1} \right] \\
 &\quad \times \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (y-c)^{\rho_2} \right] f(x, y) \\
 &\quad + (y-c)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (y-c)^{\rho_2} \right] \mathcal{J}_{\rho_1, \lambda_1, x+; \omega_1}^{\sigma_1} f(b, y) \\
 &\quad + (b-x)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-x)^{\rho_1} \right] \mathcal{J}_{\rho_2, \lambda_2, y-; \omega_2}^{\sigma_2} f(x, c) \\
 &\quad - \mathcal{J}_{\rho, \lambda, x+, y-; \omega}^{\sigma} f(b, c),
 \end{aligned}$$

and

$$\begin{aligned}
 I_4 &= \iint_{xy}^{bd} \left[ (b-t)^{\lambda_1} (d-s)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-s)^{\rho_1} \right] \right. \\
 &\quad \left. \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-s)^{\rho_2} \right] \frac{\partial^2}{\partial s \partial t} f(t, s) \right] ds dt \\
 &= (b-x)^{\lambda_1} (d-y)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-x)^{\rho_1} \right] \\
 &\quad \times \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-y)^{\rho_2} \right] f(x, y) \\
 &\quad - (d-y)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-y)^{\rho_2} \right] \\
 &\quad \times \mathcal{J}_{\rho_1, \lambda_1, x+; \omega_1}^{\sigma_1} f(b, y) \\
 &\quad - (b-x)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-x)^{\rho_1} \right] \\
 &\quad \times \mathcal{J}_{\rho_2, \lambda_2, y+; \omega_2}^{\sigma_2} f(x, d) \mathcal{J}_{\rho, \lambda, x+, y+; \omega}^{\sigma} f(b, d).
 \end{aligned}$$

Then by calculating  $I_1 - I_2 - I_3 + I_4$  we get the required identity.

In the light of this useful lemma, we can prove the following important theorems.

**Theorem 3.** Let  $f : \Delta \rightarrow \mathbb{R}$  be a partial differentiable

mapping on  $\Delta^\circ$  with  $a < b, c < d$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right| \in L_\infty(\Delta)$ , i.e.

$\left| \frac{\partial^2 f}{\partial s \partial t} \right|$  is bounded on  $\Delta$ , then the following inequality for

fractional integral operators for function in two independent variables holds

$$\begin{aligned}
 &\left[ (x-a)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (x-a)^{\rho_1} \right] \right. \\
 &\quad \left. + (b-x)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-x)^{\rho_1} \right] \right] \\
 &\quad \times \left[ (y-c)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (y-c)^{\rho_2} \right] \right. \\
 &\quad \left. + (d-y)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-y)^{\rho_2} \right] \right] f(x, y) + A(x, y) \\
 &\leq \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_\infty \left[ (x-a)^{\lambda_1+1} \mathcal{F}_{\rho_1, \lambda_1+2}^{\sigma_1} \left[ \omega_1 (x-a)^{\rho_1} \right] \right. \\
 &\quad \left. + (b-x)^{\lambda_1+1} \mathcal{F}_{\rho_1, \lambda_1+2}^{\sigma_1} \left[ \omega_1 (b-x)^{\rho_1} \right] \right] \\
 &\quad \times \left[ (y-c)^{\lambda_2+1} \mathcal{F}_{\rho_2, \lambda_2+2}^{\sigma_2} \left[ \omega_2 (y-c)^{\rho_2} \right] \right. \\
 &\quad \left. + (d-y)^{\lambda_2+1} \mathcal{F}_{\rho_2, \lambda_2+2}^{\sigma_2} \left[ \omega_2 (d-y)^{\rho_2} \right] \right]
 \end{aligned}$$

for  $(x, y) \in \Delta$ .

*Proof.* By taking modulus of Lemma 1, and using the property of boundedness of  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$ , we get

$$\begin{aligned}
 &\left[ (x-a)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (x-a)^{\rho_1} \right] \right. \\
 &\quad \left. + (b-x)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-x)^{\rho_1} \right] \right] \\
 &\quad \times \left[ (y-c)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (y-c)^{\rho_2} \right] \right. \\
 &\quad \left. + (d-y)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-y)^{\rho_2} \right] \right] f(x, y) + A(x, y) \\
 &\leq \iint_{ac}^{xy} \left\{ (t-a)^{\lambda_1} (s-c)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (t-a)^{\rho_1} \right] \right. \\
 &\quad \left. \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (s-c)^{\rho_2} \right] \left| \frac{\partial^2}{\partial s \partial t} f(t, s) \right| \right\} ds dt \\
 &\quad + \iint_{ay}^{xd} \left\{ (t-a)^{\lambda_1} (d-s)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (t-a)^{\rho_1} \right] \right. \\
 &\quad \left. \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-s)^{\rho_2} \right] \left| \frac{\partial^2}{\partial s \partial t} f(t, s) \right| \right\} ds dt \\
 &\quad + \iint_{xc}^{by} \left\{ (b-t)^{\lambda_1} (s-c)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-t)^{\rho_1} \right] \right. \\
 &\quad \left. \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (s-c)^{\rho_2} \right] \left| \frac{\partial^2}{\partial s \partial t} f(t, s) \right| \right\} ds dt \\
 &\quad + \iint_{xy}^{bd} \left\{ (b-t)^{\lambda_1} (d-s)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-s)^{\rho_1} \right] \right. \\
 &\quad \left. \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-s)^{\rho_2} \right] \left| \frac{\partial^2}{\partial s \partial t} f(t, s) \right| \right\} ds dt \\
 &\leq \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_\infty \left( \int_a^x (t-a)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (t-a)^{\rho_1} \right] dt \right) \\
 &\quad \times \left( \int_c^y (s-c)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (s-c)^{\rho_2} \right] ds \right) \\
 &\quad + \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_\infty \left( \int_a^x (t-a)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (t-a)^{\rho_1} \right] dt \right) \\
 &\quad \times \left( \int_y^d (d-s)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-s)^{\rho_2} \right] ds \right) \\
 &\quad + \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_\infty \left( \int_x^b (b-t)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-t)^{\rho_1} \right] dt \right) \\
 &\quad \times \left( \int_c^y (s-c)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (s-c)^{\rho_2} \right] ds \right) \\
 &\quad + \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_\infty \left( \int_x^b (b-t)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-t)^{\rho_1} \right] dt \right) \\
 &\quad \times \left( \int_y^d (d-s)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-s)^{\rho_2} \right] ds \right)
 \end{aligned}$$

where  $\Delta_1 = [a, x] \times [c, y]$ ,  $\Delta_2 = [a, x] \times [y, d]$ ,  $\Delta_3 = [x, b] \times [c, y]$  and  $\Delta_4 = [x, b] \times [y, d]$ . Using the fact that

$$\left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{\infty, \Delta_i} \leq \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{\infty} \text{ for } i = 1, 2, 3, 4,$$

we have

$$\begin{aligned} & \left[ (x-a)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (x-a)^{\rho_1} \right] \right. \\ & + (b-x)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-x)^{\rho_1} \right] \left. \right] \\ & \times \left[ (y-c)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (y-c)^{\rho_2} \right] \right. \\ & + (d-y)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-y)^{\rho_2} \right] \left. \right] f(x, y) + A(x, y) \Big| \\ & \leq \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{\infty} \left[ (x-a)^{\lambda_1+1} \mathcal{F}_{\rho_1, \lambda_1+2}^{\sigma_1} \left[ \omega_1 (x-a)^{\rho_1} \right] \right. \\ & + (b-x)^{\lambda_1+1} \mathcal{F}_{\rho_1, \lambda_1+2}^{\sigma_1} \left[ \omega_1 (b-x)^{\rho_1} \right] \left. \right] \\ & \times \left[ (y-c)^{\lambda_2+1} \mathcal{F}_{\rho_2, \lambda_2+2}^{\sigma_2} \left[ \omega_2 (y-c)^{\rho_2} \right] \right. \\ & + (d-y)^{\lambda_2+1} \mathcal{F}_{\rho_2, \lambda_2+2}^{\sigma_2} \left[ \omega_2 (d-y)^{\rho_2} \right] \left. \right] \end{aligned}$$

which completes the proof.

**Theorem 4.** Let  $f : \Delta \rightarrow \mathbb{R}$  be a partial differentiable mapping on  $\Delta^\circ$  with  $a < b$ ,  $c < d$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right| \in L_q(\Delta)$ , then the following inequality for fractional integral operators for function in two independent variables holds for  $(x, y) \in \Delta$

$$\begin{aligned} & \left[ (x-a)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (x-a)^{\rho_1} \right] \right. \\ & + (b-x)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-x)^{\rho_1} \right] \left. \right] \\ & \times \left[ (y-c)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (y-c)^{\rho_2} \right] \right. \\ & + (d-y)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-y)^{\rho_2} \right] \left. \right] f(x, y) + A(x, y) \Big| \\ & \leq \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{q, \Delta} \left[ (x-a)^{\lambda_1+1} B_1(x, p) + (b-x)^{\lambda_1+1} B_2(x, p) \right] \\ & \times \left[ (y-c)^{\lambda_2+1} C_1(y, p) + (d-y)^{\lambda_2+1} C_2(y, p) \right] \end{aligned}$$

where

$$\left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{q, \Delta} = \left( \int_a^b \int_c^d \left| \frac{\partial^2 f}{\partial s \partial t} \right|^q ds dt \right)^{1/q}$$

and  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $B_k$  and  $C_k$ ,  $k = 1, 2$  are defined as above.

*Proof.* By taking modulus of Lemma 1 and applying the well-known Hölder's inequality for double integrals, we obtain

$$\begin{aligned} & \left[ (x-a)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (x-a)^{\rho_1} \right] \right. \\ & + (b-x)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-x)^{\rho_1} \right] \left. \right] \\ & \times \left[ (y-c)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (y-c)^{\rho_2} \right] \right. \\ & + (d-y)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-y)^{\rho_2} \right] \left. \right] f(x, y) \\ & + A(x, y) \Big| \\ & \leq \left( \int_a^x \int_c^y \left| \frac{\partial^2 f}{\partial s \partial t} (t, s) \right|^q ds dt \right)^{\frac{1}{q}} \\ & \times \left( \int_a^x \int_c^y (t-a)^{p\lambda_1} (s-c)^{p\lambda_2} \right. \\ & \times \left[ \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (t-a)^{\rho_1} \right] \right]^p \\ & \times \left[ \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (s-c)^{\rho_2} \right] \right]^p ds dt \Big)^{\frac{1}{p}} \\ & + \left( \int_a^x \int_c^y \left| \frac{\partial^2 f}{\partial s \partial t} (t, s) \right|^q ds dt \right)^{\frac{1}{q}} \\ & \times \left( \int_a^x \int_c^y (t-a)^{p\lambda_1} (d-s)^{p\lambda_2} \left[ \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (t-a)^{\rho_1} \right] \right]^p \right. \\ & \left. \left[ \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-s)^{\rho_2} \right] \right]^p ds dt \right)^{\frac{1}{p}} \\ & + \left( \int_x^b \int_c^y \left| \frac{\partial^2 f}{\partial s \partial t} (t, s) \right|^q ds dt \right)^{\frac{1}{q}} \\ & \times \left( \int_x^b \int_c^y (b-t)^{p\lambda_1} (s-y)^{p\lambda_2} \left[ \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-t)^{\rho_1} \right] \right]^p \right. \\ & \left. \left[ \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (s-c)^{\rho_2} \right] \right]^p ds dt \right)^{\frac{1}{p}} \\ & + \left( \int_x^b \int_y^d \left| \frac{\partial^2 f}{\partial s \partial t} (t, s) \right|^q ds dt \right)^{\frac{1}{q}} \\ & \times \left( \int_x^b \int_y^d (b-t)^{p\lambda_1} (d-s)^{p\lambda_2} \left[ \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-s)^{\rho_1} \right] \right]^p \right. \\ & \left. \left[ \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-s)^{\rho_2} \right] \right]^p ds dt \right)^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned}
 &= \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{q, \Delta_1} B_1(x, p) C_1(x, p) \\
 &+ \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{q, \Delta_2} B_1(x, p) C_2(x, p) \\
 &+ \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{q, \Delta_3} B_2(x, p) C_1(x, p) \\
 &+ \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{q, \Delta_4} B_2(x, p) C_2(x, p).
 \end{aligned} \tag{2.1}$$

where  $\Delta_1 = [a, x] \times [c, y]$ ,  $\Delta_2 = [a, x] \times [y, d]$ ,  $\Delta_3 = [x, b] \times [c, y]$  and  $\Delta_4 = [x, b] \times [y, d]$ .

Using the fact that

$$\left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{\infty, \Delta_i} \leq \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{\infty} \quad \text{for } i = 1, 2, 3, 4,$$

we obtain the required result.

**Theorem 5.** Let  $f : \Delta \rightarrow \mathbb{R}$  be a partial differentiable mapping on  $\Delta^\circ$  with  $a < b$ ,  $c < d$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right| \in L_q(\Delta)$ ,

then the following inequality for fractional integral operators for function in two independent variables holds for  $(x, y) \in \Delta$

$$\begin{aligned}
 &\left[ (x-a)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(x-a)^{\rho_1} \right] \right. \\
 &+ (b-x)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(b-x)^{\rho_1} \right] \left. \right] \\
 &\times \left[ (y-c)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(y-c)^{\rho_2} \right] \right. \\
 &+ (d-y)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(d-y)^{\rho_2} \right] \left. \right] f(x, y) \\
 &+ A(x, y) \\
 &\leq \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{q, \Delta} \left[ \left[ (x-a)^{\lambda_1+1} \mathcal{F}_{\rho_1, \lambda_1+2}^{\sigma_1} \left[ \omega_1(x-a)^{\rho_1} \right] \right]^{2-\frac{1}{q}} \right. \\
 &+ \left. \left[ (b-x)^{\lambda_1+1} \mathcal{F}_{\rho_1, \lambda_1+2}^{\sigma_1} \left[ \omega_1(b-x)^{\rho_1} \right] \right]^{1-\frac{1}{q}} \right] \\
 &\times \left[ \left[ (y-c)^{\lambda_2+1} \mathcal{F}_{\rho_2, \lambda_2+2}^{\sigma_2} \left[ \omega_2(y-c)^{\rho_2} \right] \right]^{2-\frac{1}{q}} \right. \\
 &+ \left. \left[ (d-y)^{\lambda_2+1} \mathcal{F}_{\rho_2, \lambda_2+2}^{\sigma_2} \left[ \omega_2(d-y)^{\rho_2} \right] \right]^{1-\frac{1}{q}} \right]
 \end{aligned}$$

where  $q \geq 1$ .

*Proof.* Similarly by taking modulus of Lemma 1 and applying well-known power mean inequality for double integrals, we get

$$\begin{aligned}
 &\left[ (x-a)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(x-a)^{\rho_1} \right] \right. \\
 &+ (b-x)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(b-x)^{\rho_1} \right] \left. \right] \\
 &\times \left[ (y-c)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(y-c)^{\rho_2} \right] \right. \\
 &+ (d-y)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(d-y)^{\rho_2} \right] \left. \right] f(x, y) + A(x, y) \\
 &\leq \left( \iint_{ac}^{xy} (t-a)^{\lambda_1} (s-c)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(t-a)^{\rho_1} \right] \right. \\
 &\times \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(s-c)^{\rho_2} \right] \left| \frac{\partial^2}{\partial s \partial t} f(t, s) \right|^q ds dt \Big)^{\frac{1}{q}} \\
 &\times \left( \iint_{ac}^{xy} (t-a)^{\lambda_1} (s-c)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(t-a)^{\rho_1} \right] \right. \\
 &\times \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(s-c)^{\rho_2} \right] ds dt \Big)^{1-\frac{1}{q}} \\
 &+ \left( \iint_{ay}^{xd} (t-a)^{\lambda_1} (d-s)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(t-a)^{\rho_1} \right] \right. \\
 &\times \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(d-s)^{\rho_2} \right] \left| \frac{\partial^2}{\partial s \partial t} f(t, s) \right|^q ds dt \Big)^{\frac{1}{q}} \\
 &\times \left( \iint_{ay}^{xd} (t-a)^{\lambda_1} (d-s)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(t-a)^{\rho_1} \right] \right. \\
 &\times \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(d-s)^{\rho_2} \right] ds dt \Big)^{1-\frac{1}{q}} \\
 &+ \left( \iint_{xc}^{by} (b-t)^{\lambda_1} (s-y)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(b-t)^{\rho_1} \right] \right. \\
 &\times \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(s-y)^{\rho_2} \right] \left| \frac{\partial^2}{\partial s \partial t} f(t, s) \right|^q ds dt \Big)^{\frac{1}{q}} \\
 &\times \left( \iint_{xc}^{by} (b-t)^{\lambda_1} (s-y)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(b-t)^{\rho_1} \right] \right. \\
 &\times \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(s-y)^{\rho_2} \right] ds dt \Big)^{1-\frac{1}{q}} \\
 &+ \left( \iint_{xy}^{bd} (b-t)^{\lambda_1} (d-s)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(b-s)^{\rho_1} \right] \right. \\
 &\times \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(d-s)^{\rho_2} \right] \left| \frac{\partial^2}{\partial s \partial t} f(t, s) \right|^q ds dt \Big)^{\frac{1}{q}} \\
 &\times \left( \iint_{xy}^{bd} (b-t)^{\lambda_1} (d-s)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1(b-s)^{\rho_1} \right] \right. \\
 &\times \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2(d-s)^{\rho_2} \right] ds dt \Big)^{1-\frac{1}{q}}.
 \end{aligned}$$

Then by rearranging the above inequality, we have

$$\begin{aligned} & \left. \iint_{ac}^{xy} \left\{ (t-a)^{\lambda_1} (s-c)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (t-a)^{\rho_1} \right] \right. \right. \\ & \left. \left. \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (s-c)^{\rho_2} \right] \left| \frac{\partial^2}{\partial s \partial t} f(t, s) \right|^q \right\} ds dt \right. \\ & \leq \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{q, \Delta_1}^q (x-a)^{\lambda_1+1} (y-c)^{\lambda_2+1} \\ & \mathcal{F}_{\rho_1, \lambda_1+2}^{\sigma_1} \left[ \omega_1 (x-a)^{\rho_1} \right] \mathcal{F}_{\rho_2, \lambda_2+2}^{\sigma_2} \left[ \omega_2 (y-c)^{\rho_2} \right]. \end{aligned}$$

Similarly, we also have

$$\begin{aligned} & \left. \iint_{ay}^{xd} \left\{ (t-a)^{\lambda_1} (d-s)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (t-a)^{\rho_1} \right] \right. \right. \\ & \left. \left. \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-s)^{\rho_2} \right] \left| \frac{\partial^2}{\partial s \partial t} f(t, s) \right|^q \right\} ds dt \right. \\ & \leq \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{q, \Delta_2}^q (x-a)^{\lambda_1+1} (d-y)^{\lambda_2+1} \\ & \mathcal{F}_{\rho_1, \lambda_1+2}^{\sigma_1} \left[ \omega_1 (x-a)^{\rho_1} \right] \mathcal{F}_{\rho_2, \lambda_2+2}^{\sigma_2} \left[ \omega_2 (d-y)^{\rho_2} \right], \\ & \left. \iint_{xy}^{bd} \left\{ (b-t)^{\lambda_1} (d-s)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-s)^{\rho_1} \right] \right. \right. \\ & \left. \left. \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-s)^{\rho_2} \right] \left| \frac{\partial^2}{\partial s \partial t} f(t, s) \right|^q \right\} ds dt \right. \\ & \leq \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{q, \Delta_3}^q (b-x)^{\lambda_1+1} (y-c)^{\lambda_2+1} \\ & \times \left[ \mathcal{F}_{\rho_1, \lambda_1+2}^{\sigma_1} \left[ \omega_1 (b-x)^{\rho_1} \right] \mathcal{F}_{\rho_2, \lambda_2+2}^{\sigma_2} \left[ \omega_2 (y-c)^{\rho_2} \right] \right] \end{aligned}$$

and

$$\begin{aligned} & \left. \iint_{xy}^{bd} \left\{ (b-t)^{\lambda_1} (d-s)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-s)^{\rho_1} \right] \right. \right. \\ & \left. \left. \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-s)^{\rho_2} \right] \left| \frac{\partial^2}{\partial s \partial t} f(t, s) \right|^q \right\} ds dt \right. \\ & \leq \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{q, \Delta_4}^q (b-x)^{\lambda_1+1} (d-y)^{\lambda_2+1} \\ & \mathcal{F}_{\rho_1, \lambda_1+2}^{\sigma_1} \left[ \omega_1 (b-x)^{\rho_1} \right] \mathcal{F}_{\rho_2, \lambda_2+2}^{\sigma_2} \left[ \omega_2 (d-y)^{\rho_2} \right]. \end{aligned}$$

Moreover, we get

$$\begin{aligned} & \left. \iint_{ac}^{xy} \left\{ (t-a)^{\lambda_1} (s-c)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (t-a)^{\rho_1} \right] \right. \right. \\ & \left. \left. \times \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (s-c)^{\rho_2} \right] \right\} ds dt \right. \\ & = (x-a)^{\lambda_1+1} (y-c)^{\lambda_2+1} \mathcal{F}_{\rho_1, \lambda_1+2}^{\sigma_1} \left[ \omega_1 (x-a)^{\rho_1} \right] \\ & \times \mathcal{F}_{\rho_2, \lambda_2+2}^{\sigma_2} \left[ \omega_2 (y-c)^{\rho_2} \right], \end{aligned}$$

$$\begin{aligned} & \left. \iint_{ay}^{xd} \left\{ (t-a)^{\lambda_1} (d-s)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (t-a)^{\rho_1} \right] \right. \right. \\ & \left. \left. \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-s)^{\rho_2} \right] \right\} ds dt \right. \\ & = (x-a)^{\lambda_1+1} (d-y)^{\lambda_2+1} \mathcal{F}_{\rho_1, \lambda_1+2}^{\sigma_1} \left[ \omega_1 (x-a)^{\rho_1} \right] \\ & \mathcal{F}_{\rho_2, \lambda_2+2}^{\sigma_2} \left[ \omega_2 (d-y)^{\rho_2} \right], \end{aligned}$$

$$\begin{aligned} & \left. \iint_{xc}^{by} \left\{ (b-t)^{\lambda_1} (s-y)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-t)^{\rho_1} \right] \right. \right. \\ & \left. \left. \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (s-c)^{\rho_2} \right] \right\} ds dt \right. \\ & = (b-x)^{\lambda_1+1} (y-c)^{\lambda_2+1} \mathcal{F}_{\rho_1, \lambda_1+2}^{\sigma_1} \left[ \omega_1 (b-x)^{\rho_1} \right] \\ & \mathcal{F}_{\rho_2, \lambda_2+2}^{\sigma_2} \left[ \omega_2 (y-c)^{\rho_2} \right] \end{aligned}$$

and

$$\begin{aligned} & \left. \iint_{xy}^{bd} \left\{ (b-t)^{\lambda_1} (d-s)^{\lambda_2} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-s)^{\rho_1} \right] \right. \right. \\ & \left. \left. \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-s)^{\rho_2} \right] \right\} ds dt \right. \\ & = (b-x)^{\lambda_1+1} (d-y)^{\lambda_2+1} \mathcal{F}_{\rho_1, \lambda_1+2}^{\sigma_1} \left[ \omega_1 (b-x)^{\rho_1} \right] \\ & \mathcal{F}_{\rho_2, \lambda_2+2}^{\sigma_2} \left[ \omega_2 (d-y)^{\rho_2} \right]. \end{aligned}$$

Using the fact that

$$\left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{q, \Delta_i} \leq \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{q, \Delta} \quad \text{for } i = 1, 2, 3, 4,$$

we obtain the desired inequality.

**Theorem 6.** Let  $f : \Delta \rightarrow \mathbb{R}$  be a partial differentiable

mapping on  $\Delta^\circ$  with  $a < b, c < d$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$  is a co-

ordinated concave mapping on  $\Delta$ , then the following inequality for fractional integral operators for function in two independent variables holds

$$\begin{aligned} & \left| \left[ (x-a)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (x-a)^{\rho_1} \right] \right. \right. \\ & \left. \left. + (b-x)^{\lambda_1} \mathcal{F}_{\rho_1, \lambda_1+1}^{\sigma_1} \left[ \omega_1 (b-x)^{\rho_1} \right] \right] \right. \\ & \left. \times \left[ (y-c)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (y-c)^{\rho_2} \right] \right. \right. \\ & \left. \left. + (d-y)^{\lambda_2} \mathcal{F}_{\rho_2, \lambda_2+1}^{\sigma_2} \left[ \omega_2 (d-y)^{\rho_2} \right] \right] f(x, y) + A(x, y) \right| \\ & \leq ((x-a)(y-c))^{\frac{1}{q}} \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{a+x}{2}, \frac{c+y}{2} \right) \right| \\ & \times B_1(x, p) C_1(x, p) \\ & + ((x-a)(d-y))^{\frac{1}{q}} \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{a+x}{2}, \frac{y+d}{2} \right) \right| \\ & \times B_1(x, p) C_2(x, p) \end{aligned}$$



$$\begin{aligned}
 &+((b-x)(y-c))^{\frac{1}{q}} \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+b}{2}, \frac{c+y}{2} \right) \right| \\
 &\times B_2(x, p) C_1(x, p) \\
 &+((b-x)(d-y))^{\frac{1}{q}} \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+b}{2}, \frac{y+d}{2} \right) \right| \\
 &\times B_2(x, p) C_2(x, p),
 \end{aligned} \tag{2.2}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $A(x, y)$ ,  $B_1(x, p)$ ,  $B_2(x, p)$ ,  $C_1(y, p)$  and  $C_2(y, p)$  are defined as above.

*Proof.* Since  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$  is a co-ordinated concave mapping on  $\Delta$ , using the (1.2) with inequalities in reversed direction, we have

$$\begin{aligned}
 &\iint_{ac}^{xy} \left| \frac{\partial^2}{\partial s \partial t} f(t, s) \right|^q ds dt \\
 &\leq \frac{1}{2} \left[ \int_a^y \left| \frac{\partial^2}{\partial s \partial t} f \left( t, \frac{c+y}{2} \right) \right|^q dt \right. \\
 &\quad \left. + \int_c^y \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{a+x}{2}, s \right) \right|^q ds \right] \\
 &\leq (x-a)(y-c) \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{a+x}{2}, \frac{c+y}{2} \right) \right|^q.
 \end{aligned} \tag{2.3}$$

Similarly, we get

$$\begin{aligned}
 &\iint_{ay}^{xd} \left| \frac{\partial^2}{\partial s \partial t} f(t, s) \right|^q ds dt \\
 &\leq (x-a)(d-y) \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{a+x}{2}, \frac{y+d}{2} \right) \right|^q,
 \end{aligned} \tag{2.4}$$

$$\begin{aligned}
 &\iint_{xc}^{by} \left| \frac{\partial^2}{\partial s \partial t} f(t, s) \right|^q ds dt \\
 &\leq (b-x)(y-c) \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+b}{2}, \frac{c+y}{2} \right) \right|^q
 \end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
 &\iint_{xy}^{bd} \left| \frac{\partial^2}{\partial s \partial t} f(t, s) \right|^q ds dt \\
 &\leq (b-x)(d-y) \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+b}{2}, \frac{y+d}{2} \right) \right|^q.
 \end{aligned} \tag{2.6}$$

Using the inequalities (2.3)-(2.6) in (2.1), we obtain the required inequality (2.2). Thus, the proof is completed.

### 3. Concluding Remarks

In this study, first of all, we provide the practical identity for two independent variables with the help of fractional integral operators. Then by using this identity we present new upper bounds for bivariate Ostrowski type inequalities by taking advantage of co-ordinated concave mappings on closed intervals.

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