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A Cogent Argument that Supports the Conjecture of Keane in Kolakoski Sequence A000002

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Abstract The aim of our investigation is an attempt to answer two still unsolved questions about Kolakoski sequence $(K_n)_{n\geq 1}$: Is there an explicit expression of the n^{th} term K_n, and the second one, known as the conjecture of Keane, claims that the asymptotic density of twos, is $\frac{1}{2}$. In the first section of this paper, we present a new formula for K_n according to $K_1, K_2, ...K_p$ where $p \approx \frac{4}{9}n$. In the second part, we define three sequences satisfying the condition $U_i V_i = W_i$, and using the fact that (V_i) increases at least exponentially while (W_i) does not, we conclude that (U_i) should converge to zero. Our argument is inductive but so strong to insure the validity of the conjecture in concern with density of twos.

Keywords: *Kolakoski sequence, recursive formula, asymptotic density.*
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1. Introduction

The infinite Kolakoski word $K = K_1K_2K_3....$ $=122112122122112112212 11...$ as defined in Sloane's OEIS [1], is the unique fixed point, starting by \$ 1, \$ of the Run Length Encoding operator Δ:

$$(\forall n \geq 1) \;\; K_n \in \{1, 2\} \quad \text{and} \;\; \Delta(K) = K$$

Many questions about this self-generating sequence are still with no answer [2].

The two most known are:

1. Is there an explicit expression of K_n with respect to n?

2. Is the asymptotic density of twos, $\frac{1}{2}$.

Bordellès [3] gave expressions of $K_{(K_1+K_2+...+K_p)}$ and $K_{(K_1+K_2+...+K_p)+1}$.

To complete this, we add a new expression of $K_{(K_1+K_2+...+K_p)-1}$.

Steinsky [4] found a complicated formula which allows to compute K_n from the former terms $K_1, K_2, ..., K_{n-1}$.

We improve this by using only $K_1, K_2, ..., K_p$ with $p \approx \frac{4}{9}n.$

Concerning the asymptotic density question, Steinsky [4] presented a curve which disapprove the conjecture, but, in our work, we use the fact that the exponential grows very fast to highly support it.

2. Notation

The successive partial sums

For $n \geq 1$ and $i \geq 1$, we define the following very useful partial sums

$$S_{0,n} = n$$

$$S_{1,n} = S_n = \sum_{j=1}^{n} K_j$$

and

$$S_{i+1,n} = S_{S_{i,n}}.$$

For example,

$$S_{0,10} = 10, \quad S_{1,10} = \sum_{j=1}^{10} K_j = 15, \quad S_{2,2} = 7.$$

The density ρ_n of twos in $K_1K_2...K_n$ and the discrepancy $\delta(n)$

For $n \geq 1$, by definition, $n\rho_n$ and δ_n are respectively the total number of twos and the difference between the twos and the ones, in the word $K_1K_2...K_n$.

For instance,

$$K_1K_2K_3 = 122 \Rightarrow \rho_3 = \frac{2}{3}$$

and $K_1K_2K_3K_4K_5 = 12211 \Rightarrow \delta_5 = -1.$

We will also need the classical identities

$$\delta(n) = (2\rho_n - 1)n = \sum_{j=1}^{n} (-1)^{K_j}$$

$$\text{and } \delta(S_n) = (2\rho_{S_{n-1}} - 1)S_n = \sum_{j=1}^{n} (-1)^j K_j.$$

Remark 1. It is well known that

$$(\forall n \geq 1)\ \frac{4}{9} \leq \rho_n \leq \frac{5}{9}.$$

3. A First Expression of K_n

Lemma 2. *For each integer $n \geq 1$, there exists an integer $p \leq \frac{9(n+1)}{13}$ such that*

$$\left(n = S_p\right) \vee \left(n = S_p - 1\right)$$

Proof 3. *Let n be a positive integer. It is clear that*

$$(\forall p \in \mathbb{N}^*)\ S_{p+1} - S_p = K_{p+1}$$

$$\Rightarrow (\forall p \in \mathbb{N}^*)\ 1 \leq S_{p+1} - S_p \leq 2$$

So, for any integer n, There are two cases:

$$(\exists p \in \mathbb{N})\ S_p = n.$$

Or

$$(\exists p \in \mathbb{N})\ S_p < n < S_{p+1} \leq S_p + 2$$

$$\Rightarrow 0 < n - S_p < 2 \Rightarrow n = S_p + 1.$$

On the other hand, $(1+\frac{4}{9})p \leq S_p \leq (1+\frac{5}{9})p$ *and* $n \geq S_p - 1 \Rightarrow p \leq \frac{9(n+1)}{13}.$

Lemma 4. *For each integer $p \geq 2$,*

$$K_{S_p - 1} = \frac{3 + (-1)^{p+K_p}}{2}.$$

Proof 5. *We just need results in Table 1.*

Table 1. The value of K_{S_p-1} in different cases

	p even and $K_p = 1$	p even and $K_p = 2$	p even and $K_p = 3$	p even and $K_p = 4$
$K_{S_p-1}K_{S_p}$	12	22	21	11
K_{S_p-1}	1	2	2	1

Using lemmas above, one can deduce an expression of K_n in both cases.

Corollary 6. *For $n \geq 3$, let*

$$A = \sum_{j=1}^{\left\lfloor \frac{9(n+1)}{13} \right\rfloor} j0^{n-S_j},$$

and

$$B = \sum_{j=1}^{\left\lfloor \frac{9(n+1)}{13} \right\rfloor} j0^{n-S_j+1}.$$

Then

$$\boxed{K_n = \frac{3 + (-1)^{A+B.0^A}}{2}.}$$

In the next section, we give a improved expression of K_n according to $K_1, K_2, ...K_{p \approx \frac{4}{9}n}$ instead of $K_1, K_2, ..., K_{p \approx \frac{2}{3}n}$.

4. A Second Expression of K_n

Lemma 7. *For each integer $n \geq 1$, there exists an integer p such that*

$$\left(n = S_{2,p}\right) \vee \left(n = S_{2,p} - 1\right) \vee \left(n = S_{2,p} - 2\right) \vee \left(n = S_{2,p} - 3\right)$$

Proof 8. *It is a simple consequence of the fact that*

$$(\forall i \geq 2)\ S_{2,i} - S_{2,i-1} = K_i.\frac{3 + (-1)^i}{2} \in \{1, 2, 3, 4\}$$

Corollary 9. *For $n \geq 3$, let*

$$A = \sum_{j=1}^{\left\lfloor \frac{81(n+3)}{269} \right\rfloor} 0^{n-S_{2,j}},\ A_1 = \sum_{j=1}^{\left\lfloor \frac{81(n+3)}{269} \right\rfloor} j.0^{n-S_{2,j}},$$

$$B = \sum_{j=1}^{\left\lfloor \frac{81(n+3)}{269} \right\rfloor} 0^{n-S_{2,j}+1}, B_1 = \sum_{j=1}^{\left\lfloor \frac{81(n+3)}{269} \right\rfloor} j.0^{n-S_{2,j}+1},$$

$$C = \sum_{j=1}^{\left\lfloor \frac{81(n+3)}{269} \right\rfloor} 0^{n-S_{2,j}+2}, C_1 = \sum_{j=1}^{\left\lfloor \frac{81(n+3)}{269} \right\rfloor} j.0^{n-S_{2,j}+2},$$

and

$$E = \sum_{j=1}^{\left\lfloor \frac{81(n+3)}{269} \right\rfloor} 0^{n-S_{2,j}+3}, E_1 = \sum_{j=1}^{\left\lfloor \frac{81(n+3)}{269} \right\rfloor} j.0^{n-S_{2,j}+3},$$

then

$$\boxed{K_n = \frac{3}{2} + \frac{(-1)^{S_p}}{2}\left(2A - 1 - (1 + (-1)^p)B\right)}$$

with

$$p = A_1 + 0^A\left[B_1 + 0^B\left(C_1 + 0^C.D_1\right)\right].$$

Proof 10.

It is easy to check that

If $A = 1$, *then* $n = S_{2,p}$ *and* $K_n = \frac{3+(-1)^{S_p}}{2}$.

If $A = 0, B = 1$ *then* $n = S_{2,p} - 1$ *and*

$$K_n = \frac{3+(-1)^{S_p+p}}{2}.$$

If $A = B = 0$ *then* $K_n = \frac{3-(-1)^{S_p}}{2}$.

We can replace the three cases be the next unique formula and using the fact that $A + 0^A\left[B + 0^B\left(C + 0^C E\right)\right] = 1$, *we get the desired expression for* K_n.

$$\boxed{K_n = \frac{3+(-1)^{S_p}}{2}A + 0^A\left[\frac{3+(-1)^{p+S_p}}{2}B + 0^B\frac{3-(-1)^{S_p}}{2}\left(C + 0^C E\right)\right]}$$

This formula has been validated by the code below, written in Maple language.

$$N := 100 : K_1 := 1 : K_2 := 2 : p := 2 :$$

for n *from* 3 *to* N *do*

$$A := 0 : B := 0 : C := 0 : E := 0 :$$

$$A_1 := 0 : B_1 := 0 : C_1 := 0 : E_1 := 0 : S_1 := 1 : S_2 := 1 :$$

for j *from* 2 *to* $\left\lfloor \frac{81(n+3)}{269} \right\rfloor$ *do*

$S_1 := S_1 + K_j :$

$S_2 := S_2 + K_j * (3 + (-1)^j) / 2 :$

$A := A + j * 0^{|n-S_2|} : \ A_1 := A_1 + 0^{|n-S_2|} :$

$B := B + j * 0^{|n-S_2+1|} : \ B_1 := B_1 + 0^{|n-S_2+1|} :$

$C := C + j * 0^{|n-S_2+2|} : \ C_1 := C_1 + 0^{|n-S_2+2|} :$

$E := E + j * 0^{|n-S_2+3|} : E_1 := E_1 + 0^{|n-S_2+3|} :$

end do :

$p := A_1 + 0^A * (B_1 + 0^B * (C_1 + 0^C * E)) :$

$S_p := add(K_m, m = 1..p) :$

$p_1 := (3 + (-1)^{S_p}) / 2 : p_2 := (3 + (-1)^{p+S_p}) / 2 :$

$p_3 := (3 - (-1)^{S_p}) / 2 :$

$\boxed{K_n := A_1 * p_1 + 0^A * \left(B_1 * p_2 + 0^B * p_3 * (C_1 + 0^C * E_1)\right) :}$

end do:

print$(seq(K_j, j = 1..N))$

1221121221221121122121121 2...

5. A Great Support for the Conjecture of Keane

Proposition 11. For an even natural number $n \geq 2$ satisfying $\sum_{j=1}^{n}(-1)^j = 0$,

$$\delta(n) = (2\rho_n - 1)n$$

$$= \left[\sum_{j=1}^{n}(-1)^{K_j}\frac{1+(-1)^j}{2} + \sum_{j=1}^{n}(-1)^{K_j}\frac{1-(-1)^j}{2}\right]$$

$$\delta(S_n) = (2\rho_{S_n} - 1)S_n$$

$$= \frac{1}{2}\left[\sum_{j=1}^{n}(-1)^{K_j}\frac{1+(-1)^j}{2} - \sum_{j=1}^{n}(-1)^{K_j}\frac{1-(-1)^j}{2}\right]$$

Proof 12. *By definition of the density* ρ_n,

$$(2\rho_n - 1)n = \sum_{j=1}^{n}(-1)^{K_j}$$

$$= \left[\sum_{j=1}^{n}(-1)^{K_j}\frac{1+(-1)^j}{2} + \sum_{j=1}^{n}(-1)^{K_j}\frac{1-(-1)^j}{2}\right]$$

and

$$(2\rho_{S_n} - 1)S_n = \sum_{j=1}^{n}(-1)^j K_j$$

$$= \frac{1}{2}\left(\sum_{j=1}^{n}(-1)^j(3 + (-1)^{K_j})\right) = \frac{1}{2}\left(\sum_{j=1}^{n}(-1)^{j+K_j}\right)$$

$$= \frac{1}{2}\left[\sum_{j=1}^{n}(-1)^{K_j}\frac{1+(-1)^j}{2} - \sum_{j=1}^{n}(-1)^{K_j}\frac{1-(-1)^j}{2}\right]$$

Corollary 13. From the proposition 11 just above, we see that the left hand sides contain a sequence which grows exponentially from n to $S_n \approx \frac{3}{2}n$ while, in the right hand side, there is probable change of sign indicating no exponential increasing as illustrated in Figure 1.

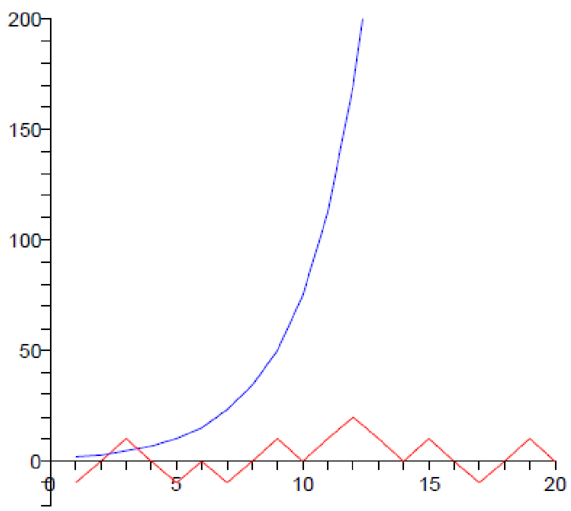

Figure 1. An exponential increasing versus a sign changing

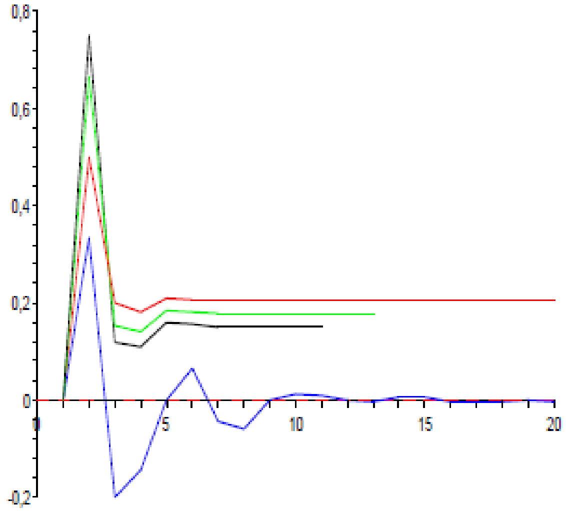

Figure 2. The density in $Kol(1,2)$ (blue curve) goes to $\frac{1}{2}$ while it does not in Kol $(1,3)$, $Kol(1,5)$ and $Kol(1,7)$

6. Concluding Remarks

We presented an optimal expression of the form $K_n = f(K_1, K_2, ..., K_p)$ with $p \approx \frac{4}{9}n$, improving so some former results. About the asymptotic density of twos, We have a strong reason to support Keane's conjecture: Our argument is based on Proposition 12. It uses the fact that if the product U_iV_i of two sequences have a changing sign and if V_i increases exponentially to $+\infty$, then (U_i) should converge to zero. This reasoning has been applied to other Kolakoski sequences: $Kol(1,3)$, $Kol(1,5)$, and $Kol(1,7)$ and we obtained the results predicted by Hammam [5] as illustrated in Figure 2. The blue curve shows clearly that the density of twos, in $Kol(1,2)$ goes to $\frac{1}{2}$.

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