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(m, r)-Convex Functions

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Abstract In this paper, we introduce a new class of extended (m, r)-convex function and we establish the Hermite-Hadamard inequality for (m, r)-convex functions. Some special cases are discussed. Results represent significant refinement and improvement of the previous results. The definition of (m, r)-convex function is given for the first time in the literature and moreover, the results obtained in special cases coincide with the well-known results in the literature.

Keywords: *convex function, r-convex function, m-convex function, (m, r)-convex, Hermite-Hadamard inequality*

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1. Introduction

Definition 1: A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature. Denote by $C(I)$ the set of the convex functions on the interval I.

Definition 2: $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions.

The inequality (1.1) is well known as the Hermite Hadamard integral inequality. Readers can find more information in [1]. Some refinements of the Hermite-Hadamard integral inequality on convex functions have been extensively investigated by a number of authors (see [2,3,4]).

Definition 3: [5] The function $f: [0, b] \to \mathbb{R}$ is said to be m-convex, where $m \in [0,1]$, if for every $x, y \in [0, b]$ and $t \in [0,1]$ we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).$$

Denote by $K_m(b)$ the set of the m-convex functions on $[0, b]$ for which $f(0) \leq 0$.

In [5], the author gave the following theorem about the inequalities of Hermite-Hadamard type for m-convex functions.

Theorem 1: Let $f: [0, \infty) \to \mathbb{R}$ be a m-convex functions with $m \in (0,1]$. If $0 \leq a < b < \infty$ and $f \in L_1[a, b]$, then one has the inequality:

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2}, \frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\}.$$

Definition 4: [6] A positive function f *is* called r-convex on interval $[a, b]$, if for each $x, y \in [a, b]$ and $t \in [0,1]$,

$$f(tx+(1-t)y) \leq \begin{cases} \left[tf^r(x)+(1-t)f^r(y)\right]^{\frac{1}{r}}, & r \neq 0 \\ \left[f(x)\right]^t\left[f(y)\right]^{1-t}, & r = 0. \end{cases}$$

If the equality is reversed, then the function f is said to be r-concave.

The definition of r-convexity naturally complements the concept of r-concavity, in which the inequality is reversed [7] and which plays an important role in statistics. It is obvious 0-convex functions are simply log-convex functions, 1-convex functions are ordinary convex functions and -1-convex functions are arithmetically harmonically convex. If f is r-convex in the interval $[a, b]$, then f^r is a convex function $(r > 0)$ and If f is r-concave in the interval $[a, b]$, then f^r is a concave function $(r < 0)$. We note that if f and g are convex and g is increasing, then gof is convex; moreover, since $f = exp(log\ f)$, it follows that a log-convex function is convex.

Some refinements of the Hadamard integral inequality for r-convex functions could be found in [8,9]. In [10], Bessenyei studied Hermite-Hadamard-type inequalities for generalized 3-convex functions. In [8], the authors showed that if f is r-convex in $[a,b]$ and $0<r\le 1$, then

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{r}{r+1}\left[f^r(a)+f^r(b)\right]^{\frac{1}{r}}.$$

In [11], the authors show that f is r-convex in the interval $[a,b]$ and $r\ge 1$, then

$$\frac{1}{b-a}\int_a^b f(x)dx \le \left[\frac{1}{2}\left(f^r(a)+f^r(b)\right)\right]^{\frac{1}{r}}$$

and they prove the following inequality for r-convex functions:

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{r}{r+1}\frac{f^{r+1}(b)-f^{r+1}(a)}{f^r(b)-f^r(a)} (r>0).$$

2. Main Results for (m,r)-convex Functions

The main aim of this paper is to establish new inequalities of Hermite-Hadamard type for the class of functions whose derivatives in absolutely value at certain powers are (m,r)-convex.

Definition 5: A positive function f is called (m,r)-convex on interval $[0,b]$, if for each $x,y\in[0,b]$ and $t\in[0,1]$,

$$f(tx+m(1-t)y) \le \left[tf^r(x)+m(1-t)f^r(y)\right]^{\frac{1}{r}}, r\ne 0 \quad (2.1)$$

If the equality is reversed, then the function f is said to be r-concave.

Theorem 2: Let $f:[0,\infty)\to(0,\infty)$ be a (m,r)-convex and $r\in\mathbb{R}\backslash\{0\}$ and $m\in(0,1]$. If $0\le a<b<\infty$ and $f\in L[a,b]$, then the following inequalities hold:

$$\frac{1}{mb-a}\int_a^{mb} f(x)dx \le \begin{cases} \frac{r}{r+1}\frac{m^{\frac{r+1}{r}}f^{r+1}(b)-f^{r+1}(a)}{mf^r(b)-f^r(a)} & (f(a)\ne m^{\frac{1}{r}}f(b), r\ne 0,-1), \\ f(a) & (f(a)=m^{\frac{1}{r}}f(b), r\ne 0,-1), \\ \frac{f(a)f(b)}{mf(a)-f(b)}\ln\frac{mf(a)}{f(b)} & (f(a)\ne m^{-1}f(b), r=-1), \\ f(a) & (f(a)=m^{-1}f(b), r=-1) \end{cases} \quad (2.2)$$

Proof: Suppose that $r\ne 0,-1$.

i. $f(a)\ne m^{\frac{1}{r}}f(b)$. Using the definition of (m,r)-convexity and changing variable as $u=tf^r(a)+m(1-t)f^r(b)$, we get

$$\frac{1}{mb-a}\int_a^{mb} f(x)dx = \int_0^1 f(ta+m(1-t)b)dt$$
$$\le \int_0^1 \left[tf^r(a)+m(1-t)f^r(b)\right]^{\frac{1}{r}}dt$$
$$= \int_{f^r(a)}^{mf^r(b)} \frac{u^{\frac{1}{r}}du}{mf^r(b)-f^r(a)}$$
$$= \frac{r}{r+1}\frac{m^{\frac{r+1}{r}}f^{r+1}(b)-f^{r+1}(a)}{mf^r(b)-f^r(a)}.$$

For $f(a)=m^{\frac{1}{r}}f(b)$, we obtain,

$$\frac{1}{mb-a}\int_a^{mb} f(x)dx \le \int_0^1 \left[tf^r(a)+m(1-t)f^r(b)\right]^{\frac{1}{r}}dt$$
$$= \int_0^1 f(a)dt = f(a).$$

ii. Let $r=-1$. For $f(a)\ne m^{-1}f(b)$ we obtain,

$$\frac{1}{mb-a}\int_a^{mb} f(x)dx \le \int_0^1 \left[tf^{-1}(a)+m(1-t)f^{-1}(b)\right]^{-1}dt$$
$$= \frac{1}{\frac{m}{f(b)}-\frac{1}{f(a)}}\int_{\frac{1}{f(a)}}^{\frac{m}{f(b)}} u^{-1}du$$
$$= \frac{f(a)f(b)}{mf(a)-f(b)}\left[\ln\frac{m}{f(b)}-\ln\frac{1}{f(a)}\right]$$
$$= \frac{f(a)f(b)}{mf(a)-f(b)}\ln\frac{mf(a)}{f(b)}.$$

For $f(a)=m^{-1}f(b)$, we have

$$\frac{1}{mb-a}\int_a^{mb} f(x)dx$$
$$\le \int_0^1 \left[tf^{-1}(a)+m(1-t)f^{-1}(b)\right]^{-1}dt$$
$$= \int_0^1 \left[f^{-1}(a)\right]^{-1}dt = f(a).$$

This completes the proof of theorem.

Corollary 1: Suppose that all the assumptions of Theorem 2 are satisfied. In the inequality 2.2, If we choose $m=1$, we obtain the inequality in [11].

In [12], İşcan obtained main results using the following lemma. We will use the same lemma to obtain the main results for (m,r)-convex functions.

Lemma 1: Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I°, $a, b \in I$ with $a < b$. If $f' \in L[a,b]$ and $\lambda, \mu \in [0, \infty)$, $\lambda + \mu > 0$, then following equality holds:

$$\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{b-a}{\lambda+\mu}\int_0^1 \left[(\lambda+\mu)t-\lambda\right] f'(tb+(1-t)a)\,dt. \quad (2.3)$$

Theorem 3: Let $f: I \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $\frac{a}{m} < b$. If $|f'|^q$ is (m,r)-convex on $[a,b]$, for some fixed $m \in (0,1]$, $\lambda, \mu \in [0,\infty)$ with $\lambda+\mu > 0$, $q \geq 1$ and $r \in \mathbb{R}\backslash\{-1,0\}$, then the following inequalities holds for $|f'(b)| \neq m^{\frac{1}{qr}} \left|f'\left(\frac{a}{m}\right)\right|$:

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \frac{b-a}{\lambda+\mu}\left(\frac{\lambda^2+\mu^2}{2(\lambda+\mu)}\right)^{\frac{q-1}{q}} \left\{ \frac{r}{r+1}\frac{\mu|f'(b)|^{q(r+1)} - \lambda m^{\frac{r+1}{r}}\left|f'\left(\frac{a}{m}\right)\right|^{q(r+1)}}{|f'(b)|^{qr} - m\left|f'\left(\frac{a}{m}\right)\right|^{qr}} \right. \\
&\left. + \frac{2}{(\lambda+\mu)^{\frac{r+1}{r}}}\frac{r^2}{(r+1)(2r+1)}\frac{\left[\lambda|f'(b)|^{qr} + m\mu\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{2r+1}{r}}}{\left[|f'(b)|^{qr} - m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^2} - \frac{r^2(\lambda+\mu)}{(r+1)(2r+1)}\frac{\left[\begin{array}{c}|f'(b)|^{q(2r+1)} \\ +m^{\frac{2r+1}{r}}\left|f'\left(\frac{a}{m}\right)\right|^{q(2r+1)}\end{array}\right]}{\left[|f'(b)|^{qr} - m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^2} \right\}^{\frac{1}{q}}. \quad (2.4)
\end{aligned}$$

Proof: Let $q = 1$. From Lemma 1 and the definition of (m,r)-convexity of $|f'|$, that is,

$$\left|f'\left(tb+m(1-t)\frac{a}{m}\right)\right| \leq \left[t|f'(b)|^r + m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^r\right]^{\frac{1}{r}}$$

we have

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| = \left|\frac{b-a}{\lambda+\mu}\int_0^1 \left[(\lambda+\mu)t-\lambda\right] f'(tb+(1-t)a)\,dt\right| \\
&\leq \frac{b-a}{\lambda+\mu}\int_0^1 |(\lambda+\mu)t-\lambda|\,|f'(tb+(1-t)a)|\,dt \\
&\leq \frac{b-a}{\lambda+\mu}\int_0^1 |(\lambda+\mu)t-\lambda|\left|f'\left(tb+m(1-t)\frac{a}{m}\right)\right|dt \\
&\leq \frac{b-a}{\lambda+\mu}\int_0^1 |(\lambda+\mu)t-\lambda|\left[t|f'(b)|^r + m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^r\right]^{\frac{1}{r}} dt \\
&= \frac{b-a}{\lambda+\mu}\int_0^{\frac{\lambda}{\lambda+\mu}} \left[\lambda-(\lambda+\mu)t\right]\left[t|f'(b)|^r + m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^r\right]^{\frac{1}{r}} dt \\
&+ \frac{b-a}{\lambda+\mu}\int_{\frac{\lambda}{\lambda+\mu}}^1 \left[(\lambda+\mu)t-\lambda\right]\left[t|f'(b)|^r + m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^r\right]^{\frac{1}{r}} dt
\end{aligned}$$

$$=\frac{b-a}{\lambda+\mu}\left\{\frac{r}{r+1}\frac{[\lambda-(\lambda+\mu)t]\left[t|f'(b)|^r+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^r\right]^{\frac{r+1}{r}}}{|f'(b)|^r-m\left|f'\left(\frac{a}{m}\right)\right|^r}+\frac{r^2(\lambda+\mu)}{(r+1)(2r+1)}\frac{\left[t|f'(b)|^r+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^r\right]^{\frac{2r+1}{r}}}{\left[|f'(b)|^r-m\left|f'\left(\frac{a}{m}\right)\right|^r\right]^2}\right\}\Bigg|_0^{\frac{\lambda}{\lambda+\mu}}$$

$$+\frac{b-a}{\lambda+\mu}\left\{\frac{r}{r+1}\frac{[(\lambda+\mu)t-\lambda]\left[t|f'(b)|^r+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^r\right]^{\frac{r+1}{r}}}{|f'(b)|^r-m\left|f'\left(\frac{a}{m}\right)\right|^r}-\frac{r^2(\lambda+\mu)}{(r+1)(2r+1)}\frac{\left[t|f'(b)|^r+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^r\right]^{\frac{2r+1}{r}}}{\left[|f'(b)|^r-m\left|f'\left(\frac{a}{m}\right)\right|^r\right]^2}\right\}\Bigg|_{\frac{\lambda}{\lambda+\mu}}^1$$

$$=\frac{b-a}{\lambda+\mu}\left\{-\frac{\lambda r}{r+1}\frac{m^{\frac{r+1}{r}}\left|f'\left(\frac{a}{m}\right)\right|^{r+1}}{|f'(b)|^r-m\left|f'\left(\frac{a}{m}\right)\right|^r}+\frac{r^2(\lambda+\mu)}{(r+1)(2r+1)}\frac{\left[\frac{\lambda|f'(b)|^r}{\lambda+\mu}+\frac{m\mu\left|f'\left(\frac{a}{m}\right)\right|^r}{\lambda+\mu}\right]^{\frac{2r+1}{r}}-m^{\frac{2r+1}{r}}\left|f'\left(\frac{a}{m}\right)\right|^{2r+1}}{\left[|f'(b)|^r-m\left|f'\left(\frac{a}{m}\right)\right|^r\right]^2}\right\}$$

$$+\frac{b-a}{\lambda+\mu}\left\{\frac{\mu r}{r+1}\frac{|f'(b)|^{r+1}}{|f'(b)|^r-m\left|f'\left(\frac{a}{m}\right)\right|^r}+\frac{r^2(\lambda+\mu)}{(r+1)(2r+1)}\frac{\left[\frac{\lambda|f'(b)|^r}{\lambda+\mu}+\frac{m\mu\left|f'\left(\frac{a}{m}\right)\right|^r}{\lambda+\mu}\right]^{\frac{2r+1}{r}}-|f'(b)|^{2r+1}}{\left[|f'(b)|^r-m\left|f'\left(\frac{a}{m}\right)\right|^r\right]^2}\right\}$$

$$=\frac{b-a}{\lambda+\mu}\frac{r}{r+1}\frac{\mu|f'(b)|^{r+1}-\lambda m^{\frac{r+1}{r}}\left|f'\left(\frac{a}{m}\right)\right|^{r+1}}{|f'(b)|^r-m\left|f'\left(\frac{a}{m}\right)\right|^r}-\frac{(b-a)r^2}{(r+1)(2r+1)}\frac{|f'(b)|^{2r+1}+m^{\frac{2r+1}{r}}\left|f'\left(\frac{a}{m}\right)\right|^{2r+1}}{\left[|f'(b)|^r-m\left|f'\left(\frac{a}{m}\right)\right|^r\right]^2}$$

$$+\frac{2(b-a)}{(\lambda+\mu)^{\frac{2r+1}{r}}}\frac{r^2}{(r+1)(2r+1)}\frac{\left[\lambda|f'(b)|^r+m\mu\left|f'\left(\frac{a}{m}\right)\right|^r\right]^{\frac{2r+1}{r}}}{\left[|f'(b)|^r-m\left|f'\left(\frac{a}{m}\right)\right|^r\right]^2}.$$

Let $q\in(1,\infty)$. From Lemma 1, the Power-mean integral inequality and the (m,r)-convexity of $|f'|^q$, we can write

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|=\left|\frac{b-a}{\lambda+\mu}\int_0^1\left[(\lambda+\mu)t-\lambda\right]f'(tb+(1-t)a)dt\right|$$

$$\leq\frac{b-a}{\lambda+\mu}\left(\int_0^1\left|(\lambda+\mu)t-\lambda\right|dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|(\lambda+\mu)t-\lambda\right|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \tag{2.5}$$

$$\leq\frac{b-a}{\lambda+\mu}\left(\int_0^1\left|(\lambda+\mu)t-\lambda\right|dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|(\lambda+\mu)t-\lambda\right|\left[t\left|f'(b)\right|^{qr}+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}.$$

Let calculate the integral in (2.5) respectively: Firstly, sample calculation give us

$$\int_0^1\left|(\lambda+\mu)t-\lambda\right|dt=\int_0^{\frac{\lambda}{\lambda+\mu}}\left[\lambda-(\lambda+\mu)t\right]dt+\int_{\frac{\lambda}{\lambda+\mu}}^1\left[(\lambda+\mu)t-\lambda\right]dt=\frac{\lambda^2+\mu^2}{2(\lambda+\mu)}. \tag{2.6}$$

Using the partial integration, we get

$$\int_0^1\left|(\lambda+\mu)t-\lambda\right|\left[t\left|f'(b)\right|^{qr}+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{1}{r}}dt$$

$$=\int_0^{\frac{\lambda}{\lambda+\mu}}\left[\lambda-(\lambda+\mu)t\right]\left[t\left|f'(b)\right|^{qr}+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{1}{r}}dt+\int_{\frac{\lambda}{\lambda+\mu}}^1\left[(\lambda+\mu)t-\lambda\right]\left[t\left|f'(b)\right|^{qr}+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{1}{r}}dt$$

$$=\left\{\frac{r}{r+1}\frac{\left[\lambda-(\lambda+\mu)t\right]\left[t\left|f'(b)\right|^{qr}+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{r+1}{r}}}{\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}}+\frac{r^2(\lambda+\mu)}{(r+1)(2r+1)}\frac{\left[t\left|f'(b)\right|^{qr}+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{2r+1}{r}}}{\left[\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^2}\right\}\Bigg|_0^{\frac{\lambda}{\lambda+\mu}}$$

$$+\left\{\frac{r}{r+1}\frac{\left[(\lambda+\mu)t-\lambda\right]\left[t\left|f'(b)\right|^{qr}+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{r+1}{r}}}{\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}}-\frac{r^2(\lambda+\mu)}{(r+1)(2r+1)}\frac{\left[t\left|f'(b)\right|^{qr}+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{2r+1}{r}}}{\left[\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^2}\right\}\Bigg|_{\frac{\lambda}{\lambda+\mu}}^1$$

$$=\frac{r}{r+1}\frac{\mu\left|f'(b)\right|^{q(r+1)}-\lambda m^{\frac{r+1}{r}}\left|f'\left(\frac{a}{m}\right)\right|^{q(r+1)}}{\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}}+\frac{2}{(\lambda+\mu)^{\frac{r+1}{r}}}\frac{r^2}{(r+1)(2r+1)}\frac{\left[\lambda\left|f'(b)\right|^{qr}+m\mu\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{2r+1}{r}}}{\left[\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^2}$$

$$-\frac{r^2(\lambda+\mu)}{(r+1)(2r+1)}\frac{\left|f'(b)\right|^{q(2r+1)}+m^{\frac{2r+1}{r}}\left|f'\left(\frac{a}{m}\right)\right|^{q(2r+1)}}{\left[\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^2}. \tag{2.7}$$

Substituting (2.6) and (2.7) inequalities in (2.5), we obtain

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{\lambda+\mu}\left(\frac{\lambda^2+\mu^2}{2(\lambda+\mu)}\right)^{\frac{q-1}{q}}\left\{\frac{r}{r+1}\frac{\mu|f'(b)|^{q(r+1)}-\lambda m^{\frac{r+1}{r}}\left|f'\left(\frac{a}{m}\right)\right|^{q(r+1)}}{|f'(b)|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}}\right.$$

$$\left.+\frac{2}{(\lambda+\mu)^{\frac{r+1}{r}}}\frac{r^2}{(r+1)(2r+1)}\frac{\left[\lambda|f'(b)|^{qr}+m\mu\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{2r+1}{r}}}{\left[|f'(b)|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^2}-\frac{r^2(\lambda+\mu)}{(r+1)(2r+1)}\frac{|f'(b)|^{q(2r+1)}+m^{\frac{2r+1}{r}}\left|f'\left(\frac{a}{m}\right)\right|^{q(2r+1)}}{\left[|f'(b)|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^2}\right\}^{\frac{1}{q}}.$$

Corollary 2: Under the conditions of Theorem 3,

(i) In the inequality (2.4), for $r=1$, we get the following inequality:

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{2}\left(\frac{1}{3}\right)^{\frac{1}{q}}\frac{\left(\lambda^2+\mu^2\right)^{\frac{q-1}{q}}}{(\lambda+\mu)^{\frac{2q+1}{q}}}\left[\begin{array}{l}\left(\lambda^3+3\lambda\mu^2+2\mu^3\right)|f'(b)|^q\\+m\left(2\lambda^3+3\lambda^2\mu+\mu^3\right)\left|f'\left(\frac{a}{m}\right)\right|^q\end{array}\right]^{\frac{1}{q}}.$$

(ii) In the inequality (2.4), for $\lambda=\mu$, $r=1$ and $m=1$, we get the following:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(b)|^q\right),$$

where A is the arithmetic mean.

(iii) In the inequality (2.4), for $\lambda=\mu$, $r=1$, $m=1$ and $q=1$ we get:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}A\left(|f'(a)|,|f'(b)|\right),$$

where A is the arithmetic mean. This inequality coincides with in [12] for $\alpha=1$.

Theorem 4: Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $\frac{a}{m}<b$. *If* $|f'|^q$ is (m,r)-convex on the interval $[a,b]$, for some fixed $m\in(0,1]$, $\lambda,\mu\in[0,\infty)$ with $\lambda+\mu>0, q>1$, and and $r\in\mathbb{R}\backslash\{-1,0\}$, then the following inequalities holds for $|f'(b)|\neq m^{\frac{1}{qr}}\left|f'\left(\frac{a}{m}\right)\right|$:

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\frac{b-a}{\lambda+\mu}\left(\frac{\lambda^{p+1}}{(p+1)(\lambda+\mu)}\right)^{\frac{1}{p}}\left(\frac{\frac{r}{r+1}}{|f'(b)|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}}\right)^{\frac{1}{q}}\left(\left[\frac{\lambda|f'(b)|^{qr}}{\lambda+\mu}+m\frac{\mu\left|f'\left(\frac{a}{m}\right)\right|^{qr}}{\lambda+\mu}\right]^{\frac{r+1}{r}}-\left[m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{r+1}{r}}\right)^{\frac{1}{q}}$$

$$+\frac{b-a}{\lambda+\mu}\left(\frac{\mu^{p+1}}{(p+1)(\lambda+\mu)}\right)^{\frac{1}{p}}\left(\frac{\frac{r}{r+1}}{|f'(b)|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}}\right)^{\frac{1}{q}}\left(\left[|f'(b)|^{qr}\right]^{\frac{r+1}{r}}-\left[\frac{\lambda|f'(b)|^{qr}}{\lambda+\mu}+m\frac{\mu\left|f'\left(\frac{a}{m}\right)\right|^{qr}}{\lambda+\mu}\right]^{\frac{r+1}{r}}\right)^{\frac{1}{q}}, \tag{2.8}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof: Using Lemma 1, the Hölder inequality and (m,r)-convexity of $|f'|^q$, we get

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\frac{b-a}{\lambda+\mu}\left(\int_0^{\frac{\lambda}{\lambda+\mu}}\left[\lambda-(\lambda+\mu)t\right]^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\lambda}{\lambda+\mu}}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}+\frac{b-a}{\lambda+\mu}\left(\int_{\frac{\lambda}{\lambda+\mu}}^1\left[(\lambda+\mu)t-\lambda\right]^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{\lambda}{\lambda+\mu}}^1\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq\frac{b-a}{\lambda+\mu}\left(\int_0^{\frac{\lambda}{\lambda+\mu}}\left[\lambda-(\lambda+\mu)t\right]^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\lambda}{\lambda+\mu}}\left[t\left|f'(b)\right|^{qr}+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}$$

$$+\frac{b-a}{\lambda+\mu}\left(\int_{\frac{\lambda}{\lambda+\mu}}^1\left[(\lambda+\mu)t-\lambda\right]^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{\lambda}{\lambda+\mu}}^1\left[t\left|f'(b)\right|^{qr}+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}$$

$$=\frac{b-a}{\lambda+\mu}\left(\frac{\lambda^{p+1}}{(p+1)(\lambda+\mu)}\right)^{\frac{1}{p}}\left(\frac{\frac{r}{r+1}}{\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}}\right)^{\frac{1}{q}}\left(\left[\frac{\lambda\left|f'(b)\right|^{qr}}{\lambda+\mu}+m\frac{\mu\left|f'\left(\frac{a}{m}\right)\right|^{qr}}{\lambda+\mu}\right]^{\frac{r+1}{r}}-\left[m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{r+1}{r}}\right)^{\frac{1}{q}}$$

$$+\frac{b-a}{\lambda+\mu}\left(\frac{\mu^{p+1}}{(p+1)(\lambda+\mu)}\right)^{\frac{1}{p}}\left(\frac{\frac{r}{r+1}}{\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}}\right)^{\frac{1}{q}}\left(\left[\left|f'(b)\right|^{qr}\right]^{\frac{r+1}{r}}-\left[\frac{\lambda\left|f'(b)\right|^{qr}}{\lambda+\mu}+m\frac{\mu\left|f'\left(\frac{a}{m}\right)\right|^{qr}}{\lambda+\mu}\right]^{\frac{r+1}{r}}\right)^{\frac{1}{q}}$$

where we use the fact that

$$\int_0^{\frac{\lambda}{\lambda+\mu}}\left[\lambda-(\lambda+\mu)t\right]^p dt=\frac{\lambda^{p+1}}{(p+1)(\lambda+\mu)},\quad\int_{\frac{\lambda}{\lambda+\mu}}^1\left[(\lambda+\mu)t-\lambda\right]^p dt=\frac{\mu^{p+1}}{(p+1)(\lambda+\mu)},$$

$$\left|f'\left(tb+m(1-t)\frac{a}{m}\right)\right|\leq\left[t\left|f'(b)\right|^r+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^r\right]^{\frac{1}{r}}.$$

Corollary 3: Under the conditions of Theorem 4,

(i) In the inequality (2.8), for $r=1$, we get the following inequality:

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\frac{b-a}{2^{\frac{1}{q}}}\left(\frac{1}{\lambda+\mu}\right)^{2+\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\lambda^{\frac{p+1}{p}}\left(\lambda^2\left|f'(b)\right|^q+m\left(\lambda^2+2\lambda\mu\right)\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}+\mu^{\frac{p+1}{p}}\left(\left(2\lambda\mu+\mu^2\right)\left|f'(b)\right|^q+m\mu^2\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}\right].$$

(ii) In the inequality (2.8), for $\lambda=\mu$ and $r=1$, we get the following inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{b-a}{2^{2+\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[A^{\frac{1}{q}}\left(\left|f'(b)\right|^q,3m\left|f'\left(\frac{a}{m}\right)\right|^q\right)+A^{\frac{1}{q}}\left(3\left|f'(b)\right|^q,m\left|f'\left(\frac{a}{m}\right)\right|^q\right)\right],$$

where A is the arithmetic mean.

(iii) In the inequality (2.8), for $\lambda=\mu$, $r=1$ and $m=1$, we get the following inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{b-a}{2^{2+\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[A^{\frac{1}{q}}\left(\left|f'(b)\right|^q,3\left|f'(a)\right|^q\right)+A^{\frac{1}{q}}\left(3\left|f'(b)\right|^q,\left|f'(a)\right|^q\right)\right].$$

where A is the arithmetic mean.

Theorem 5: Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $\frac{a}{m}<b$. If $|f'|^q$ is (m,r)-convex on the interval $[a,b]$, for some fixed $m\in(0,1]$, $\lambda,\mu\in[0,\infty)$ with $\lambda+\mu>0, q\ge 1$, and $r\in\mathbb{R}\setminus\{-1,0\}$, then the following inequalities holds for $\left|f'(b)\right|\neq m^{\frac{1}{qr}}\left|f'\left(\frac{a}{m}\right)\right|$:

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le\frac{b-a}{\lambda+\mu}\left(\frac{\lambda^2}{2(\lambda+\mu)}\right)^{1-\frac{1}{q}}\left\{\frac{r^2(\lambda+\mu)}{(r+1)(2r+1)}\frac{\left[\frac{\lambda\left|f'(b)\right|^{qr}+m\mu\left|f'\left(\frac{a}{m}\right)\right|^{qr}}{\lambda+\mu}\right]^{\frac{2r+1}{r}}}{\left[\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^2}-\frac{r}{r+1}\frac{\lambda\left[m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{r+1}{r}}}{\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}}-\frac{r^2}{(r+1)(2r+1)}\frac{(\lambda+\mu)\left[m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{2r+1}{r}}}{\left[\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^2}\right\}^{\frac{1}{q}}$$

$$+\frac{b-a}{\lambda+\mu}\left(\frac{\mu^2}{2(\lambda+\mu)}\right)^{1-\frac{1}{q}}\left\{\frac{r^2(\lambda+\mu)}{(r+1)(2r+1)}\frac{\left[\frac{\lambda\left|f'(b)\right|^{qr}+m\mu\left|f'\left(\frac{a}{m}\right)\right|^{qr}}{\lambda+\mu}\right]^{\frac{2r+1}{r}}}{\left[\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^2}+\frac{r}{r+1}\frac{\mu\left[\left|f'(b)\right|^{qr}\right]^{\frac{r+1}{r}}}{\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}}-\frac{r^2}{(r+1)(2r+1)}\frac{(\lambda+\mu)\left[\left|f'(b)\right|^{qr}\right]^{\frac{2r+1}{r}}}{\left[\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^2}\right\}^{\frac{1}{q}}. \qquad (2.9)$$

Proof: From Lemma 1 and using the Power-mean integral inequality and (m,r)-convexity of $|f'|^q$, we get

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$=\left|\frac{b-a}{\lambda+\mu}\int_0^1\left[(\lambda+\mu)t-\lambda\right]f'\left(tb+(1-t)a\right)dt\right|$$

$$\leq\frac{b-a}{\lambda+\mu}\left(\int_0^{\frac{\lambda}{\lambda+\mu}}\left[\lambda-(\lambda+\mu)t\right]dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{\lambda}{\lambda+\mu}}\left[\lambda-(\lambda+\mu)t\right]\left|f'\left(tb+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{b-a}{\lambda+\mu}\left(\int_{\frac{\lambda}{\lambda+\mu}}^1\left[(\lambda+\mu)t-\lambda\right]dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{\lambda}{\lambda+\mu}}^1\left[(\lambda+\mu)t-\lambda\right]\left|f'\left(tb+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq\frac{b-a}{\lambda+\mu}\left(\int_0^{\frac{\lambda}{\lambda+\mu}}\left[\lambda-(\lambda+\mu)t\right]dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{\lambda}{\lambda+\mu}}\left[\lambda-(\lambda+\mu)t\right]\left[t\left|f'(b)\right|^{qr}+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}$$

$$+\frac{b-a}{\lambda+\mu}\left(\int_{\frac{\lambda}{\lambda+\mu}}^1\left[(\lambda+\mu)t-\lambda\right]dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{\lambda}{\lambda+\mu}}^1\left[(\lambda+\mu)t-\lambda\right]\left[t\left|f'(b)\right|^{qr}+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}$$

$$=\frac{b-a}{\lambda+\mu}\left(\frac{\lambda^2}{2(\lambda+\mu)}\right)^{1-\frac{1}{q}}\left(\frac{r^2(\lambda+\mu)}{(r+1)(2r+1)}\frac{\left[\left[\frac{\lambda\left|f'(b)\right|^{qr}+m\mu\left|f'(a/m)\right|^{qr}}{\lambda+\mu}\right]\right]^{\frac{2r+1}{r}}}{\left[\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^2}\right.$$

$$\left.-\frac{r}{r+1}\frac{\lambda\left[m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{r+1}{r}}}{\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}}-\frac{r^2}{(r+1)(2r+1)}\frac{(\lambda+\mu)\left[m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^{\frac{2r+1}{r}}}{\left[\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^2}\right)^{\frac{1}{q}}$$

$$+\frac{b-a}{\lambda+\mu}\left(\frac{\mu^2}{2(\lambda+\mu)}\right)^{1-\frac{1}{q}}\left(\frac{r^2(\lambda+\mu)}{(r+1)(2r+1)}\frac{\left[\frac{\lambda\left|f'(b)\right|^{qr}+m\mu\left|f'(a/m)\right|^{qr}}{\lambda+\mu}\right]^{\frac{2r+1}{r}}}{\left[\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^2}\right.$$

$$\left.+\frac{r}{r+1}\frac{\mu\left[\left|f'(b)\right|^{qr}\right]^{\frac{r+1}{r}}}{\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}}-\frac{r^2}{(r+1)(2r+1)}\frac{(\lambda+\mu)\left[\left|f'(b)\right|^{qr}\right]^{\frac{2r+1}{r}}}{\left[\left|f'(b)\right|^{qr}-m\left|f'\left(\frac{a}{m}\right)\right|^{qr}\right]^2}\right)^{\frac{1}{q}}$$

Corollary 4: Under the conditions of the Theorem 5,

(i) In the inequality (2.9), if we choose $r = 1$, we obtain the following inequality:

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{2}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left(\frac{\lambda}{\lambda+\mu}\right)^2\left[\frac{\lambda|f'(b)|^q+m(2\lambda+3\mu)\left|f'\left(\frac{a}{m}\right)\right|^q}{\lambda+\mu}\right]^{\frac{1}{q}}$$

$$+\frac{b-a}{2}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left(\frac{\mu}{\lambda+\mu}\right)^2\left[\frac{(2\mu+3\lambda)|f'(b)|^q+m\mu\left|f'\left(\frac{a}{m}\right)\right|^q}{\lambda+\mu}\right]^{\frac{1}{q}}.$$

(ii) In the inequality (2.9), if we choose $\lambda = \mu$ and $r = 1$, we obtain the following inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{8}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left\{\left[\frac{|f'(b)|^q+5m\left|f'\left(\frac{a}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}+\left[\frac{5|f'(b)|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}\right\}.$$

(iii) In the inequality (2.9), if we choose $\lambda = \mu$, $r = 1$ and $m = 1$, we obtain the following inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{8}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left\{\left[\frac{|f'(b)|^q+5|f'(a)|^q}{2}\right]^{\frac{1}{q}}+\left[\frac{5|f'(b)|^q+|f'(a)|^q}{2}\right]^{\frac{1}{q}}\right\}.$$

Corollary 5: Suppose that all the assumptions of Theorem 5 are satisfied. In the inequality (2.9), if we choose $\lambda = \mu$, $r = 1$, $m = 1$ and $q = 1$ we get the following inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{4}A\left(|f'(a)|,|f'(b)|\right)$$

where A is the arithmetic mean.

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