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Some New Fixed Point Theorems of Expanding Mappings in Complete *G*-metric Spaces

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Abstract In this paper, we obtain some fixed point theorems of expanding mappings in *G*-metric spaces. And the existence and uniqueness of the fixed point and common fixed point of some expansive mapping in the complete *G*-metric space are discussed. The results not only directly improve and generalize some fixed point results in *G*-metric spaces, but also expand and complement some previous results in the papers by Asadi, et al. [1] and Lei et al. [2].

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1. Introduction and Preliminaries

Fixed point theory plays a basic role in applications of many branches mathematics. Finding the fixed point of contractive mappings becomes the center of strong research activity [3,4]. In 2007, Mustafa and Sims [3] introduced the notion of *G*-metric and investigated the topology of such spaces. Mustafa [5] provided many examples of *G*-metric spaces and developed some of their properties. Samet et al. [6] and Jleli and Samet [7] reported that some published results can be considered as a straight consequence of the existence theorem in the setting of the usual metric spaces. Asadi et al. [1] stated and proved some fixed point theorems in the framework of *G*-metric space. At the same time, the authors of those papers established some fixed point results for expansive mappings.

The object of this paper is to get some fixed point results in the complete *G*-metric space and some of the results are different from [1].

First, we recollect some necessary definitions and results in this direction. The notion of *G*-metric spaces is defined as follows.

Definition 1.1. *(See [3]) A G-metric space is a pair* (X,G) *where* X *is a nonempty set and* $G: X\times X\times X\to R^+$ *is a function such that, for all* $x,y,z,a\in X$*, the following conditions are fulfilled*:

(G_1) $G(x,y,z)=0$ if $x=y=z$;

(G_2) $G(x,x,y)>0$ for all $x,y\in X$ with $x\neq y$;

(G_3) $G(x,x,y)\le G(x,y,z)$ for all $x,y,z\in X$ with $z\neq y$;

(G_4) $G(x,y,z)=G(x,z,y)=G(y,z,x)=\dots$ (symmetry in all three variables);

(G_5) $G(x,y,z)\le G(x,a,a)+G(a,y,z)$ (rectangle inequality).

Then the function G is called generalized metric or, more specifically, a G-metric in X, and the pair (X,G) is called a G-metric space.

Remark 1.2. Throughout this paper we denote R^+ the set of all positive real numbers and N the set of all natural numbers.

For a better understanding of the subject, we give the following example of G-metric.

Example 1.3. *If* X *is a non-empty subset of* R*, then the function* $G: X\times X\times X\to R^+$*, given by*

$$G(x,y,z)=|x-y|+|x-z|+|y-z| \quad \text{for all } x,y,z\in X,$$

is a G-metric on X.

Example 1.4. Let $X=[0,\infty)$ be the interval of nonnegative real numbers and let G be defined by:

$$G(x,y,z)=\begin{cases}0, & \text{if } x=y=z;\\ max\{x,y\}+max\{y,z\}+max\{x,z\}, & \text{otherwise.}\end{cases}$$

Then G is a complete G-metric on X.

Definition 1.5. *(See [3]) Let* (X,G) *be a* G*-metric space, let* $\{x_n\}$ *be a sequence of points of* X*, a point* $x\in X$ *is said to be the limit of the sequence* $\{x_n\}$ *if* $\lim_{n,m\to\infty} G(x,x_n,x_m)=0$*, and one say that the sequence*

$\{x_n\}$ is G-convergent to x. That is, for any $\varepsilon>0$, there exists $n_0\in N$ such that $G(x,x_n,x_m)<\varepsilon$ for all $n,m\geq n_0$. We call x is the limit of the sequence and write $x_n\to x$ or $\lim_{n\to\infty} x_n = x$.

Proposition 1.6. *(See [3]) Let (X,G) be a G-metric space. The following are equivalent*:

(1) $\{x_n\}$ is G-convergent to x;

(2) $G(x_n,x_n,x)\to 0$ as $n\to\infty$;

(3) $G(x_n,x,x)\to 0$ as $n\to\infty$;

(4) $G(x_n,x_m,x)\to\infty$ as $n,m\to\infty$.

Definition 1.7. *(See [3]) Let (X,G) be a G-metric space. Sequence $\{x_n\}$ is called a G-Cauchy sequence if, for any $\varepsilon>0$, there exists $n_0\in N$ such that $G(x_n,x_m,x_l)<\varepsilon$ for all $n,m,l\geq n_0$, that is $G(x_n,x_m,x_l)\to 0$, as $n,m,l\to\infty$.*

Proposition 1.8. *(See [3]) Let (X,G) be a G-metric space. Then the following are equivalent*:

(1) The sequence $\{x_n\}$ is G-Cauchy;

(2) For any $\varepsilon>0$, there exists $n_0\in N$ such that $G(x_n,x_m,x_m)<\varepsilon$ for all $n,m\geq n_0$.

Definition 1.9. *(See [3]) A G-metric space (X,G) is called G-complete if every G-Cauchy sequence is G-convergent in (X,G).*

The following fixed point theorem for a contractive mapping on G-metric space has proved in [5].

Theorem 1.10. *(See [5]) Let (X,G) be a complete G-metric space and $T:X\to X$ be a mapping satisfying the following condition for all $x,y,z\in X$:*

$$G(Tx,Ty,Tz)\leq kG(x,y,z), \tag{1.1}$$

where $k\in[0,1)$. Then T has a unique fixed point.

Theorem 1.11. *Let (X,G) be a complete G-metric space and $T:X\to X$ be a mapping satisfying the following condition for all $x,y\in X$:*

$$G(Tx,Ty,Ty)\leq kG(x,y,y), \tag{1.2}$$

where $k\in[0,1)$. Then T has a unique fixed point.

Remark 1.12. *We notice that condition (1.1) implies condition (1.2). The converse is true only if $k\in[0,\frac{1}{2})$. For detail see* [5].

Lemma 1.13. *(See [5]) By the rectangle inequality together with the symmetry (G_4), we have*

$$\begin{aligned} &G(x,y,y)=G(y,y,x)\\ &\leq G(y,x,x)+G(x,y,x)=2G(y,x,x). \end{aligned} \tag{1.3}$$

Definition 1.14. A (c)-comparison function is a non-decreasing function $\phi:[0,\infty)\to[0,\infty)$ such that there exist $k_0\in N$, $a\in(0,1)$ and a convergent series of nonnegative terms $\sum_{k\geq 1} v_k$ verifying

$$\phi^{k+1}(t)\leq a\phi^k(t)+v_k \text{ for all } k\geq k_0 \text{ and all } t\geq 0.$$

Let $F_{com}^{(c)}$ denote the family of all (c)-comparison functions. Consider the family

$$F_{Kr}=\{\phi \text{ is continuous and } \phi^{-1}(\{0\})=\{0\}\}.$$

Lemma 1.15. *(See [5]) Let $\{x_n\}$ be a sequence in a G-metric space (X,G) and assume that there exist a function $\varphi\in F_{Kr}$ and $n_0\in N$ such that, at least, one of the following conditions holds:*

(a) $G(x_{n+1},x_{n+2},x_{n+2})\leq\varphi(G(x_n,x_{n+1},x_{n+1}))$ for all $n\geq n_0$;

(b) $G(x_{n+1},x_{n+1},x_{n+2})\leq\varphi(G(x_n,x_n,x_{n+1}))$ for all $n\geq n_0$.

Then $\{x_n\}$ is a Cauchy sequence in (X,G).

If we take $\varphi_\lambda(t)=\lambda t$ for all $t\in[0,\infty)$, where $\lambda\in[0,1)$, then $\varphi_\lambda\in F_{com}^{(c)}$ and the above result can be stated as follows.

Lemma 1.16. *(See [5]) Let $\{x_n\}$ be a sequence in a G-metric space (X,G) and assume that there exist a constant $\lambda\in[0,1)$ and $n_0\in N$ such that , at least, one of the following conditions holds:*

(a) $G(x_{n+1},x_{n+2},x_{n+2})\leq\lambda G(x_n,x_{n+1},x_{n+1})$ for all $n\geq n_0$;

(b) $G(x_{n+1},x_{n+1},x_{n+2})\leq\lambda G(x_n,x_n,x_{n+1})$ for all $n\geq n_0$.

Then $\{x_n\}$ is a Cauchy sequence in (X,G).

Definition 1.17. *(See [8])* A mapping $T:X\to X$ from a G-metric space (X,G) into itself is said to be:

- expansive of type I if there exists $\lambda>1$ such that

$$G(Tx,Ty,Tz)\geq\lambda G(x,y,z) \text{ for all } x,y,z\in X. \tag{1.4}$$

- expansive of type *II* if there exists $\lambda>1$ such that

$$G(Tx,Tx,Ty)\geq\lambda G(x,x,y) \text{ for all } x,y\in X. \tag{1.5}$$

2. Main Results

In this section, we start our work by proving the following theorem:

Theorem 2.1. Let (X,G) be a complete G-metric space and $T:X\to X$ be a onto mapping. Suppose that there exists $\lambda>1$ such that

$$G(Tx,Ty,Ty)\geq\lambda G(x,y,y) \text{ for all } x,y\in X. \tag{2.1}$$

Then T has a unique fixed point.

Proof. Let $x_0\in X$ be arbitrary. Since T is onto, then there exists $x_1\in X$ such that $x_0=Tx_1$. By continuing this process, we get $x_n=Tx_{n+1}$ for all $n\in N$. If there exists some $n_0\in N$ such that $x_{n_0}=x_{n_0+1}$, then x_{n_0+1} is a fixed point of T. Now assume that $x_n\neq x_{n+1}$ for all $n\in N$. For (2.1) with $x=x_{n+1}$ and $y=x_{n+2}$, we have

$$\begin{aligned} G(x_n,x_{n+1},x_{n+1}) &= G(Tx_{n+1},Tx_{n+2},Tx_{n+2})\\ &\geq\lambda G(x_{n+1},x_{n+2},x_{n+2}), \end{aligned}$$

which implies that

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \le \alpha G(x_n, x_{n+1}, x_{n+1}),$$

where $\alpha = \frac{1}{\lambda} < 1$. Form Lemma 1.16, $\{x_n\}$ is a Cauchy sequence. Since, (X,G) is complete, there exists $z \in X$ such that $\{x_n\} \to z$. As T is onto, there exists $w \in X$ such that $z = Tw$. From (2.1) with $x = x_{n+1}$ and $y = w$ we have that, for all $n \in N$,

$$G(x_n, z, z) = G(Tx_{n+1}, Tw, Tw) \ge \lambda G(X_{n+1}, w, w).$$

Taking the limit as $n \to \infty$ in the above inequality we get,

$$G(z, w, w) = \lim_{n\to\infty} G(x_n, z, z) = 0,$$

that is, $z = w$. Then, z is a fixed point of T because $z = Tw = Tz$. We shall show that z is the unique fixed point of T. Suppose, on the contrary, that there exists another fixed point $v \in X$ such that $v = Tv$. If $v \ne z$, then $G(z, v, v) > 0$. From (2.1) and $\lambda > 1$ we have that

$$G(z, v, v) = G(Tz, Tv, Tv) \ge \lambda G(z, v, v),$$

which is a contradiction. Hence, z is the unique fixed point of T.

Remark 2.2. *Condition* (2.1) *was inspired by* (1.4) *in the definition* 1.17. *If z = y, then condition* (2.1) *implies condition* (1.4), *and if x = y, then condition* (2.1) *implies condition* (1.5).

Example 2.3. Let $X = [0, \infty)$ be the interval of nonnegative real numbers and let G the complete G-metric on X defined by

$$G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z; \\ max\{x, y, z\}, & otherwise. \end{cases}$$

Defined $T : X \to X$ *by* $Tx = 2x$ *for all* $x \in X$. *Then, all the hypotheses of Theorem 2.1 hold. In fact,*

$$G(Tx, Ty, Tz) = \begin{cases} 0, & \text{if } x = y = 0; \\ 2max\{x, y\}, & otherwise, \end{cases}$$

and

$$G(x, y, y) = \begin{cases} 0, & \text{if } x = y = 0; \\ max\{x, y\}, & otherwise, \end{cases}$$

Therefore,

$$G(Tx, Ty, Ty) \ge 2G(x, y, y)$$

for all $x, y \in X$. Then T has a unique fixed point on X, which is $z = 0$.

Based on theorem 2.1, the following result considered two nonnegative real numbers and four nonnegative real numbers can be proved.

Theorem 2.4. *Let* (X,G) *be a complete* G*-metric space and* $T : X \to X$ *be a onto mapping. Suppose that there exist nonnegative real numbers a, b, with* $a + b > 1$, *such that, for all* $x, y, z \in X$

$$G(Tx, Ty, Tz) \ge aG(x, Ty, z) + bG(x, Ty, y). \quad (2.2)$$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Since T is onto, then there exists $x_1 \in X$ such that $x_0 = Tx_1$. By continuing this process, we can find a sequence $\{x_n\}$ such that $x_n = Tx_{n+1}$ for all $n \in N$. If there exists some $n_0 \in N$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0+1} is a fixed point of T. Now assume that $x_n \ne x_{n+1}$ for all $n \in N$. For (2.2) with $x = x_n$ and $y = z = x_{n+1}$, we have that, for all $n \ge 1$,

$$\begin{aligned} G(x_{n-1}, x_n, x_n) &= G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\ge G(x_n, Tx_{n+1}, x_{n+1}) + bG(x_n, Tx_{n+1}, x_{n+1}) \\ &= aG(x_n, x_n, x_{n+1}) + bG(x_n, x_n, x_{n+1}) \\ &= (a+b)G(x_n, x_n, x_{n+1}), \end{aligned}$$

which implies that

$$G(x_n, x_n, x_{n+1}) \le kG(x_{n-1}, x_n, x_n),$$

where $k = \frac{1}{a+b} < 1$. Using (G_4), that is,

$$G(x_{n+1}, x_n, x_n) \le kG(x_{n-1}, x_n, x_n)$$

where $k = \frac{1}{a+b} < 1$. Then we have,

$$G(x_{n+1}, x_n, x_n) \le k^n G(x_0, x_1, x_1). \quad (2.3)$$

From Lemma 1.13 we get,

$$G(x_n, x_{n+1}, x_{n+1}) \le 2G(x_{n+1}, x_n, x_n).$$

Then by (2.3), we have

$$G(x_n, x_{n+1}, x_{n+1}) \le 2k^n G(x_0, x_1, x_1).$$

Moreover, for all $n, m \in N$, $n < m$, we have by rectangle inequality that

$$\begin{aligned} &G(x_n, x_m, x_m) \\ &\le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &+G(x_{n+2}, x_{n+3}, x_{n+3}) + \dots + G(x_{m-1}, x_m, x_m) \\ &\le 2\left(k^n + k^{n+1} + k^{n+2} + \dots + k^{m-1}\right) G(x_0, x_1, x_1) \\ &= \frac{2k^n}{1-k} G(x_0, x_1, x_1), \end{aligned}$$

and so, $G(x_n, x_m, x_m) \to 0$, as $n, m \to \infty$. Thus, $\{x_n\}$ is G-Cauchy sequence. Due to (X,G) is complete, there exists $u \in X$ such that $\{x_n\}$ is G-convergent to u, Since T is onto, there exists $w \in X$ such that $u = Tw$. Form (2.2) with $x = x_n$ and $z = y = w$, we have that, for all $n \ge 1$,

$$\begin{aligned} G(x_{n-1}, u, u) &= G(Tx_n, Tw, Tw) \\ &\ge aG(x_n, Tw, w) + bG(x_n, Tw, w) \\ &= (a+b)G(x_n, Tw, w). \end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality we get,

$$G(u,Tw,w)\le \frac{1}{a+b}G(u,u,u)=0.$$

That is, $G(u,Tw,w)=0.$ Then $u=Tw=w.$ So, u is a fixed point of T because $u=Tw=Tu.$ To prove uniqueness, suppose that v is another fixed point of T such that $v=Tv$. If $u\neq v$, again by (2.2), we get

$$\begin{aligned}&G(u,v,v)=G(Tu,Tv,Tv)\\&\ge aG(u,Tv,v)+bG(u,Tv,v)\\&=(a+b)G(u,Tv,v)=(a+b)G(u,v,v),\end{aligned}$$

which is a contradiction. Hence $u=v$. Therefore, T has a unique fixed point.

Theorem 2.5. Let (X,G) be a complete G-metric space and let $T:X\to X$ be a onto mapping. Assume that there exist nonnegative real numbers a, b, c and d, with $a>1$ and $b+c<1,$ such that, for all $x,y,z\in X$,

$$\begin{aligned}&G(Tx,Ty,Tz)\\&\ge aG(x,y,z)+bG(x,Tx,Tx)\\&\quad+cG(y,Ty,Ty)+dG(z,Tz,Tz).\end{aligned}\tag{2.4}$$

Then T has a unique fixed point.

Proof. Let $x_0\in X$, since T is onto, then there exists $x_1\in X$ such that $x_0=Tx_1$. Continuing in this way, we get a sequence $\{x_n\}$ such that $x_n=Tx_{n+1}$ for all $n\in N.$ If there exists some $n_0\in N$ such that $x_{n_0}=x_{n_0+1},$ then x_{n_0+1} is a fixed point of T because $x_{n_0}=x_{n_0+1}=Tx_{n_0+1}$. On the contrary case, assume that $x_n\neq x_{n+1},$ for all $n\in N.$ By taking $x=y=x_n$ and $z=x_{n+1}$ in the (2.4), we have that, for all $n\ge 1$,

$$\begin{aligned}&G(x_{n-1},x_{n-1},x_n)=G(Tx_n,Tx_n,Tx_{n+1})\\&\ge aG(x_n,x_n,x_{n+1})+bG(x_n,Tx_n,Tx_n)\\&+cG(x_n,Tx_n,Tx_n)+dG(x_{n+1},Tx_{n+1},Tx_{n+1}),\end{aligned}$$

which implies that

$$\begin{aligned}&G(x_{n-1},x_{n-1},x_n)\\&\ge aG(x_n,x_n,x_{n+1})+bG(x_n,x_{n-1},x_{n-1})\\&\quad+cG(x_n,x_{n-1},x_{n-1})+dG(x_{n+1},x_n,x_n),\end{aligned}$$

and so,

$$G(x_{n+1},x_n,x_n)\le hG(x_{n-1},x_{n-1},x_n),\tag{2.5}$$

where $h=\dfrac{1-b-c}{a+d}<1$. Proceeding in this way, we get

$$G(x_{n+1},x_n,x_n)\le h^nG(x_0,x_1,x_1).\tag{2.6}$$

From Lemma 1.13 we get,

$$G(x_n,x_{n+1},x_{n+1})\le 2G(x_{n+1},x_n,x_n).$$

Then by (2.6), we have

$$G(x_n,x_{n+1},x_{n+1})\le 2h^nG(x_0,x_1,x_1).$$

Moreover, for all $n,m\in N,$ $n<m,$ we have by rectangle inequality that

$$\begin{aligned}&G(x_n,x_m,x_m)\le G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},x_{n+2},x_{n+2})\\&+G(x_{n+2},x_{n+3},x_{n+3})+\cdots+G(x_{m-1},x_m,x_m)\\&\le 2(h^n+h^{n+1}+h^{n+2}+\cdots+h^{m-1})G(x_0,x_1,x_1)\\&\le\frac{2h^n}{1-h}G(x_0,x_1,x_1).\end{aligned}$$

So, $G(x_n,x_m,x_m)\to 0,$ as $n,m\to\infty$ and $\{x_n\}$ is G-Cauchy sequence. Due to the completeness of $(X,G),$ there exists $u\in X$ such that $\{x_n\}$ is G-convergent to $u.$ As T is onto, there exists $w\in X$ such that $u=Tw$. Form (2.4) with $x=x_{n+1}$ and $y=x_n,$ we have that, for all $n\ge 1,$

$$\begin{aligned}&G(x_n,x_{n-1},u)=G(Tx_{n+1},Tx_n,Tw)\\&\ge aG(x_{n+1},x_n,w)+bG(x_{n+1},Tx_{n+1},Tx_{n+1})\\&\quad+cG(x_n,Tx_n,Tx_n)+dG(w,Tw,Tw),\end{aligned}$$

which implies that

$$\begin{aligned}G(x_n,x_{n-1},u)&\ge aG(x_{n+1},x_n,w)+bG(x_{n+1},x_n,x_n)\\&\quad+cG(x_n,x_{n-1},x_{n-1})+dG(w,u,u).\end{aligned}$$

Taking the limit as $n\to\infty$ in the above inequality we get,

$$G(u,u,w)\le\frac{1-b-c}{a+d}G(u,u,u)=0.$$

So, $G(u,u,w)=0$, then $u=w$. Therefore, u is a fixed point of T because $u=Tw=Tu.$ Suppose there is another fixed point v of T such that $v=Tv.$ If $u\neq v$, again by (2.4), we get

$$\begin{aligned}&G(v,u,u)=G(Tv,Tu,Tu)\\&\ge aG(v,u,u)+bG(v,Tv,Tv)\\&\quad+cG(u,Tu,Tu)+dG(u,Tu,Tu).\end{aligned}$$

Then

$$\begin{aligned}G(v,u,u)&\ge aG(v,u,u)+bG(v,v,v)\\&\quad+cG(u,u,u)+dG(u,u,u).\end{aligned}$$

That is $G(v,u,u)\ge aG(v,u,u),$ which is a contradiction because of $a>1.$ Hence $u=v.$ Therefore, T has a unique fixed point.

Corollary 2.6. *Let (X,G) be a complete G-metric space and let $T:X\to X$ be a onto mapping. Assume that there exist nonnegative real numbers a, b, c and d, with $a>1$ and $b+c<1$, such that, for all $x,y,z\in X$, $m\in N,$*

$$\begin{aligned}&G(T^mx,T^my,T^mz)\\&\ge aG(x,y,z)+bG(x,T^mx,T^mx)\\&\quad+cG(y,T^my,T^my)+dG(z,T^mz,T^mz)\end{aligned}$$

Then T has a unique fixed point.

Proof. From the previous theorem, we see that T^m has a unique fixed point (say u), that is, $T^mu=u$. But $Tu=T(T^mu)=T^{m+1}u=T^m(Tu)$, so \$Tu\$ is another fixed point for T^m and by uniqueness $Tu = u$. Therefore, T has a unique fixed point.

Theorem 2.7. *Let (X,G) be a symmetric complete G-metric space and let $S,T: X \to X$ be two continuous onto mappings. Suppose that there exist nonnegative real numbers a, b, c, d, e with $a+b+c+d+e>1$ and $b+c \le 1$, $d+e \le 1$ such that, for all $x,y,z \in X$,*

$$\begin{aligned} &G(Sx,Ty,Ty) \ge aG(x,y,y)+bG(x,x,Sx) \\ &+cG(x,Sx,Tx)+dG(y,y,Ty)+eG(y,Sy,Ty). \end{aligned} \tag{2.7}$$

Then S and T have a common fixed point; Specially, if $a>1$, then S and T have a unique common fixed point.

Proof. Suppose x_0 is an arbitrary point in X. Since S, T are onto, there exist x_1, x_2 such that $x_0 = Tx_1$, $x_1 = Sx_2$. Continuing this process, we can define $\{x_n\}$ by $x_{2n} = Tx_{2n+1}$, $x_{2n+1} = Sx_{2n+2}$, for all $n \in N$. By (2.7), we have

$$\begin{aligned} &G(x_{2n+1},x_{2n},x_{2n})=G(Sx_{2n+2},Tx_{2n+1},Tx_{2n+1}) \\ &\ge aG(x_{2n+2},x_{2n+1},x_{2n+1})+bG(x_{2n+2},x_{2n+2},Sx_{2n+2}) \\ &+cG(x_{2n+2},Sx_{2n+2},Tx_{2n+2})+dG(x_{2n+1},x_{2n+1},Tx_{2n+1}) \\ &+eG(x_{2n+1},Sx_{2n+1},Tx_{2n+1}) \\ &= aG(x_{2n+2},x_{2n+1},x_{2n+1})+bG(x_{2n+2},x_{2n+2},x_{2n+1}) \\ &+cG(x_{2n+2},x_{2n+1},x_{2n+1})+dG(x_{2n+1},x_{2n+1},x_{2n}) \\ &+eG(x_{2n+1},x_{2n},x_{2n}). \end{aligned}$$

Apply to the symmetric of (X,G), we have

$$\begin{aligned} &G(x_{2n+1},x_{2n+1},x_{2n}) \\ &\ge aG(x_{2n+2},x_{2n+2},x_{2n+1})+bG(x_{2n+2},x_{2n+2},x_{2n+1}) \\ &+cG(x_{2n+2},x_{2n+2},x_{2n+1})+dG(x_{2n+1},x_{2n+1},x_{2n}) \\ &+eG(x_{2n+1},x_{2n+1},x_{2n}), \end{aligned}$$

which implies that

$$\begin{aligned} &(1-d-e)G(x_{2n},x_{2n+1},x_{2n+1}) \\ &\ge (a+b+c)G(x_{2n+1},x_{2n+2},x_{2n+2}). \end{aligned} \tag{2.8}$$

Similarly, it can be shown that

$$\begin{aligned} &G(x_{2n-1},x_{2n},x_{2n}) \\ &=G(Sx_{2n},Tx_{2n+1},Tx_{2n+1}) \\ &\ge aG(x_{2n},x_{2n+1},x_{2n+1})+bG(x_{2n},x_{2n},Sx_{2n}) \\ &+cG(x_{2n},Sx_{2n},Tx_{2n})+dG(x_{2n+1},x_{2n+1},Tx_{2n+1}) \\ &+eG(x_{2n+1},Sx_{2n+1},Tx_{2n+1}) \\ &= aG(x_{2n},x_{2n+1},x_{2n+1})+bG(x_{2n},x_{2n},x_{2n-1}) \\ &+cG(x_{2n},x_{2n-1},x_{2n-1}) \\ &+dG(x_{2n+1},x_{2n+1},x_{2n})+eG(x_{2n+1},x_{2n},x_{2n}) \\ &= aG(x_{2n},x_{2n+1},x_{2n+1})+bG(x_{2n},x_{2n},x_{2n-1}) \\ &+cG(x_{2n},x_{2n},x_{2n-1}) \\ &+dG(x_{2n+1},x_{2n+1},x_{2n})+eG(x_{2n+1},x_{2n+1},x_{2n}), \end{aligned}$$

which implies that

$$\begin{aligned} &(1-b-c)G(x_{2n-1},x_{2n},x_{2n}) \\ &\ge (a+d+e)G(x_{2n},x_{2n+1},x_{2n+1}). \end{aligned} \tag{2.9}$$

Let $P = \dfrac{1-d-e}{a+b+c}$, $Q = \dfrac{1-b-c}{a+d+e}$. From $a+b+c+d+e>1$ and $b+c \le 1$, $d+e \le 1$, we know $a+b+c > 1-d-e \ge 0$ and $a+d+e > 1-b-c \ge 0$. Thus, let $h = max\{P,Q\}$, then $h \in [0,1)$. So, from (2.8) and (2.9), for all $n \in N$, we get

$$G(x_n,x_{n+1},x_{n+1}) \le hG(x_{n-1},x_n,x_n).$$

Hence, for $n \in N$ it follows that

$$\begin{aligned} &G(x_n,x_{n+1},x_{n+1}) \le hG(x_{n-1},x_n,x_n) \\ &\le h^2G(x_{n-2},x_{n-1},x_{n-1}) \le \cdots \le h^nG(x_0,x_1,x_1). \end{aligned}$$

Moreover, for all $n,m \in N$, $n<m$, we have by rectangle inequality that

$$\begin{aligned} &G(x_n,x_m,x_m) \\ &\le G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},x_{n+2},x_{n+2}) \\ &+G(x_{n+2},x_{n+3},x_{n+3})+\cdots+G(x_{m-1},x_m,x_m) \\ &\le (h^n+h^{n+1}+h^{n+2}+\cdots+h^{m-1})G(x_0,x_1,x_1) \\ &\le \frac{h^n}{1-h}G(x_0,x_1,x_1). \end{aligned}$$

So, $G(x_n,x_m,x_m) \to 0$, as $n,m \to \infty$ and $\{x_n\}$ is G-Cauchy sequence. Due to the completeness of (X,G), there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. It's equivalent to $x_{2n} = Sx_{2n+1} \to u$, $x_{2n+1} = Tx_{2n+2} \to u$ as $n \to \infty$. Since S, T are continuous, then we have $u = Su$ and $u = Tu$, that is, $u = Su = Tu$. Therefore, u is a common fixed point of S and T.

If $a>1$, assume that v is another common fixed point of S and T, then we hav

$$\begin{aligned} &G(v,u,u) \\ &= G(Sv,Tu,Tu) \\ &\ge aG(v,u,u)+bG(v,v,Sv) \\ &+cG(v,Sv,Tv)+dG(u,u,Tu)+eG(u,Su,Tu) \\ &= aG(v,u,u)+bG(v,v,v)+cG(v,v,v) \\ &+dG(u,u,u)+eG(u,u,u) \\ &= aG(v,u,u), \end{aligned}$$

so $G(v,u,u)=0$, that is, $v=u$. Therefore, when $a>1$, S and T have a unique common fixed point.

Remark 2.8. *Theorem 2.7 of this paper extends Theorem 1 of [2] from metric spaces to G-metric spaces, but we add to the continuity of the mappings.*

Corollary 2.9. *Let (X,G) be a symmetric complete G-metric space and let $S,T: X \to X$ be two continuous onto mappings. Suppose that there exist nonnegative real numbers α,β,γ with $\alpha+2\beta+2\gamma>1$ and $2\beta \le 1$, $2\gamma \le 1$ such that, for all $x,y,z \in X$,*

$$G(Sx,Ty,Ty) \geq \alpha G(x,y,y)+\beta[G(x,x,Sx)+G(x,Sx,Tx)] + \gamma[G(y,y,Ty)+G(y,Sy,Ty)].$$

Then S and T have a common fixed point.

Corollary 2.10. *Let* (X,G) *be a complete* G *-metric space and let* $S,T: X \to X$ *be two onto mappings. Suppose that there exists* $k>1$ *such that, for all* $x,y,z \in X$,

$$G(Sx,Ty,Ty) \geq kG(x,y,y).$$

Then S and T have a unique common fixed point.

Corollary 2.11. *Let* (X,G) *be a complete* G *-metric space and let* $T: X \to X$ *is onto mappings. Suppose that there exist p, q are positive integers and* $k>1$ *such that, for all* $x,y,z \in X$,

$$G(T^p x, T^q y, T^q y) \geq kG(x,y,y).$$

Then T *has a unique common fixed point.*

Proof. Let $S=T^p$, $T=T^q$. Since T is an onto mapping, then $S=T^p$, $T=T^q$ are onto mappings, the conditions of Corollary 2.10 are satisfied.

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