Turkish Journal of Analysis and Number Theory, 2018, Vol. 6, No. 5, 149-151
Available online at http://pubs.sciepub.com/tjant/6/5/5

DOI:10.12691/tjant-6-5-5

Transitivity and Minimality of Sets

Dana Mawlood Mohammed*

Institute of Training and Educational Development, Sulaimani
*Corresponding author: Danamath82@gmail.com

Received August 11, 2018; Revised September 15, 2018; Accepted October 22, 2018

Abstract In this paper, we have study some concepts of minimal open, closed sets and minimal functions. Further, we have shown that these properties preserved under conjugate maps.

Keywords: *minimal open sets, minimal functions, transitivity*

Cite This Article: Dana Mawlood Mohammed, "Transitivity and Minimality of Sets." *Turkish Journal of Analysis and Number Theory*, vol. 6, no. 5 (2018): 149-151. doi: 10.12691/tjant-6-5-5.

1. Introduction

Let (X,τ) be a compact topological space. All maps under consideration are supposed to be continuous. The set of all continuous maps f: X → X is denoted by C(X). By a system (X,f), we mean a compact topological space (phase space) X and $f \in C(X)$. In a topological space a trajectory consists of a sequence of points $(x, f(x), f^2(x),...)$, and can possibly contain additional attributes a measured at each point. Trajectories can be generated by moving objects but also by moving phenomena, e.g. measurement points on a hill slide. The pointscan be captured at regular intervals or irregularly A point $x \in X$ "moves," its trajectory [1] being the sequence $x, f(x), f^2(x),...,$ where f^n is the nth iteration of f. The point $f^n(x)$ is the position of the point x after n units of time. The set of points of the trajectory of x under f is called the orbit of x, denoted by $O_f(x)$. A map f ∈ C(X) is (topologically) transitive if for any two nonempty open sets U and V in X, there is a nonnegative integer n such that $f^n(U) \cap V \neq \varphi$. If X has no isolated points then this definition is equivalent to the existence of a dense orbit, i.e. $Cl(O_f(x)) = X$. If every orbit of f is dense, the map f is called minimal. Denote by $T(X)$ the set of transitive self-maps of the space X. A minimal map f is necessarily surjective if X is assumed to be Hausdorff and compact.

Now, to study the existence of minimal sets, given a system (X,f), a set $A \subseteq X$ is called a *minimal set* if it is non-empty, closed and invariant and if no proper subset of A has these three properties. So, $A \subseteq X$ is a minimal set if and only if $(A, f|A)$ is a minimal system. A system (X,f) is minimal if and only if X is a minimal set in (X,f). The basic fact discovered by G. D. Birkhoff is that in any compact system (X,f) there are minimal sets. This follows immediately from the Zorn's lemma. Since any orbit closure is invariant, we get that *any compact orbit closure contains a minimal set*. This is how compact minimal sets may appear in non-compact spaces. Two minimal sets in (X,f) either are disjoint or coincide. A minimal set A is strongly $f-invariant$, i.e. $f(A) = A$. Provided it is compact Hausdorff

2. Preliminaries and Definitions

Definition 2.1

1. (Minimal Hausdorff spaces) [3]

A topological space (X,τ) is said to be minimal Hausdorff if τ is Hausdorff and there exists no Hausdorff topology on X strictly weaker than τ. Thus this minimality property is topological.

2. Two topological spaces (X,τ) and (Y,τ_1) are called homeomorphic [5] if there exists a one-to-one onto function $f:(X,\tau) \to (Y,\tau_1)$ such that f and f^{-1} are both continuous.

3. Two topological systems $f: X \to X$ and $g: Y \to Y$ are said to be topologically conjugate if there is a homeomorphism $h: X \to Y$ such that $h \circ f = g \circ h$. We will call h topological Conjugacy. Thus, the two topological systems with their respective function acting on them share the same dynamics(see the following diagram)

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h\downarrow & & \downarrow h \\ Y & \xrightarrow[g]{} & Y \end{array}$$

Definition 2.2 (minimal open set). Recall that a proper non empty open subset U of a topological space X is said to be a minimal open set, if any open set which is contained in U is empty set or U.

Definition 2.3 Recall that a proper non empty closed subset F of a topological space X is said to be a minimal closed set if any closed set which is contained in F is empty set or F.

Definition 2.4

A system (X, f) is called *minimal* if X does not contain any non-empty, proper, closed $f-inveriant$ subset. In such a case we also say that the map f itself is minimal. Thus, one cannot simplify the study of the dynamics of a minimal system by finding its non- trivial closed subsystems and studying first the dynamics restricted to them.

Proposition 2.5

Let $f: X \to X$ be continuous function. The following are equivalent:

1. f is minimal.
2. The only closed invariant sets of X are X itself and the empty set.
3. For any non-empty open subset $U \subset X$, then $X = \bigcup_{k=0}^{\infty} f^{-k}(U)$.

Proof:

(1) $\Rightarrow$ (2)

Suppose that $C \subset X$ is a non-empty closed invariant set. Let $x \in C$. Then since C is invariant, $O_f(x) \subset C$. Since C is closed, so $Cl(O_f(x)) \subset C$, but $X = Cl(O_f(x))$ since the orbit $O_f(x)$ is dense, this means $X \subset C$. Thus X=C.

(2) $\Rightarrow$ (3), let $U \subset X$ be a non-empty open set. Put $C = X \setminus \bigcup_{k=0}^{\infty} f^{-k}(U)$ then C is closed and invariant. Since $C \neq X$, by (2) we must have $C = \varphi$.

(3) $\Rightarrow$ (1), let $x \in X$ and let U be an arbitrary non-empty open subset. Then by (3), $x \in f^{-k}(U)$ for some $k \geq 0$. Thus $f^k(x) \in U$, and hence $O_f(x) \cap U \neq \varphi$. Since U was arbitrary, $O_f(x)$ is dense, i.e. f is minimal.

Definition 2.6 (minimal) Let X be a topological space

And f be continuous map on X. Then (X, f) is called minimal system (or f is called minimal map on X) if one of the three equivalent conditions hold:

(1) The orbit of each point in X is dense in X

(2) $Cl(O_f(x)) = X$ for each x $\in$ X.

(3) Given x $\in$ X and a nonempty open U in X, there exists $n \in \mathbf{N}$ such that $f^n(x) \in U$.

Definition 2.7 A subset M of X is said to be minimal under provided that M is non-empty, closed and invariant, that is $f(M) \subset M$, and no proper subset of M has all these properties.

Theorem 2.8 [4] Any two minimal sets must have empty intersection.

Proof: Let M_1 *and* M_2 be two distinct minimal sets, and suppose that $A = M_1 \cap M_2 \neq \varphi$. Then A is closed, and fore very a ∈ A and every n ∈ N, $f^n(a) \in M_1 \cap M_2$, so A is invariant. But then A is a proper subset of both M_1 and M_2 which is closed, invariant and non-empty, contradicting the fact that M_1 *and* M_2 are minimal.

Definition 2.9 Two topological systems $f: X \to X$ and $g: Y \to Y$ are said to be topologically conjugate if there is a homeomorphism $h: X \to Y$ such that $h \circ f = g \circ h$. We will call h a topological Conjugacy.

We have stated a new proposition as follows:

Proposition 2.10 if $f: X \to X$ *and* $g: Y \to Y$ are topologically conjugated by $h: X \to Y$. Then A is a minimal open set in X if and only if $h(A)$ is a minimal open set in Y.

Definition 2.11 Let X be a topological space, and $f: X \to X$ a continuous map. We say f is (topologically) transitive if for any nonempty open sets $U.V \subset X$ there exists n > 0 such that $V \cap f^n(U) \neq \varphi$. We say f is strongly transitive [3] if for any nonempty open set $U \subset X$, $X = \bigcup_{k=0}^{s} f^k(U)$ for some s > 0. For more knowledge see [4].

Definition 2.12 A subset S of X is called $\wedge$-set if it is the intersection of open sets containing S.

Definition 2.13 Recall that a subset of a topological space (X, τ) is called λ-closed set if $A = S \cap C$ where S is $\wedge$-set and C is a closed set.

Proposition 2.14 Let (X, τ) be topological space and A be a nonempty λ-closed $f-$invariant set of X. Then A is a λ-type transitive set of (X, f) if and only if (A, f) is λ-type transitive.

Proof:

$\Rightarrow$) Let V_1 *and* U_1 be two nonempty λ-open subsets of A. For a nonempty λ-open subset U_1 of A, there exists a λ- open set U of X such that $U_1 = U \cap A$ Since A is a λ-type transitive set of (X, f), there exists n ∈ N such that $f(V_1) \cap U \neq \varphi$. Moreover, A is invariant, i.e., $f(A) \subset A$, which implies that $f(A) \subset A$. Therefore, $f(V_1) \cap A \cap U \neq \varphi$, i.e. $f(V_1) \cap U_1 \neq \varphi$. This shows that (A, f) is λ- type transitive.

$\Leftarrow$) Let V_1 be a nonempty λ-open set of A and U be a nonempty λ-open set of X with $A \cap U \neq \varphi$, Since U is an λ-open set of X and $A \cap U \neq \varphi$, , it follows that U ∩ A is a nonempty λ-open set of A. Since (A, f) is topologically λ-type transitive, there exists n ∈ N such that $f(V_1) \cap (A \cap U) \neq \varphi$, which implies that $f(V_1) \cap U \neq \varphi$.. This shows that A is a λ-type transitive set of (X, f).

Theorem 2.15 Let X be a non-empty λ-compact Hausdorff space. Then the intersection of a countable collection of λ-open λ-dense subsets of X is λ -dense in X.

Definition 2.16 Let (X,τ) be a topological space. Recall that subset A of X is called λ -dense in X if λCl(A)=X.

Corollary 2.17 A subset A of a space (X,τ) is λ -dense if and only if $A \cap U \neq \varphi$ for all $U \in \tau^{\lambda}$ other than $U = \varphi$.

Proof: If A is λ -dense set in X, then by definition, $\lambda Cl(A) = X$, and let U be a non-empty λ-open set in X. Suppose that A∩U=ϕ. Therefore $B = U^c$ is λ-closed and $A \subset U^C = B$. So, $\lambda Cl(A) \subset \lambda Cl(B)$, i.e. $\lambda Cl(A) \subset B$, but $\lambda Cl(A) = X$, so X ⊂ B, this contradicts U ≠ φ.

Theorem 2.18 Let (X,τ) be a non-empty λ-compact Hausdorff space and $f : X \to X$ is λ -irresolute map and that X is λ-type separable. Suppose that f is topologically λ -type transitive. Then there is an element $x \in X$ such that the orbit $O_f(x) = \{x, f(x), f^2(x), ...f^n(x),.....\}$ is λ -dense in X.

Proof: Let $B = \{U_i\}$ $i = 1, 2, 3, ...$ be a countable basis for the λ -topology of X. For each i, let $O_i = \{x \in X : f^n(x) \in U_i \text{ for some } n \geq 0\}$

Then, clearly O_i is λ -open and λ -dense. It is λ -open since f is λ -irresolute, so, $O_i = \bigcup_{i=1}^{\infty} f^{-1}(U_i)$ is λ - open and λ -dense since f is topological λ- transitive map. Further, for every λ -open set V, there is n>0, such that $f^n(V) \cap U_i \neq \varphi,$ since f is λ- transitive.

Now, apply theorem 2.15 to the countable λ -dense set $\{O_i\}$ to say that $\bigcap_{i=0}^{\infty} O_i$ is λ -dense and so non-empty. Let $y \in \bigcap_{i=0}^{\infty} O_i$. This means that, for each i, there is a positive integer n such that $f^n(y) \in U_i$ for each i. By corollary 2.17, this implies that $O_f(x)$ is λ -dense in *X*.

References

[1] Bruckner A. M., Hu Thakyin On scrambled sets for chaotic functions,Trans. Amer. Math. Soc. 301.1 (1987), 289-297.

[2] Anuradha N[1] and Baby Chacko[2], On Minimal Regular Open Sets and Maps in Topological Spaces, Journal of Computer and Mathematical Sciences, *Vol.6(4),* April 2015. Pp. 182-192.

[3] A. KAMEYAMA, TOPOLOGICAL TRANSITIVITY AND STRONG TRANSITIVITY, Acta Math. Univ. Comenianae Vol. LXXI, 2(2002), pp. 139-145.

[4] Mohammed Nokhas Murad Kaki, Introduction to Topological Dynamical Systems II, Lambert academic publisher, Germany.

[5] P.S. Aleksandrov, H. HopfTopologieSpringer Verlag, Berlin (1935).