

Notes on a Double Inequality for Ratios of any Two Neighbouring Non-zero Bernoulli Numbers

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Abstract In the paper, the author notes on a double inequality published in “Feng Qi, *A double inequality for the ratio of two non-zero neighbouring Bernoulli numbers*, Journal of Computational and Applied Mathematics 351 (2019), 1-5; Available online at <https://doi.org/10.1016/j.cam.2018.10.049>.”

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1. Introduction

We recall from [[1], p. 804, 23.1.1] and [[2], p. 3, (1.1)] that the Bernoulli numbers B_n can be generated by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!}$$

for $|z| < 2\pi$. It is easy to verify that the function

$$\frac{x}{e^x - 1} - 1 + \frac{x}{2}$$

is even in $x \in \mathbb{R}$. Consequently, all the Bernoulli numbers B_{2k+1} for $k \in \mathbb{N}$ equal 0.

To discover explicit formulas, recurrent formulas, closed expressions, and integral representations of the Bernoulli numbers B_{2k} for $k \in \mathbb{N}$ is a classical topic. For recently published results, please refer to the papers [3-12] and closely related references therein.

To bound the Bernoulli numbers B_{2k} for $k \in \mathbb{N}$ by inequalities is an alternative topic. In [[1], p. 805, 23.1.15], [[13], Theorem 1.1], [[2], p. 14, (1.23) and p. 23, Exercise 1.2], and the papers [14,15,16], some inequalities for bounding the Bernoulli numbers B_{2k} were established and collected. Most of these inequalities have been refined or sharpened in [17] by the double inequality

$$\frac{2(2k)!}{(2\pi)^{2k}} \frac{1}{1-2^{\alpha-2k}} \leq B_{2k} \leq \frac{2(2k)!}{(2\pi)^{2k}} \frac{1}{1-2^{\beta-2k}} \quad (1)$$

for $k \in \mathbb{N}$, where $\alpha = 0$ and

$$\beta = 2 + \frac{\ln(1-6/\pi^2)}{\ln 2} = 0.649\dots$$

are the best possible in the sense that they can not be replaced respectively by any bigger and smaller constants in the double inequality (1).

To study the differences $|B_{2k+2}| - |B_{2k}|$ and the ratios $\frac{|B_{2k+2}|}{|B_{2k}|}$ for $k \in \mathbb{N}$ is also an interesting topic. In

the newly published paper [18], the ratios $\frac{|B_{2k+2}|}{|B_{2k}|}$

for $k \in \mathbb{N}$, which is equivalent to the differences $\ln |B_{2k+2}| - \ln |B_{2k}|$, were bounded by the double inequality

$$\begin{aligned} \frac{2^{2k-1} - 1}{2^{2k+1} - 1} \frac{(2k+1)(2k+2)}{\pi^2} &< \frac{|B_{2k+2}|}{|B_{2k}|} \\ &< \frac{2^{2k} - 1}{2^{2k+2} - 1} \frac{(2k+1)(2k+2)}{\pi^2}. \end{aligned} \quad (2)$$

Motivated by the double inequality (2) and by the fact that the function $\frac{2^{2k+x} - 1}{2^{2k+2+x} - 1}$ is strictly increasing in $x \neq -2(k+1)$ for all $k \in \mathbb{N}$, we naturally pose a problem: what are the best constants α and β such that the double inequality

$$\begin{aligned} \frac{2^{2k+\beta} - 1}{2^{2k+2+\beta} - 1} \frac{(2k+1)(2k+2)}{\pi^2} &< \frac{|B_{2k+2}|}{|B_{2k}|} \\ &< \frac{2^{2k+\alpha} - 1}{2^{2k+2+\alpha} - 1} \frac{(2k+1)(2k+2)}{\pi^2}. \end{aligned} \quad (3)$$

is valid for all $k \in \mathbb{N}$?

In [2], p. 5, (1.14)], it was listed that

$$B_{2k} = \frac{(-1)^{k+1} 2(2k)!}{(2\pi)^{2k}} \zeta(2k), \quad k \in \mathbb{N},$$

where the Riemann zeta function ζ can be defined

[19,20,21] by the series $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ under the condition

$\Re(z) > 1$ and by analytic continuation elsewhere.

$$\frac{|B_{2k+2}|}{|B_{2k}|} = \frac{(2k+1)(2k+2)}{\pi^2} \frac{1}{4} \frac{\zeta(2k+2)}{\zeta(2k)} \quad (4)$$

for $k \in \mathbb{N}$. By virtue of (4), the double inequality (3) can be rewritten as

$$\frac{2^{2k+\beta} - 1}{2^{2k+2+\beta} - 1} < \frac{1}{4} \frac{\zeta(2k+2)}{\zeta(2k)} < \frac{2^{2k+\alpha} - 1}{2^{2k+2+\alpha} - 1} \quad (5)$$

which can be further reformulated as

$$\left(1 - \frac{1}{2^{2k+\beta}}\right) \zeta(2k) < \left(1 - \frac{1}{2^{2k+2+\beta}}\right) \zeta(2k+2)$$

and

$$\left(1 - \frac{1}{2^{2k+2+\alpha}}\right) \zeta(2k+2) < \left(1 - \frac{1}{2^{2k+\alpha}}\right) \zeta(2k).$$

Let

$$S_{\theta}(x) \triangleq \left(1 - \frac{1}{2^{2x+\theta}}\right) \zeta(2x), \quad x \in [1, \infty), \quad \theta \in \mathbb{R}.$$

Then

$$S_{\theta}'(x) = \frac{1}{2^{2x+\theta-1}} \left[(2^{2x+\theta} - 1) \zeta'(2x) + (\ln 2) \zeta(2x) \right].$$

In order that the function $S_{\theta}(x)$ is strictly increasing (or strictly decreasing, respectively) on $[1, \infty)$, it is necessary and sufficient that

$$(2^{2x+\theta} - 1) \zeta'(2x) + (\ln 2) \zeta(2x) \geq 0$$

on $[1, \infty)$, which can be rearranged as

$$2^{\theta} \begin{cases} \leq \left[1 - \frac{(\ln 2) \zeta(2x)}{\zeta'(2x)} \right] \frac{1}{2^{2x}} \\ \geq 1, & x \rightarrow \infty; \\ \rightarrow \left\{ \frac{1}{4} - \frac{\pi^2 \ln 2}{24 \zeta'(2)} = 0.55\dots, x \rightarrow 1^+. \right. \end{cases}$$

Consequently, in order that the function $S_{\theta}(x)$ for $x \in [1, \infty)$ and the sequence $S_{\theta}(k)$ with $k \in \mathbb{N}$ are strictly increasing (or strictly decreasing, respectively), it is necessary that $\theta \geq 0$ (or

$$\theta \leq \frac{\ln \left[\frac{1}{4} - \frac{\pi^2 \ln 2}{24 \zeta'(2)} \right]}{\ln 2} = -0.85\dots,$$

respectively). The double inequality (5) can also be reformulated as

$$2^{2+\beta} < \frac{1}{2^{2k}} \frac{4 - \zeta(2k+2) / \zeta(2k)}{1 - \zeta(2k+2) / \zeta(2k)}$$

and

$$2^{2+\alpha} > \frac{1}{2^{2k}} \frac{4 - \zeta(2k+2) / \zeta(2k)}{1 - \zeta(2k+2) / \zeta(2k)}.$$

Since

$$\frac{1}{2^{2k}} \frac{4 - \zeta(2k+2) / \zeta(2k)}{1 - \zeta(2k+2) / \zeta(2k)} \rightarrow \begin{cases} \frac{\pi^2 - 60}{4(\pi^2 - 15)} = 2.44\dots, & k \rightarrow 1, \\ 4, & k \rightarrow \infty, \end{cases}$$

It follows that the necessary conditions are $\alpha \geq 0$ and

$$\beta \leq \frac{\ln \frac{\pi^2 - 60}{\pi^2 - 15}}{\ln 2} - 4 = -0.711\dots$$

This implies that the right-hand side inequality in (2) is sharp, but the left-hand side inequality in (2) perhaps can be improved. In conclusion, we guess that the double inequality (3) is valid if and only if $\alpha \geq 0$ and

$$\beta \leq \frac{\ln \left[\frac{1}{4} - \frac{\pi^2 \ln 2}{24 \zeta'(2)} \right]}{\ln 2} = -0.85\dots$$

Since

$$\lim_{x \rightarrow 1^+} \left\{ \left[1 - (\ln 2) \frac{\zeta(x)}{\zeta'(x)} \right] \frac{1}{2^x} \right\} = \frac{1}{2}$$

and

$$\lim_{x \rightarrow \infty} \left\{ \left[1 - (\ln 2) \frac{\zeta(x)}{\zeta'(x)} \right] \frac{1}{2^x} \right\} = 1,$$

we guess that the function

$$\left(1 - \frac{1}{2^{x+\theta}}\right) \zeta(x), \quad x \in (1, \infty)$$

is strictly increasing (or strictly decreasing, respectively) if and only if $\theta \leq -1$ (or $\theta \geq 0$, respectively).

The double inequality (2) has been cited and applied in the papers [22-29].

Can one generalize the inequality (2) to the case for the Bernoulli polynomials?

This paper and [18] are respectively extracted from the preprints [30,31,32,33].

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