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On New Trapezoid Type Inequalities for *h*-convex Functions via Generalized Fractional Integral

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Abstract In this paper, some new inequalities of the trapezoid type for h-convex functions via generalized fractional integral are given. The results also provide new estimates on these types of trapezoid inequalities for Riemann-Liouville type fractional operators.

Keywords: *Riemann-Liouville fractional integral, h-convex function, Convex functions, integral inequalities*

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1. Introduction

First, we recall some necessary definitions and mathematical preliminaries of the generalized fractional integrals which are defined by Sarikaya and Ertugral [1].

Let which satisfies the following condition:

$$\int_0^1 \frac{\varphi(t)}{t} dt < \infty.$$

We define the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$$_{a^+}I_\varphi f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t)dt, \quad x > a, \qquad (1.1)$$

$$_{b^-}I_\varphi f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} f(t)dt, \quad x < b. \qquad (1.2)$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integral, *k*-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc. These important special cases of the integral operators (1.1) and (1.2) are mentioned below.

a) If we take $\varphi(t) = t$, the operator (1.1) and (1.2) reduce to the Riemann integral as follows:

$$I_{a^+} f(x) = \int_a^x f(t)dt, \, x > a,$$

$$I_{b^-} f(x) = \int_x^b f(t)dt, \, x < b.$$

b) If we take $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, the operator (1.1) and (1.2) reduce to the Riemann-Liouville fractional integral as follows:

$$I_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

$$I_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b.$$

c) If we take $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ the operator (1.1) and (1.2) reduce to the k-Riemann-Liouville fractional integral as follows:

$$I_{a^+,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a,$$

$$I_{b^-,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad x < b.$$

where

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt, \, \Re\mathfrak{e}(\alpha) > 0$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1} \Gamma\left(\frac{\alpha}{k}\right), \quad \Re\mathfrak{e}(\alpha) > 0; k > 0$$

are given by Mubeen and Habibullah in [2].

Recently, in [1], Sarikaya and Ertugral established the following Trapezoid inequalities for generalized fractional integrals:

Theorem 1.1. *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a < b$. *If* $|f'|$ *is convex on* $[a,b]$. *then the following inequality for generalized fractional integrals holds*:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_\varphi f(b)+{}_{b^-}I_\varphi f(a)\right]\right|$$
$$\leq \frac{b-a}{\Lambda(1)}\int_0^1 t\,|\Lambda(1-t)-\Lambda(t)|\,dt\left(\frac{|f'(a)|+|f'(b)|}{2}\right),$$

where

$$\Lambda(t)=\int_0^t \frac{\varphi((b-a)u)}{u}du<\infty.$$

Theorem 1.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is convex on $[a,b]$, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality for generalized fractional integrals holds*:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_\varphi f(b)+{}_{b^-}I_\varphi f(a)\right]\right|$$
$$\leq \frac{b-a}{2\Lambda(1)}\left(\int_0^1|\Lambda(1-t)-\Lambda(t)|^p\,dt\right)^{\frac{1}{p}}$$
$$\times\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Recently, in [3], Ertugral and Sarikaya established the following Trapezoid inequalities for generalized fractional integrals:

Theorem 1.3. *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be an absolutely continuous mapping on I^0 such that $f'\in L_1([a,b])$, where $a,b\in I^0$ with $a<b$. If the mapping $|f'|$ is*

$$\left|\frac{\nabla(0)f(b)+\Delta(0)f(a)}{b-a}-\frac{1}{b-a}\left[\begin{array}{c}{}_{a^+}I_\varphi f(b)\\ +{}_{b^-}I_\varphi f(a)\end{array}\right]\right|$$
$$\leq\frac{b-x}{b-a}|f'(x)|\int_0^1|\nabla(t)|\,tdt+\frac{x-a}{b-a}|f'(x)|\int_0^1|\Delta(t)|\,tdt$$
$$+\frac{b-x}{b-a}|f'(b)|\int_0^1|\nabla(t)|(1-t)dt$$
$$+\frac{x-a}{b-a}|f'(a)|\int_0^1|\Delta(t)|(1-t)dt.$$

Theorem 1.4. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If the mapping $|f'|^q, q>1$*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_\varphi f(b)+{}_{b^-}I_\varphi f(a)\right]\right|$$
$$\leq\frac{b-x}{b-a}\left(\int_0^1|\nabla(t)|^p\,dt\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}$$
$$+\frac{b-x}{b-a}\left(\int_0^1|\Delta(t)|^p\,dt\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

In [4], Varošanec introduced the following class of functions.

Definition 1.1. Let $f:J\subseteq\mathbb{R}\to\mathbb{R}$ be a positive function. We say that $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is h-convex, or that f belongs to the class $SX(h,I)$, if f is nonnegative and for all $x,y\in I$ and $t\in(0,1)$ we have

$$f(tx+(1-t)y)\leq h(t)f(x)+h(1-t)f(y). \tag{1.3}$$

If inequality (1.3) is reversed, then f is said to be h-concave, i.e. $f\in SV(h,I)$.

The systematic study of h-convex functions with their various applications has been given by many authors, see [6-10].

In this paper, we establish some trapezoid type inequalities via generalized fractional integrals for h-convex functions.

2. New Integral Inequalities via *P*-preinvexity

For our results, we need the following important fractional integrtal identity [3]:

Lemma 2.1. *Let $f:I\to\mathbb{R}$ be an absolutely continuous mapping on I^o such that $f'\in L_1([a,b])$, where $a,b\in I^0$ with $a<b$. Then the following equality holds*:

$$\frac{\nabla(0)f(b)+\Delta(0)f(a)}{b-a}-\frac{1}{b-a}[x^+I_\varphi f(b)+x^-I_\varphi f(a)]$$
$$=\frac{b-x}{b-a}\int_0^1\nabla(t)f'(tx+(1-t)b)dt, \tag{2.1}$$

where

$$\Delta(t)=\int_t^1\frac{\varphi((x-a)u)}{u}du<\infty$$

and

$$\nabla()=\int_t^1\frac{\varphi((b-x)u)}{u}du<\infty.$$

Theorem 2.1. Let $f:I=[a,b]\subset R\to R$ be an absolutely continuous mapping on I^0 such that $f'\in L_1([a,b])$, where $a,b\in I^0$ with $a<b$. If the mapping $|f'|$ is h-convex on $[a,b]$, then we have the following inequality

$$\left|\frac{\nabla(0)f(b)+\Delta(0)f(a)}{b-a}-\frac{1}{b-a}\left[\begin{array}{c}{}_x^+I_\varphi f(b)\\ +{}_x^-I_\varphi f(a)\end{array}\right]\right|$$
$$\leq\frac{b-x}{b-a}|f'(x)|\int_0^1|\nabla(t)|\,h(t)dt$$
$$+\frac{x-a}{b-a}|f'(x)|\int_0^1|\Delta(t)|\,h(t)dt$$
$$+\frac{b-x}{b-a}|f'(b)|\int_0^1|\nabla(t)|\,h(1-t)dt$$
$$+\frac{x-a}{b-a}|f'(a)|\int_0^1|\Delta(t)|\,h(1-t)dt.$$

Proof. By Lemma 2.1 and h-convexity of $|f'|$ on $[a,b]$, we have

$$\left|\frac{\nabla(0)f(b)+\Delta(0)f(a)}{b-a} - \frac{1}{b-a}\left[{}_x^{+}I_{\varphi}f(b)+{}_x^{-}I_{\varphi}f(a)\right]\right|$$

$$\leq \left|\frac{b-x}{b-a}\int_0^1 \nabla(t)f'(tx+(1-t)b)dt - \frac{x-a}{b-a}\int_0^1 \Delta(t)f'(tx+(1-t)a)dt\right|$$

$$\leq \frac{b-x}{b-a}\int_0^1 |\nabla(t)||f'(tx+(1-t)b)|dt + \frac{x-a}{b-a}\int_0^1 |\Delta(t)||f'(tx+(1-t)a)|dt$$

$$\leq \frac{b-x}{b-a}\int_0^1 |\nabla(t)|(h(t)|f'(x)|+h(1-t)|f'(b)|)dt + \frac{x-a}{b-a}\int_0^1 |\Delta(t)|(h(t)|f'(x)|+h(1-t)|f'(a)|)dt$$

$$\leq \frac{b-x}{b-a}|f'(x)|\int_0^1 |\nabla(t)|h(t)dt + \frac{x-a}{b-a}|f'(x)|\int_0^1 |\Delta(t)|h(t)dt + \frac{b-x}{b-a}|f'(b)|\int_0^1 |\nabla(t)|h(1-t)dt + \frac{x-a}{b-a}|f'(a)|\int_0^1 |\Delta(t)|h(1-t)dt$$

this is the required result.

Remark 2.1. *Under assumptions of Theorem 2.1, if* $h(t)=t$, *then Theorem 2.1 reduces to Theorem 1 in* [3].

Remark 2.2. *Under assumptions of Theorem 2.1, if* $\varphi(t)=h(t)=t$, *then Theorem 2.1 reduces to Theorem 4 in* [5].

Corollary 2.1. *Under assumptions of Theorem* 2.1,

(1) if $\varphi(t)=\dfrac{t^{\alpha}}{\Gamma(\alpha)}$ and $t=1/2$, then

$$\left|\frac{f(b)(b-x)^{\alpha}+f(a)(x-a)^{\alpha}}{b-a} - \frac{\Gamma(\alpha+1)}{b-a}\left[I_{x^+}f(b)+I_{x^-}f(a)\right]\right|$$

$$\leq \frac{\alpha h\left(\frac{1}{2}\right)}{\alpha+2}\left[\frac{(b-x)^{\alpha+1}+(x-a)^{\alpha+1}}{b-a}\right]|f'(x)| + \frac{\left(\alpha^2+3\alpha\right)h\left(\frac{1}{2}\right)}{\alpha+2}\left[\frac{\left[(b-x)^{\alpha+1}|f'(b)| + (x-a)^{\alpha+1}|f'(a)|\right]}{b-a}\right].$$

(2) *if* $\varphi(t)=\dfrac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *and* $t=1/2$, *then*

$$\leq \frac{(b-x)^{\frac{\alpha}{k}+1}}{(b-a)k\Gamma_k(\alpha+k)}\int_0^1\left(1-t^{\frac{\alpha}{k}}\right)f(tx+(1-t)b)dt - \frac{(x-a)^{\frac{\alpha}{k}+1}}{(b-a)k\Gamma_k(\alpha+k)}\int_0^1\left(1-t^{\frac{\alpha}{k}}\right)f(tx+(1-t)a)dt$$

$$\leq \frac{2\alpha h\left(\frac{1}{2}\right)}{\alpha+2k}\left[\frac{(b-x)^{\frac{\alpha}{k}+1}+(x-a)^{\frac{\alpha}{k}+1}}{(b-a)\Gamma_k(\alpha+k)}\right]|f'(x)| + 2h\left(\frac{1}{2}\right)\left(\frac{1}{2}-\frac{k}{\alpha+k}+\frac{k}{\alpha+2k}\right)\times\left[\frac{(b-x)^{\frac{\alpha}{k}+1}|f'(b)|+(x-a)^{\frac{\alpha}{k}+1}|f'(a)|}{(b-a)\Gamma_k(\alpha+k}\right].$$

Remark 2.3. *Under assumptions of Theorem 2.1,*

(1) if $\varphi(t)=\dfrac{t^{\alpha}}{\Gamma(\alpha)}$ and $h(t)=t$, then Theorem 2.1 reduces to Corollary 2 in [3].

(2) if $\varphi(t)=\dfrac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $h(t)=t$, then Theorem 2.1 reduces to Corollary 3 in [3].

Theorem 2.2. *Let* $f:I=[a,b]\subset R\to R$ *be an absolutely continuous mapping on* I^0 *such that* $f'\in L_1([a,b])$, *where* $a,b\in I^0$ *with* $a<b$. *If the mapping* $|f'|^q$, $q>1$ *is h-convex on* $[a,b]$, *then we have the following inequality*

$$\left|\frac{\nabla(0)f(b)+\Delta(0)f(a)}{b-a} - \frac{1}{b-a}\left[{}_x^{+}I_{\varphi}f(b)+{}_x^{-}I_{\varphi}f(a)\right]\right|$$

$$\leq \frac{b-x}{b-a}\left(\int_0^1|\nabla(t)|^p\,dt\right)^{\frac{1}{p}}\begin{pmatrix}|f'(x)|^q\\+|f'(b)|^q\end{pmatrix}^{\frac{1}{q}}\left(\int_0^1|h(t)|\,dt\right)^{\frac{1}{q}} + \frac{x-a}{b-a}\left(\int_0^1|\Delta(t)|^p\,dt\right)^{\frac{1}{p}}\begin{pmatrix}|f'(x)|^q\\+|f'(a)|^q\end{pmatrix}^{\frac{1}{q}}\left(\int_0^1|h(t)|\,dt\right)^{\frac{1}{q}}.$$

Proof. Using the h-convexity of $|f'|^q$ on $[a,b]$, Lemma 2.1, and Hölder's inequality, we have

$$\left|\frac{\nabla(0)f(b)+\Delta(0)f(a)}{b-a} - \frac{1}{b-a}\left[{}_x^{+}I_{\varphi}f(b)+{}_x^{-}I_{\varphi}f(a)\right]\right|$$

$$\leq \frac{b-x}{b-a}\left(\int_0^1|\nabla(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(tx+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} + \frac{x-a}{b-a}\left(\int_0^1|\Delta(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}$$

$$\leq \frac{b-x}{b-a}\left(\int_0^1 |\nabla(t)|^p\, dt\right)^{\frac{1}{p}} \left(\begin{array}{l}\int_0^1 (|h(t)||f'(x)|^q \\ +|h(1-t)||f'(b)|^q)dt\end{array}\right)^{\frac{1}{q}}$$

$$+\frac{x-a}{b-a}\left(\int_0^1 |\Delta(t)|^p\, dt\right)^{\frac{1}{p}} \left(\begin{array}{l}\int_0^1 (|h(t)||f'(x)|^q \\ +|h(1-t)||f'(a)|^q)dt\end{array}\right)^{\frac{1}{q}}$$

$$\leq \frac{b-x}{b-a}\left(\int_0^1 |\nabla(t)|^p\, dt\right)^{\frac{1}{p}} \left(\begin{array}{l}|f'(x)|^q \int_0^1 |h(t)|\, dt \\ +|f'(b)|^q \int_0^1 |h(1-t)|\, dt\end{array}\right)^{\frac{1}{q}}$$

$$+\frac{x-a}{b-a}\left(\int_0^1 |\Delta(t)|^p\, dt\right)^{\frac{1}{p}} \left(\begin{array}{l}|f'(x)|^q \int_0^1 |h(t)|\, dt \\ +|f'(a)|^q \int_0^1 |h(1-t)|\, dt\end{array}\right)^{\frac{1}{q}},$$

where we have used the fact that

$$\int_0^1 |h(t)|\, dt = \int_0^1 |h(1-t)|\, dt.$$

Therefore

$$\left|\begin{array}{l}\dfrac{\nabla(0) f(b)+\Delta(0) f(a)}{b-a} \\ -\dfrac{1}{b-a}\left[{}_{x}^{+}I_{\varphi} f(b) + {}_{x}^{-} I_{\varphi} f(a)\right]\end{array}\right|$$

$$\leq \frac{b-x}{b-a}\left(\int_0^1 |\nabla(t)|^p\, dt\right)^{\frac{1}{p}} \left(\begin{array}{l}|f'(x)|^q \\ +|f'(b)|^q\end{array}\right)^{\frac{1}{q}} \left(\int_0^1 |h(t)|\, dt\right)^{\frac{1}{q}}$$

$$+\frac{x-a}{b-a}\left(\int_0^1 |\Delta(t)|^p\, dt\right)^{\frac{1}{p}} \left(\begin{array}{l}|f'(x)|^q \\ +|f'(a)|^q\end{array}\right)^{\frac{1}{q}} \left(\int_0^1 |h(t)|\, dt\right)^{\frac{1}{q}}.$$

Remark 2.4. *Under assumptions of Theorem 2.2, if* $h(t)=t$, *then Theorem 2.2 reduces to Theorem 2 in* [3].

Remark 2.5. *Under assumptions of Theorem 2.2, if* $\varphi(t)=h(t)=t$, *then Theorem 2.2 reduces to Theorem 5 in [5].*

Corollary 2.2. *Under assumptions of Theorem 2.2, if* $\varphi(t)=\dfrac{t^{\alpha}}{\Gamma(\alpha)}$ *and* $t=1/2$, *then*

$$\left|\begin{array}{l}\dfrac{f(b)(b-x)^{\alpha}+f(a)(x-a)^{\alpha}}{b-a} \\ -\dfrac{\Gamma(\alpha+1)}{b-a}\left[I_{x^+} f(b)+I_{x^-} f(a)\right]\end{array}\right|$$

$$\leq 2h\left(\frac{1}{2}\right)\left(\frac{\alpha+2}{\alpha+2}\right)\frac{(b-x)^{\alpha+1}}{(b-a)}\left(\frac{|f'(x)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}$$

$$+2h\left(\frac{1}{2}\right)\left(\frac{\alpha+2}{\alpha+2}\right)\frac{(x-a)^{\alpha+1}}{(b-a)}\left(\frac{|f'(x)|^q+|f'(a)|^q}{2}\right)^{\frac{1}{q}}$$

Remark 2.6. *Under assumptions of Theorem 2.2,* $\varphi(t)=\dfrac{t^{\alpha}}{\Gamma(\alpha)}$ *and* $h(t)=t$, *then Theorem 2.2 reduce to Corollary 4 in [3].*

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