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On the *p*-adic Gamma Function and Changhee Polynomials

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Abstract The *p*-adic gamma function is considered to obtain its derivative and to evaluate its the fermionic *p*-adic integral. Furthermore the relationship between the *p*-adic gamma function and Changhee polynomials and also between the Changhee polynomials and *p*-adic Euler constants is obtained. In addition, the *p*-adic Euler constants are expressed in term of Mahler coefficients of the *p*-adic gamma function.

Keywords: *p-adic number, p-adic gamma function, the fermionic p-adic integral, Mahler coefficients, p-adic Euler constant, Changhee Polynomials*

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1. Introduction

The *p*-adic numbers introduced by the German mathematician Kurt Hensel (1861-1941), are widely used in mathematics: in number theory, algebraic geometry, representation theory, algebraic and arithmetical dynamics, and cryptography. The *p*-adic numbers have been used to applying fields with successfully applying in super.eld theory of *p*-adic numbers by Vladimirov and Volovich. In addition, the *p*-adic model of the universe, the *p*-adic quantum theory, the *p*-adic string theory such as areas occurred in physics (for detail see [1,2]).

Special numbers and polynomials plays an important role in almost all areas of mathematics, in mathematical physics, computer science, engineering problems and other areas of science. The q-calculus (or quantum calculus) appeared in the 18th century and it continues to develop rapidly and has been studied by many scientists (cf. [3-8]). Many generalizations of special functions with a q-parameter recently were obtained using *p*-adic *q*-integral on $\mathbb{Z}_p$ (cf. [9-15]).

Let p be chosen as a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote the ring of *p*-adic integers, the field of *p*-adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively.

In the year 1975, Morita [16] defined the gamma function over *p*-adic fields, denoted by Γ_p, by the following formula:

$$\Gamma_p(x) = \lim_{n\to x} (-1)^n \prod_{\substack{1\le j<n \\ (p,j)=1}} j \quad \left(x \in \mathbb{Z}_p\right)$$

where n approaches x through positive integers. The *p*-adic gamma function Γ_p is analytic on $\mathbb{Z}_p$ and satisfies the functional relation:

$$\Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x) & |x|_p = 1 \\ -\Gamma_p(x) & |x|_p < 1 \end{cases} \tag{1.1}$$

The *p*-adic Euler constant γ_p is defined by the formula:

$$\gamma_p := \frac{\Gamma'_p(1)}{\Gamma_p(1)} = \Gamma'_p(1) = -\Gamma'_p(0). \tag{1.2}$$

The *p*-adic gamma function $\Gamma_p(x)$ has a great interest and has a great interest and has been studied by Diamond (1977) [17], Barsky (1977) [18], Dwork (1983) [19] and cited references therein.

For $x \in \mathbb{Z}_p$, the symbol $\binom{x}{n}$ is defined by $\binom{x}{0} = 1$ and $\binom{x}{n} = \frac{x(x-1)\dots(x-n+1)}{n!} = \frac{(x)_n}{n!}\left(x \in \mathbb{N}\right)$. The functions $x \to \binom{x}{n}\left(x \in \mathbb{Z}_p, n \in \mathbb{N}\right)$ form an orthonormal base of the space $C\left(\mathbb{Z}_p \to \mathbb{C}_p\right)$ with respect the norm $\|.\|_\infty$. This orthonormal base have the following property:

$$\binom{x}{n}' = \sum_{j=0}^{n-1} \frac{(-1)^{n-j-1}}{n-j} \binom{x}{j} \quad \text{[[20], p162]} \tag{1.3}$$

In 1958, Mahler introduced an expansion for continuous functions of a *p*-adic variable using special polynomials as binomial coefficient polynomial [21]. Means that for any

$f \in C(\mathbb{Z}_p \to \mathbb{C}_p)$, there exist unique elements $a_0, a_1, \ldots$ of $\mathbb{C}_p$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} \quad (x \in \mathbb{Z}_p).$$

The base $\left\{\binom{*}{x} : n \in \mathbb{N}\right\}$ is called Mahler base of the space $C(\mathbb{Z}_p \to \mathbb{C}_p)$, and the elements $\{a_n : n \in \mathbb{N}\}$ in $f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$ are called Mahler coefficients of $f \in C(\mathbb{Z}_p \to \mathbb{C}_p)$.

The Mahler expansion of the *p*-adic gamma function Γ_p and its Mahler coefficients are determined by the following proposition:

Proposition 1. ([20,22]) *Let*

$$\Gamma_p(x+1) = \sum_{n=0}^{\infty} a_n \binom{x}{n} (x \in \mathbb{Z}_p) \tag{1.4}$$

and

$$\exp\left(x + \frac{x^p}{p}\right)\frac{1-x^p}{1-x} = \sum_{n=0}^{\infty} (-1)^{n+1} a_n \frac{x^n}{n!} (x \in E) \tag{1.5}$$

where E *is the region of convergence of the power series* $\sum \frac{x^n\%}{n!}$.

For $f \in (\mathbb{Z}_p \to \mathbb{C}_p)$, the fermionic *p*-adic integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) := \lim_{N\to\infty} \sum_{j=0}^{p^N-1} f(j)(-1)^j \tag{1.6}$$

(see [10,11]). For any $f \in (\mathbb{Z}_p \to \mathbb{C}_p)$, by (1.6), the following relation holds:

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0) \tag{1.7}$$

where $f_1(x) = f(x+1)$.

The Changhee numbers and polynomials which are derived umbral calculus are defined by Kim et al. as the generating function to be

$$\frac{2}{t+2}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x) \frac{t^n}{n!}.$$

In the case when $x = 0$, $Ch_n(0) = Ch_n$ stands for Changhee numbers, see [23] for details. In [24], Kim et al. obtained following theorems which will be useful in deriving the main results of this paper:

Theorem 1. *For* $n \in \mathbb{N}_0$, *one has*

$$\int_{\mathbb{Z}_p} (x)_n d\mu_{-1}(x) = Ch_n.$$

Theorem 2. *For* $n \in \mathbb{N}_0$, *one has*

$$\int_{\mathbb{Z}_p} (x+y)_n d\mu_{-1}(y) = Ch_n(x).$$

Theorem 3. *For* $n \in \mathbb{N}_0$, *one has*

$$\int_{\mathbb{Z}_p} \binom{x}{n} d\mu_{-1}(x) = \left(\frac{-1}{2}\right)^n.$$

2. Main Results

In the present work, the fermionic *p*-adic integral of *p*-adic gamma function and of derivative of *p*-adic gamma function are evaluated. The *p*-adic Euler constants are expressed in term of Mahler coefficients of the *p*-adic gamma function. The relationship between the Changhee polynomials and the *p*-adic Euler constants are obtained.

Theorem 4. *Then the equality holds*:

$$\int_{\mathbb{Z}_p} \Gamma_p(x+1) d\mu_{-1}(x) = \sum_{n=0}^{\infty} a_n \frac{Ch_n}{n!}.$$

for $x \in \mathbb{Z}_p$, *where* a_n *is defined by Proposition* 1.

Proof. Let $x \in \mathbb{Z}_p$. From Proposition 1, we have

$$\begin{aligned}\int_{\mathbb{Z}_p} \Gamma_p(x+1) d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} a_n \binom{x}{n} d\mu_{-1}(x) \\ &= \sum_{n=0}^{\infty} a_n \int_{\mathbb{Z}_p} \binom{x}{n} d\mu_{-1}(x).\end{aligned} \tag{2.1}$$

From Theorem 1, we get

$$\int_{\mathbb{Z}_p} \Gamma_p(x+1) d\mu_{-1}(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} Ch_n.$$

Using Theorem 3 we can rewrite (2.1) and we have the following corollory:

Corollary 1. *For* $n \in \mathbb{N}$ *and* $x \in \mathbb{Z}_p$.

$$\int_{\mathbb{Z}_p} \Gamma_p(x+1) d\mu_{-1}(x) = \sum_{n=0}^{\infty} a_n \left(\frac{-1}{2}\right)^n$$

where a_n *is defined by Proposition 1.*

Lemma 1. *For* $n \in \mathbb{N}$ *and* $x \in \mathbb{Z}_p$, *the following equality holds*:

$$\int_{\mathbb{Z}_p} \binom{x-1}{n} d\mu_{-1}(x) = \left(2^{n+1} - 1\right)\left(\frac{-1}{2}\right)^n.$$

Proof. When $f(x) = \binom{x-1}{n}$ in (1.7), we have

$$\int_{\mathbb{Z}_p}\binom{x}{n}d\mu_{-1}(x)+\int_{\mathbb{Z}_p}\binom{x-1}{n}d\mu_{-1}(x)=2\binom{0-1}{n}.$$

Form Theorem 3 we prove the theorem.

Theorem 5. *The following relation is holds*

$$\int_{\mathbb{Z}_p}\Gamma_p(x+s)d\mu_{-1}(x)=\sum_{n=0}^{\infty}a_n\frac{Ch_n(s-1)}{n!}$$

where a_n *is defined by Proposition* 1.

Proof. Let $x\in\mathbb{Z}_p$. By Proposition 1, we have

$$\int_{\mathbb{Z}_p}\Gamma_p(x+s)d\mu_{-1}(x)=\sum_{n=0}^{\infty}a_n\int_{\mathbb{Z}_p}\frac{(x+s-1)_n}{n!}d\mu_{-1}(x).$$

By using Theorem 2 we can write

$$\int_{\mathbb{Z}_p}\Gamma_p(x+s)d\mu_{-1}(x)=\sum_{n=0}^{\infty}a_n\frac{Ch_n(s-1)}{n!}.$$

Theorem 6. *If* $x\in\mathbb{Z}_p$, *then*

$$\int_{\mathbb{Z}_p}\Gamma_p(x)d\mu_{-1}(x)=\sum_{n=0}^{\infty}a_n\left(2^{n+1}-1\right)\left(\frac{-1}{2}\right)^n.$$

Proof. By using Proposition 1, we have

$$\int_{\mathbb{Z}_p}\Gamma_p(x)d\mu_{-1}(x)=\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}a_n\binom{x-1}{n}d\mu_{-1}(x)$$
$$=\sum_{n=0}^{\infty}a_n\int_{\mathbb{Z}_p}\binom{x-1}{n}d\mu_{-1}(x).$$

From Lemma 1, fermionic *p*-adic integral of $\Gamma_p(x)$ is evaluated.

From Theorem 5 and Theorem 6, the following corollary is obtained.

Corollary 2. *For* $n\in\mathbb{N}$,

$$Ch_n(-1)=n!\left(2^{n+1}-1\right)\left(\frac{-1}{2}\right)^n.$$

From Proposition 1 and (1.3), derivative of *p*-adic Gamma functions, Γ'_p, is obtained as

$$\Gamma'_p(x+1)=\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}a_n\frac{(-1)^{n-j-1}}{n-j}\binom{x}{j} \tag{2.2}$$

where a_n is defined by Proposition 1.

Theorem 7. *The p-adic Euler constants have the expansion*

$$\gamma_p=\sum_{n=1}^{\infty}a_n\frac{(-1)^{n-1}}{n}.$$

Proof. When $f(x)=\Gamma'_p(x)$ in (1.7), we get

$$\int_{\mathbb{Z}_p}\Gamma'_p(x+1)d\mu_{-1}(x)+\int_{\mathbb{Z}_p}\Gamma'_p(x)d\mu_{-1}(x)=2\Gamma'_p(0).$$

From (2.2) and (1.2), we can write

$$\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}a_n\frac{(-1)^{n-j-1}}{n-j}\int\binom{x}{j}d\mu_{-1}(x)$$
$$+\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}a_n\frac{(-1)^{n-j-1}}{n-j}\int_{\mathbb{Z}_p}\binom{x-1}{j}d\mu_{-1}(x)$$
$$=-2\gamma_p.$$

Using Theorem 3 and Lemma 1 we can rewrite (2.3) as

$$\gamma_p=\sum_{n=1\%}^{\infty}\sum_{j=0}^{n-1}a_n\frac{(-1)^n}{n-j}.$$

By some computing steps, the proof is completed.

Theorem 8. *Relationship between the Changhee polynomials and the p-adic Euler constants is as*

$$\gamma_p=\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}a_n\frac{(-1)^{n-j}}{j!2(n-j)}\left(Ch_j+Ch_j(-1)\right)$$

and

$$\gamma_p=\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}a_n\frac{(-1)^{n-j}Ch_j}{j!2(n-j)}$$
$$+\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\frac{(-1)^n}{n-j}a_n\left(1-2^{-j-1}\right).$$

Proof. we can rewrite (2.3) by

$$\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}a_n\frac{(-1)^{n-j-1}}{n-j}\int_{\mathbb{Z}_p}\frac{(x)_j}{j!}d\mu_{-1}(x)$$
$$+\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}a_n\frac{(-1)^{n-j-1}}{n-j}\int_{\mathbb{Z}_p}\frac{(x-1)_j}{j!}d\mu_{-1}(x)$$
$$=-2\gamma_p.$$

From Theorem 1 and Theorem 2, it is obtained

$$\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}a_n\frac{(-1)^{n-j}}{j!2(n-j)}\left(Ch_j+Ch_j(-1)\right)=\gamma_p.$$

In addition, by using Corollary 2, we get

$$\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}a_n\frac{(-1)^{n-j}Ch_j}{j!2(n-j)}+\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\frac{(-1)^n}{n-j}a_n\left(1-2^{-j-1}\right)=\gamma_p.$$

Theorem 9. *If* $x,s\in\mathbb{Z}_p$ *then*

$$\int_{\mathbb{Z}_p}\Gamma'_p(x+s)d\mu_{-1}(x)=\sum_{n=0}^{\infty}\sum_{j=0}^{n-1}a_n\frac{(-1)^{n-j-1}Ch_j(s-1)}{(n-j)j!}.$$

Proof. Let $x,s\in\mathbb{Z}_p$. We have

$$\int_{\mathbb{Z}_p} \Gamma'_p(x+s)\, d\mu_{-1}(x)$$
$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n-1} a_n \frac{(-1)^{n-j-1}}{n-j} \int_{\mathbb{Z}_p} \Gamma'_p \binom{x+s-1}{j} d\mu_{-1}(x).$$

By using Theorem 3 we can write

$$\int_{\mathbb{Z}_p} \Gamma'_p(x+s)\, d\mu_{-1}(x)$$
$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n-1} a_n \frac{(-1)^{n-j-1} Ch_j(s-1)}{(n-j)\, j!}.$$

In the case $s = 1$ in Theorem 9 we obtain the following conclusion

Corollary 3. *For* $x \in \mathbb{Z}_p$,

$$\int_{\mathbb{Z}_p} \Gamma'_p(x+1)\, d\mu_{-1}(x) = \sum_{n=0}^{\infty} \sum_{j=0}^{n-1} a_n \frac{(-1)^{n-j-1} Ch_j}{(n-j)\, j!}.$$

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