Turkish Journal of Analysis and Number Theory, 2018, Vol. 6, No. 4, 116-119
Available online at http://pubs.sciepub.com/tjant/6/4/2
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DOI:10.12691/tjant-6-4-2

Simplification of Coefficients in Two Families of Nonlinear Ordinary Differential Equations

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Received June 29, 2018; Revised July 30, 2018; Accepted September 17, 2018

Abstract In the paper, in terms of the Stirling numbers of the first and second kinds, by three approaches, the author derives simple, meaningful, and significant forms for coefficients in two families of nonlinear ordinary differential equations.

Keywords: *coefficient, nonlinear ordinary differential equation, Stirling number, derivative polynomial*

Cite This Article: Feng Qi, and Bai-Ni Guo, "Simplification of Coefficients in Two Families of Nonlinear Ordinary Differential Equations." *Turkish Journal of Analysis and Number Theory*, vol. 6, no. 4 (2018): 116-119. doi: 10.12691/tjant-6-4-2.

1. Motivation and Main Results

In [[6], Theorems 2.1 and 2.2], it was established inductively and recursively that the function

$$F(t)=\frac{1}{2-e^t} \tag{1}$$

satisfies the families of nonlinear ordinary differential functions

$$F^{(n)}(t)=\sum_{k=0}^{n}(-1)^{n-k}a_k(n)F^{k+1}(t) \tag{2}$$

and

$$2^n n!F^{n+1}(t)=\sum_{k=0}^{n}b_k(n)F^{(k)}(t) \tag{3}$$

for $n\in\mathbb{N}$, where $a_0(n)=1$, $a_n(n)=2^n n!$, $b_0(n)=n!$, $b_n(n)=1$,

$$\begin{aligned}a_k(n)=2^{k-1}k!\sum_{i_k=0}^{n-k}\sum_{i_{k-1}=0}^{n-k-i_k}\cdots\\ \cdots\sum_{i_1=1}^{n-k+1-i_k-\cdots-i_2}2^{i_1}\cdots k^{i_{k-1}}(k+1)^{i_k},\end{aligned} \tag{4}$$

and

$$b_k(n)=\sum_{i_k=0}^{n-k}\sum_{i_{k-1}=0}^{i_k}\cdots\sum_{i_1=0}^{i_2}\frac{n!}{\prod_{j=1}^{k}(i_j+j)} \tag{5}$$

for $0<k<n$.

In this paper, since

(1) the original proofs of [[6], Theorems 2.1 and 2.2] are long and tedious,
(2) the expressions in (4) and (5) are too complex to be remembered, understood, and computed easily,

we will provide three simple and standard proofs for [[6], Theorems 2.1 and 2.2] and, more importantly, derive simple, meaningful, and significant expressions for the quantities $a_k(n)$ and $b_k(n)$.

Our main results can be stated as the following theorem.

Theorem 1. *For* $k\in\{0\}\cup\mathbb{N}$, *the function* $F(t)$ *defined by* (1) *satisfies*

$$F^{(k)}(t)=(-1)^k\sum_{m=0}^{k}(-1)^m 2^m m!S(k+1,m+1)F^{m+1}(t) \tag{6}$$

and

$$F^{k+1}(t)=\frac{(-1)^k}{2^k k!}\sum_{m=0}^{k}(-1)^m s(k+1,m+1)F^{(m)}(t), \tag{7}$$

where $s(n,k)$ *and* $S(n,k)$ *stand for the Stirling numbers of the first and second kinds.*

2. Proofs of Theorem 1

In this section, we provide three proofs for Theorem 1 as follows.

First proof. It is well known [[1], Theorem 11.4] and [[2], p. 139, Theorem C] that the Faàdi Bruno formula can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\cdots,x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} f\circ h(x) = \sum_{k=0}^{n} f^{(k)}(h(x)) \times \mathrm{B}_{n,k}\left(h'(x), h''(x), \dots, h^{(n-k+1)}(x)\right). \quad (8)$$

The identities

$$\mathrm{B}_{n,k}\left(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\right) = a^k b^n \mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \quad (9)$$

and

$$\mathrm{B}_{n,k}(1,1,\dots,1) = S(n,k) \quad (10)$$

for $n \ge k \ge 0$ and $a,b \in \mathbb{C}$ can be found in [[1],p. 412] and [[2], p. 135]. Applying (8), (9), and (10) in sequence and denoting $u = u(x) = 2 - e^x$ yield

$$\begin{aligned}
F^{(n)}(t) &= \sum_{k=0}^{n} \left(\frac{1}{u}\right)^{(k)} \mathrm{B}_{n,k}\left(-e^x, -e^x, \dots, -e^x\right) \\
&= \sum_{k=0}^{n} \frac{(-1)^k k!}{u^{k+1}} \left(-e^x\right)^k \mathrm{B}_{n,k}(1,1,\dots,1) \\
&= \sum_{k=0}^{n} \frac{(-1)^k k!}{(2-e^x)^{k+1}} \left[\left(2-e^x\right) - 2\right]^k S(n,k) \\
&= \sum_{k=0}^{n} \frac{(-1)^k k!}{(2-e^x)^{k+1}} S(n,k) \sum_{\ell=0}^{k} \binom{k}{\ell} \left(2-e^x\right)^{\ell} (-2)^{k-\ell} \\
&= \sum_{k=0}^{n} k! S(n,k) \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} F^{k-\ell+1}(x) 2^{k-\ell} \\
&= \sum_{k=0}^{n} (-1)^k k! S(n,k) \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} F^{\ell+1}(x) 2^{\ell} \\
&= \sum_{\ell=0}^{n} (-1)^{\ell} F^{\ell+1}(x) 2^{\ell} \sum_{k=\ell}^{n} (-1)^k \binom{k}{\ell} k! S(n,k).
\end{aligned}$$

Combining this with the identity

$$\sum_{k=\ell}^{n} (-1)^k \binom{k}{\ell} k! S(n,k) = (-1)^n \ell! S(n+1, \ell+1) \quad (11)$$

in [[24], Theorem 2.2] results in (6).

The identity (7) follows from applying [[2], p. 213, eq. (5c)] or [[34], p. 171, Theorem 12.1], which reads that

$$a_n = \sum_{\alpha=0}^{n} S(n,\alpha) b_\alpha \text{ if and only if } b_n = \sum_{k=0}^{n} s(n,k) a_k$$

for a collection of constants b_α and a_k independent of n, to (6). The first proof of Theorem 1 is complete.

Second proof. A sequence of polynomials P_n of order n are called [5,9,22] the derivative polynomials of a function $f(x)$ if and only if $f^{(n)}(x) = P_n(f(x))$ for $n \ge 0$. In [[38], Theorem 1.1], it was obtained that the derivative polynomials of the function $F(t,\lambda) = \frac{1}{e^t + \lambda}$ can be computed by

$$P_n(x) = \sum_{k=0}^{n} (-1)^{n-k} k! S(n+1, k+1) \lambda^k x^{k+1} \quad (12)$$

for $\lambda \ne 0$ and $n \in \mathbb{N}$. In [[38], Theorem 1.3], it was obtained that the nonlinear differential equations

$$\sum_{k=0}^{n} (-1)^k s(n+1, k+1) \frac{\mathrm{d}^k F(t,\lambda)}{\mathrm{d}t^k} = (-1)^n n! \lambda^n F^{n+1}(t,\lambda) \quad (13)$$

have a common solution $F(t,\lambda) = \frac{1}{e^t + \lambda}$ for $\lambda \ne 0$ and $n \in \mathbb{N}$. Letting $\lambda = -2$ in (12) and (13) gives

$$P_n(x) = (-1)^n \sum_{k=0}^{n} k! S(n+1, k+1) 2^k x^{k+1}$$

and

$$\sum_{k=0}^{n} (-1)^k s(n+1, k+1) F^{(k)}(t) = n! 2^n F^{n+1}(t).$$

The second proof of Theorem 1 is thus complete.

Third proof. By virtue of Theorem 2.1 in [3], Theorems 3.1 and 3.2 in [37], and Lemma 2.1 in [38], it follows that

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\frac{1}{\beta e^{\alpha t} - 1}\right) = (-1)^k \alpha^k \sum_{m=1}^{k+1} (m-1)! S(k+1, m) \left(\frac{1}{\beta e^{\alpha t} - 1}\right)^m \quad (14)$$

and

$$\left(\frac{1}{\beta e^{\alpha t} - 1}\right)^k = \frac{1}{(k-1)!} \sum_{m=1}^{k} \frac{(-1)^{m-1}}{\alpha^{m-1}} \times s(k,m) \frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}}\left(\frac{1}{\beta e^{\alpha t} - 1}\right), \quad (15)$$

where $\alpha, \beta \ne 0$ are real constants, $k \in \mathbb{N}$, either $\beta > 0$ and $t \ne -\frac{\ln \beta}{\alpha}$ or $\beta < 0$ and $t \in \mathbb{R}$. See also [4,35,36].

Taking $\alpha = 1$ and $\beta = \frac{1}{2}$ in (14) and (15) leads to

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\frac{1}{e^t/2 - 1}\right) = (-1)^k \sum_{m=1}^{k+1} (m-1)! S(k+1, m) \left(\frac{1}{e^t/2 - 1}\right)^m$$

and

$$\left(\frac{1}{e^t/2 - 1}\right)^k = \frac{1}{(k-1)!} \sum_{m=1}^{k} (-1)^{m-1} s(k,m) \frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}}\left(\frac{1}{e^t/2 - 1}\right).$$

These two identities can be further rearranged as

$$-2\frac{d^k}{dt^k}\Big(\frac{1}{2-e^t}\Big)=(-1)^k\sum_{m=1}^{k+1}(-1)^m2^m(m-1)!\times S(k+1,m)\Big(\frac{1}{2-e^t}\Big)^m$$

and

$$(-1)^k2^k\Big(\frac{1}{2-e^t}\Big)^k=\frac{2}{(k-1)!}\sum_{m=1}^{k}(-1)^m s(k,m)\frac{d^{m-1}}{dt^{m-1}}\Big(\frac{1}{2-e^t}\Big).$$

The third proof of Theorem 1 is thus complete.

3. Remarks

Finally, we list several remarks on our main results and closely related things.

Remark 1. Comparing (2) and (3) with (6) and (7) figures out that

$$a_m(k)=2^m m!S(k+1,m+1),\quad k\ge m\ge 0$$

and

$$b_m(k)=(-1)^{k+m}s(k+1,m+1),\quad k\ge m\ge 0$$

which are simpler, more meaningful, and more significant than the expressions in (4) and (5).

Remark 2. In [[34], p. 118, Eq. (9.18)], it is listed that

$$\alpha!S(n,\alpha)=\sum_{j=\alpha}^{n}(-1)^{n-j}\binom{j-1}{\alpha-1}j!S(n,j).$$

This identity is different from (11) and

$$\sum_{\ell=k+1}^{n}(-1)^{\ell}\binom{\ell-1}{k}\ell!S(n,\ell)=(-1)^n(k+1)!S(n,k+1)\quad(16)$$

for $n>k\ge 0$ in [[24], Theorem 2.2].

Remark 3. Any one among three proofs is simpler and shorter than the one in the paper [6].

Remark 4. The motivations in the papers [3,4,7,8,10,11, 12,14-33,38] are same as the one in this paper.

Remark 5. This paper is a slightly modified version of the preprint [13].

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