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First-order Boundedly Solvable Singular Differential Operators

Pembe Ipek Al[1,*], Zameddin I. Ismailov[2]

[1]Institute of Natural Sciences, Karadeniz Technical University, Turkey
[2]Department of Mathematics, Karadeniz Technical University, Turkey
*Corresponding author: ipekpembe@gmail.com

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Abstract For the expression of all boundedly solvable extensions of the minimal operator generated by linear singular differential-operator expression for first order it has been applied Operator Theory Methods. Lastly, geometry of spectrum of these extensions is investigated.

Keywords: *differential operator, boundedly solvable operator, spectrum*

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1. Introduction

It is known that many solvability problems arising in life sciences can be expressed as boundary value problems for linear functional equations in corresponding functional spaces.

The solvability of the considered problems may be seen as boundedly solvability of linear differential operators in corresponding functional Banach spaces. Note that the theory of boundedly solvable extensions of a linear densely defined closed operator in Hilbert spaces was presented in the important works of Vishik in [1,2].

Let us recall that an operator $S:D(S)\subset H\to H$ on any Hilbert space H is called boundedly solvable, if S is one-to-one and onto, and $S^{-1}\in L(H)$.

The main aim of this work is to describe of all boundedly solvable extensions of the minimal operator generated by first-order linear quasi differential-operator expression in the Hilbert space of vector-functions at right semi-axis in terms of boundary conditions. Lastly, the structure of spectrum of these extensions will be investigated.

2. Statement of the Problem

Let H be a separable Hilbert space and $\alpha:(0,\infty)\to(0,\infty)$, $\alpha\in C(0,\infty)$, $\frac{1}{\alpha}\in L^1(0,\infty)$. In the weighted Hilbert space $L^2_\alpha(H,(0,\infty))$ of H-valued vector-functions defined at the interval $(0,\infty)$ consider the following linear quasi-differential expression with operator coefficient for first order in a form

$$l(u)=(\alpha u)'(t)+A(t)u(t),$$

where for operator-function $A(\cdot):(0,\infty)\to L(H)$ is satisfied $\sup\limits_{t>0}\|A(t)\|<\infty$.

By the standard way the minimal L_0 and maximal L operators corresponding to differential expression $l(\cdot)$ in $L^2_\alpha(H,(0,\infty))$ can be defined (see [3]).

In this case $KerL_0=\{0\}$ and $\overline{Im(L_0)}\neq L^2_\alpha(H,(0,\infty))$ (see sec.3).

In this work, firstly all boundedly solvable extensions of the minimal operator generated by first order linear singular differential-operator expression in the weighted Hilbert space of vector-functions at right semi-axis in terms of boundary conditions. Later on, the structure of spectrum of these type extensions will be investigated.

3. Description of Boundedly Solvable Extensions

In this section using the Vishik's methods all boundedly solvable extensions of the minimal operator L_0 in weighted Hilbert spaces $L^2_\alpha(H,(0,\infty))$.

Before of all note that using the knowing standard way the minimal M_0 and the maximal M operators generated by differential expression

$$m(v)=(\alpha v)'(t)$$

in Hilbert space $L^2_\alpha(H,(0,\infty))$ can be defined (see [3]).

Later on, by $U(t,s)$, $t,s\in[0,\infty)$ will be defined the family of evolution operators corresponding to the homogeneous differential-operator equation

$$\alpha(t)\frac{\partial}{\partial t}U(t,s)f+A(t)U(t,s)f=0, t,s\geq 0$$

with boundary condition

$$U(s,s)f=f, f\in H.$$

The operator $U(t,s)$, $t,s\in[0,\infty)$ is linear continuous and boundedly solvable in H. And also for any $t,s\geq 0$

$$U^{-1}(t,s)=U(t,s)$$

(for detail analysis see [4]).

If introduce the following operator

$$U_z(t)=U(t,0)z(t),$$

$$U:L_\alpha^2(H,(0,\infty))\to L_\alpha^2(H,(0,\infty)),$$

then it is easily to check that

$$\begin{aligned} l(Uz)&=(\alpha Uz)'(t)+A(t)Uz(t)\\ &=U(\alpha z)'(t)+U_t'(\alpha z)(t)+A(t)Uz(t)\\ &=U(\alpha z)'(t)+\left[\alpha(t)U_t'z(t)+A(t)Uz(t)\right]\\ &=U(\alpha z)'(t)\\ &=Um(z).\end{aligned}$$

Therefore

$$U^{-1}l(Uz)=m(z).$$

Hence it is clear that if $\tilde{L}$ is some extension of the minimal operator L_0, that is, $L_0\subset\tilde{L}\subset L$, then $U^{-1}L_0U=M_0$, $M_0\subset U^{-1}\tilde{L}U=\widetilde{M}\subset M$, $U^{-1}LU=M$.

Now we prove the following assertion.

Theorem 3.1. *$KerL_0=\{0\}$ and $\overline{Im(L_0)}\neq L_\alpha^2(H,(0,\infty))$.*

Proof. Consider the following boundary value problem in $L_\alpha^2(H,(0,\infty))$

$$(\alpha u)'(t)+A(t)u(t)=0, t>0,$$
$$(\alpha u)(0)=(\alpha u)(\infty)=0.$$

Then the general solution of above differential equation is in form

$$(\alpha u)(t)=\exp\left(-\int_0^t\frac{A(s)}{\alpha(s)}ds\right)f_0, f_0\in H.$$

From this and boundary conditions we have

$$u(t)=0, t>0.$$

Consequently, $Ker(L_0)=\{0\}$.

On the other hand it is clear that the general solution of following differential equation in $L_\alpha^2(H,(0,\infty))$

$$-(\alpha v)'(t)+A^*(t)v(t)=0$$

in form

$$v(t)=\frac{1}{\alpha(t)}\exp\left(\int_0^t\frac{A^*(s)}{\alpha(s)}ds\right)g, g\in H.$$

This means that

$$\dim KerL_0^*=\dim H.$$

So

$$\overline{Im(L_0)}\neq L_\alpha^2(H,(0,\infty)).$$

Theorem 3.2. *Each solvable extension $\tilde{L}$ of the minimal operator L_0 in $L_\alpha^2(H,(0,\infty))$ is generated by the differential-operator expression $l(\cdot)$ with boundary condition*

$$(B+E)U^{-1}(\alpha u)(0)=BU^{-1}(\alpha u)(\infty),$$

where $B\in L(H)$, E is a identity operator in H. The operator B is determined uniquely by the extension $\tilde{L}$, i.e $\tilde{L}=L_B$.

On the contrary, the restriction of the maximal operator L to the manifold of vector-functions satisfy the above boundary condition for some bounded operator $B\in L(H)$ is a boundedly solvable extension of the minimal operator L_0 in $L_\alpha^2(H,(0,\infty))$.

Proof. Firstly, all boundedly solvable extensions $\widetilde{M}$ of the minimal operator L_0 in $L_\alpha^2(H,(0,\infty))$ in terms of boundary conditions will be described.

Consider the following so-called Cauchy extension M_c,

$$M_cu=(\alpha u)'(t),$$
$$M_c:D(M_c)\subset L_\alpha^2(H,(0,\infty))\to L_\alpha^2(H,(0,\infty)),$$
$$D(M_c)=\{u\in D(L):(\alpha u)(0)=0\}.$$

of the minimal operator M_0. It is clear that M_c is a boundedly solvable extension of minimal operator M_0 and

$$M_c^{-1}f(t)=\frac{1}{\alpha(t)}\int_0^t f(s)ds,$$
$$f\in L_\alpha^2(H,(0,\infty)),$$
$$M_c^{-1}:L_\alpha^2(H,(0,\infty))\to L_\alpha^2(H,(0,\infty)).$$

Indeed, for any $f\in L_\alpha^2(H,(0,\infty))$ we have

$$\left\| \frac{1}{\alpha(t)} \int_0^t f(s) ds \right\|^2_{L^2_\alpha(H,(0,\infty))}$$
$$= \int_0^\infty \alpha(t) \frac{1}{\alpha^2(t)} \left\| \int_0^t f(s) ds \right\|^2_H dt$$
$$\le \int_0^\infty \frac{1}{\alpha(t)} \left(\int_0^t \frac{1}{\sqrt{\alpha(s)}} \sqrt{\alpha(s)} \| f(s) \|_H ds \right)^2 dt$$
$$\le \int_0^\infty \frac{dt}{\alpha(t)} \left(\int_0^\infty \frac{ds}{\alpha(s)} \right) \left(\int_0^\infty \| f(s) \|^2_H \alpha(s) ds \right)$$
$$= \left(\int_0^\infty \frac{ds}{\alpha(s)} \right)^2 \| f \|^2_{L^2_\alpha(H,(0,\infty))}.$$

Now assumed that $\widetilde{M}$ is a solvable extension of the minimal operator M_0 in $L^2_\alpha(H,(0,\infty))$. In this case it is known that the domain of $\widetilde{M}$ can be written as a direct sum

$$D(\widetilde{M}) = D(M_0) \oplus \left(M_c^{-1} + K \right) V,$$

where $V = KerM_0^*, K \in L(H)$ (see [1,2]).

It is easily to see that

$$KerM_0^* = \left\{ \frac{1}{\alpha(t)} f : f \in H \right\}.$$

Therefore each function $u \in D(\widetilde{M})$ can be written in following form

$$u(t) = u_0(t) + M_c^{-1} \left(\frac{1}{\alpha(t)} f \right) + \frac{1}{\alpha(t)} Kf,$$
$$u_0 \in D(M_0), f \in H.$$

And from this we have

$$(\alpha u)(t) = (\alpha u_0)(t) + \int_0^t \frac{ds}{\alpha(s)} f + Kf, f \in H.$$

Hence

$$(\alpha u)(0) = Kf,$$
$$(\alpha u)(\infty) = \left(\int_0^\infty \frac{ds}{\alpha(s)} E + K \right) f.$$

From these relations it is obtained that

$$\left(\int_0^\infty \frac{ds}{\alpha(s)} E + K \right) (\alpha u)(0) = K(\alpha u)(\infty).$$

Then the last equality can be written in form

$$(B+E)(\alpha u)(0) = B(\alpha u)(\infty),$$

where

$$B = \left(\int_0^\infty \frac{ds}{\alpha(s)} \right)^{-1} K.$$

On the other hand note that the uniquenses of the operator $B \in L(H)$ is clear from [1,2]. Therefore, $\widetilde{M} = M_B$. This completes of necessary part of assertion.

On the contrary, if M_B is a operator generated by $m(\cdot)$ and boundary condition

$$(B+E)(\alpha u)(0) = B(\alpha u)(\infty),$$

then M_B is boundedly invertible and

$$M_B^{-1} : L^2_\alpha(H,(0,\infty)) \to L^2_\alpha(H,(0,\infty)),$$
$$M_B^{-1} f(t) = \frac{1}{\alpha(t)} \int_0^t f(s) ds + B \int_0^\infty f(s) ds, f \in L^2_\alpha(H,(0,\infty)).$$

Consequently, assertion of theorem for the boundedly solvable extension of the minimal operator M_0 is true.

The extension $\widetilde{L}$ of the minimal operator L_0 is boundedly solvable in $L^2_\alpha(H,(0,\infty))$ if and only if the operator $\widetilde{M} = U^{-1} \widetilde{L} U$ is a boundedly solvable extension of the minimal operator M_0 in $L^2_\alpha(H,(0,\infty))$. Then $u \in D(\widetilde{L})$ if and only if $U^{-1} u \in D(\widetilde{M})$.

Since $\widetilde{M} = M_B$ for some $B \in L(H)$, then we have

$$(B+E) U^{-1} (\alpha u)(0) = B U^{-1} (\alpha u)(\infty).$$

This completes the proof of theorem.

4. Spectrum of Boundedly Solvable Extensions

In this section the structure of spectrum of boundedly solvable extensions of the minimal operator L_0 in $L^2_\alpha(H,(0,\infty))$ will be investigated.

Firstly, prove the following result.

Theorem 4.1. *If L_B is a boundedly solvable extension of the minimal operator L_0 and $M_B = U^{-1} L_B U$ corresponding boundedly solvable extension of the minimal operator M_0, then it is true $\sigma(L_B) = \sigma(M_B)$.*

Proof. Consider the following problem to spectrum for any boundedly solvable extension L_B in $L^2_\alpha(H,(0,\infty))$, that is

$$L_B u = \lambda u + f, \lambda \in \mathbb{C}, f \in L^2_\alpha(H,(0,\infty)).$$

From this it is obtained that

$$(L_B - \lambda E) u = f \text{ or } \left(U M_B U^{-1} - \lambda E \right) u = f.$$

Then we have

$$U(M_B-\lambda)U^{-1}(u)=f.$$

Therefore, the validity of the theorem is clear.

Now prove the main theorem on the structure of spectrum.

Theorem 4.2. *The spectrum of the boundedly solvable extension L_B of the minimal operator L_0 in $L^2_\alpha(H,(0,\infty))$ has the form*

$$\sigma(L_B)=\left\{\begin{array}{l}\lambda\in\mathbb{C}:\lambda=\left(\int\limits_0^\infty\frac{ds}{\alpha(s)}\right)^{-1}\\ \qquad\times\left(\ln\left|\frac{\mu+1}{\mu}\right|+i\arg\left(\frac{\mu+1}{\mu}\right)+2n\pi i\right),\\ \mu\in\sigma(B)\backslash\{0,-1\},n\in\mathbb{Z}\end{array}\right\}$$

Proof. By Theorem 4.1. for this it is sufficiently the investigate the spectrum of the corresponding boundedly solvable extension $M_B=U^{-1}L_BU$ of the minimal operator M_0 in $L^2_\alpha(H,(0,\infty))$.

Now consider the following problem to spectrum for the extension M_B, that is,

$$M_Bu=\lambda u+f,\lambda\in\mathbb{C},f\in L^2_\alpha(H,(0,\infty)).$$

Then

$$(\alpha u)'(t)=\lambda u(t)+f(t),t>0$$

with boundary condition

$$(B+E)(\alpha u)(0)=B(\alpha u)(\infty).$$

It is clear that a general solution of the above differential equation has the form

$$u(t;\lambda)=\frac{1}{\alpha(t)}\exp\left\{\lambda\int\limits_0^t\frac{ds}{\alpha(s)}\right\}f_0+\frac{1}{\alpha(t)}\int\limits_0^t\exp\left\{\lambda\int\limits_s^t\frac{d\tau}{\alpha(\tau)}\right\}f(s)ds,\ f_0\in H.$$

From this and boundary condition it is obtained that

$$\left(E+B\left(1-\exp\left\{\lambda\int\limits_0^\infty\frac{ds}{\alpha(s)}\right\}\right)\right)f_0=B\left(\int\limits_0^\infty\exp\left\{\lambda\int\limits_s^\infty\frac{d\tau}{\alpha(\tau)}\right\}f(s)ds\right).$$

In case when $\lambda_m\int\limits_0^\infty\frac{ds}{\alpha(s)}=2m\pi i,\ m\in\mathbb{Z},$ from the last relation it is established that

$$f_0^{(m)}=B\left(\int\limits_0^\infty\exp\left\{\lambda_m\int\limits_s^\infty\frac{d\tau}{\alpha(\tau)}\right\}f(s)ds\right),m\in\mathbb{Z}.$$

Consequently, in this case the resolvent operator of M_B is in form

$$R_{\lambda_m}(M_B)f(t)=B\left(\frac{1}{\alpha(t)}\exp\left\{\lambda_m\int\limits_0^t\frac{ds}{\alpha(s)}\right\}\int\limits_0^\infty\exp\left\{\lambda_m\int\limits_s^\infty\frac{d\tau}{\alpha(\tau)}\right\}f(s)ds\right)+\frac{1}{\alpha(t)}\int\limits_0^t\exp\left\{\lambda_m\int\limits_s^t\frac{d\tau}{\alpha(\tau)}\right\}f(s)ds,m\in\mathbb{Z}.$$

Now assumed that $\lambda\int\limits_s^\infty\frac{ds}{\alpha(s)}\neq 2m\pi i,\ m\in\mathbb{Z}.$ Then from the mentioned above equation for $f_0\in H$ we have

$$\left(B-\left(1-\exp\left\{\lambda_m\int\limits_0^\infty\frac{ds}{\alpha(s)}\right\}\right)^{-1}E\right)f_0=\left(1-\exp\left\{\lambda\int\limits_0^\infty\frac{ds}{\alpha(s)}\right\}\right)^{-1}B\left(\int\limits_0^\infty\exp\left\{\lambda\int\limits_0^\infty\frac{d\tau}{\alpha(\tau)}\right\}f(s)ds\right),$$

$$f_0\in H,f\in L^2_\alpha(H,(0,\infty)).$$

Then $\lambda\in\sigma(M_B)$ if and only if

$$\mu=\left(1-\exp\left\{\lambda\int\limits_0^\infty\frac{ds}{\alpha(s)}\right\}\right)^{-1}\in\sigma(B).$$

In this case since $\mu\neq 0,$

$$\exp\left\{\lambda\int\limits_0^\infty\frac{ds}{\alpha(s)}\right\}=\frac{\mu+1}{\mu},\mu\in\sigma(B),\mu\neq -1.$$

Then

$$\lambda=\left(\int\limits_0^\infty\frac{ds}{\alpha(s)}\right)^{-1}\left(\ln\left|\frac{\mu+1}{\mu}\right|+i\arg\left(\frac{\mu+1}{\mu}\right)+2n\pi i\right),n\in\mathbb{Z}.$$

Remark 4.3. In finite interval case similar problems have been investigated in [5].

Example 4.4. All boundedly solvable extensions L_k of the minimal operator L_0 in $L^2_\alpha(0,\infty)$, $\alpha(t)=(1+t)^2$, $t>0$, generated by differential expression

$$l(u)=\left((1+t)^2u(t)\right)'+arctantu(t),t>0$$

are generated by differential expression $l(\cdot)$ and boundary condition

$$(k+1)U^{-1}(\alpha u)(0)=kU^{-1}(\alpha u)(\infty),k\in\mathbb{C},$$

where $U(\cdot,\cdot)$ are the corresponding evolution operators. In this case the spectrum $\sigma(L_k)$ of the extension L_k, $k\neq 0,\ -1$ is in form

$$\sigma(L_k)=\left\{\lambda\in\mathbb{C}:\lambda=\ln\left|\frac{k+1}{k}\right|+i\arg\left(\frac{k+1}{k}\right)+2\pi ni,n\in\mathbb{Z}\right\}.$$

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