Turkish Journal of Analysis and Number Theory, 2018, Vol. 6, No. 3, 107-110
Available online at http://pubs.sciepub.com/tjant/6/3/8

DOI:10.12691/tjant-6-3-8

The Log-concavity Property Associated to Hyperjacobsthal and Hyperjacobsthal-Lucas Sequences

Moussa Ahmia[1,*], Hacène Belbachir[2]

[1]Department of Mathematic, University of Mohamed Seddik Ben yahia, Jijel, Algeria
[2]Faculty of Mathematics, University of Sciences and Technology Houari Boumediene, Algiers, Algeria
*Corresponding author: ahmiamoussa@email.com

Received January 20, 2018; Revised March 26, 2018; Accepted May 12, 2018

Abstract In this paper, we show the log-concavity properties for the hyperjacobsthal, hyperjacobsthal-Lucas and associated sequences. Further, we investigate the q-log-concavity property.

Keywords: *Hyperjacobsthal numbers, hyperjacobsthal-Lucas numbers, log-concavity, q-log-concavity*

Cite This Article: Moussa Ahmia, and Hacène Belbachir, “The Log-concavity Property Associated to Hyperjacobsthal and Hyperjacobsthal-Lucas Sequences.” *Turkish Journal of Analysis and Number Theory*, vol. 6, no. 3 (2018): 107-110. doi: 10.12691/tjant-6-3-8.

1. Introduction

Let $(x_n)_n$ be a sequence of nonnegative numbers. If for all > 0, $x_j^2 \geq x_{j-1}x_{j+1}$ (respectively $x_j^2 \leq x_{j-1}x_{j+1}$), the sequence is called log-concave (respectively log-convex), which is equivalent to $x_i x_j \geq x_{i-1}x_{j+1}$ (respectively $x_i x_j \geq x_{i-1}x_{j+1}$) for $j \geq i \geq 1$.

The log-concave and log-convex sequences arise often in combinatorics, algebra, geometry, analysis, probability and statistics and have been extensively investigated. We refer the reader to [1,2,3] for log-concavity.

Let $(f_n(q))_{n\geq 0}$ be a sequences of polynomials in q. If for each $n \geq 1$, $f_n^2(q) - f_{n-1}(q)f_{n+1}(q)$ has nonnegative coefficients as a polynomials in q; we say that $(f_n(q))_{n\geq 0}$ is q-log-concave. The q-log-concavity of polynomials have been extensively studied; see for instance [4,5,6].

In [7,8], some properties of hyperfibonacci numbers $F_n^{[r]}$ and hyperlucas numbers $L_n^{[r]}$ are given. For Fibonacci numbers F_n and Lucas numbers L_n, it is well known that (F_{2n+1}) and (L_{2n}) are log-concave (see [9]).

L. Zheng and R. Liu [10] gived some properties of the hyperfibonacci numbers and hyperhucas numbers, and investigated the log-concavity and log-convexity of these numbers. Finaly, they also studied the log-concavity (log-convexity) of generalized hyperfibonacci numbers and hyperlucas numbers. In [11], we established these properties for hyperpell numbers and hyperpell-Lucas numbers.

In section two, we give the definitions and some properties of hyperjacobsthal and hyperjacobsthal-Lucas sequences. In section three, we establish the generating functions of these sequences. In section 4, we discuss their log- concavity. In addition, we investigate the q-log-concavity of some polynomials related to hyperjacobsthal and hyperjacobsthal-Lucas numbers.

2. Definitions

Dil and Mezö [8] introduced the hyperfibonacci numbers $F_n^{[r]}$ and hyperlucas numbers $L_n^{[r]}$ to be

$$F_n^{[r]} = \sum_{k=0}^{n} F_k^{[r-1]}, \text{ with } F_n^{[0]} = F_n,$$

$$L_n^{[r]} = \sum_{k=0}^{n} L_k^{[r-1]}, \text{ with } L_n^{[0]} = L_n,$$

where r is a positive integer, and F_n and L_n are Fibonacci and Lucas numbers, respectively.

Definition 2.1. Let r be positive integer. The hyperjacobsthal numbers $J_n^{[r]}$ and hyperjacobsthal-Lucas numbers $j_n^{[r]}$ are defined as follows

$$J_n^{[r]} = \sum_{k=0}^{n} J_k^{[r-1]}, \quad j_n^{[r]} = \sum_{k=0}^{n} j_k^{[r-1]}.$$

where $J_n^{[0]} = J_n$, $j_n^{[0]} = j_n$, and $(J_n)_n$ and $(j_n)_n$ are Jacobsthal and Jacobsthal-Lucas sequences, respectively.

The initial values of $\left(J_n^{[r]}\right)_n$ and $\left(j_n^{[r]}\right)_n$ are as follows

n	0	1	2	3	4	5	6	7
$J_n^{[1]}$	0	1	2	5	10	21	42	85
$j_n^{[1]}$	2	3	8	15	32	63	128	255

Now we recall some formulas for Jacobsthal and Jacobsthal-Lucas numbers. It is well know that the Binet forms of J_n and j_n are

$$J_n = \frac{2^n - (-1)^n}{3}, \; j_n = 2^n + (-1)^n. \tag{2.0}$$

See for instance [12].

The sequences $(J_n)_n$ and $(j_n)_n$ satisfy the following recurrences

$$W_n = W_{n-1} + 2W_{n-2}, n \geq 2. \tag{2.1}$$

For more details, see for instance [13].

It follows from (2.1) that the following formulas hold:

$$J_{n+2}^2 - J_{n+1}J_{n+3} = (-1)^{n+1} 2^{n+1}, \tag{2.2}$$

$$j_{n+2}^2 - j_{n+1}j_{n+3} = 9(-1)^n 2^{n+1}. \tag{2.3}$$

It is easy to see, for example by induction, that

$$J_{n+1} = 2J_n + (-1)^n, j_{n+1} = 2j_n - 3(-1)^n. \tag{2.4}$$

$$J_n \geq n-1 \text{ and } j_n \geq n \text{ for } n \geq 1. \tag{2.5}$$

The generating function of Jacobsthall numbers and Jacobsthal-Lucas numbers, denoted $G_J(t)$ and $G_j(t)$, are respectively

$$G_J(t) := \sum_{n=0}^{+\infty} J_n t^n = \frac{t}{1-t-2t^2}, \tag{2.6}$$

and

$$G_j(t) := \sum_{n=0}^{+\infty} j_n t^n = \frac{2-2t}{1-t-2t^2}. \tag{2.7}$$

So, we establish the generating function of hyperjacobsthal and hyperjacobsthal-Lucas numbers using respectively

$$J_n^{[r]} = J_{n-1}^{[r]} + J_n^{[r-1]} \text{ and } j_n^{[r]} = j_{n-1}^{[r]} + j_n^{[r-1]}. \tag{2.8}$$

The generating functions of hyperjacobsthal numbers and hyperjacobsthal-Lucas numbers are

$$G_J^{[r]}(t) := \sum_{n=0}^{+\infty} J_n^{[r]} t^n = \frac{t}{(1-t-2t^2)(1-t)^r}, \tag{2.9}$$

and

$$G_j^{[r]}(t) := \sum_{n=0}^{+\infty} j_n^{[r]} t^n = \frac{2-2t}{(1-t-2t^2)(1-t)^r}. \tag{2.10}$$

3. The Log-concavity Property

We start the section by some useful lemmas.

Lemma 3.1. [15] If the sequences $(x_n)_n$ and $(y_n)_n$ are log-concave, then so is their ordinary convolution $z_n = \sum_{k=0}^{n} x_k y_{n-k}$, $n = 0,1,2$ ....

Lemma 3.2. [15] If the sequences $(x_n)_n$ is log-concave, then so is the binomial convolution $z_n = \sum_{k=0}^{n} \binom{n}{k} x_k$, $n = 0,1,2$ ....

The following result deals with the log-concavity of hyperjacobsthal and hyperjacobsthal-Lucas sequences.

Theorem 3.3. The sequences $\left(J_n^{[r]}\right)_{n\geq 0}$ and $\left(j_n^{[r]}\right)_{n\geq 0}$ are log-concave for $r \geq 2$ and $r \geq 3$ respectively.

Proof. To prove the results, we use the following relations

$$J_n^{[2]} = \frac{1}{4}(J_{n+4} - 2n - 5) \text{ and } j_n^{[2]} = \frac{1}{4}(j_{n+4} - 2n - 9) \tag{3.1}$$

When $n = 1$, $\left(J_n^{[2]}\right)^2 - J_{n-1}^{[2]}J_{n+1}^{[2]} = 1 > 0$. When $n \geq 2$ it follows from (2.2), (2.4), (2.5) and (3.1)

$$\begin{aligned}
&\left(J_n^{[2]}\right)^2 - J_{n-1}^{[2]}J_{n+1}^{[2]} \\
&= \frac{1}{16}([J_{n+4} - (2n+5)]^2 \\
&- [J_{n+3} - (2n+3)] \times [J_{n+5} - (2n+7)] \\
&= \frac{1}{4}\left[(-1)^{n+1} 2^{n+1} + (2n-1)J_{n+1} + 2(-1)^{n+1} + 1\right] \\
&\geq \frac{1}{4}\left[(-1)^{n+1} 2^{n+1} + (2n-1) \times \frac{2^n - (-1)^n}{3} - 1\right].
\end{aligned} \tag{3.2}$$

There exist two cases. If n is even, then

$$\left(J_n^{[2]}\right)^2 - J_{n-1}^{[2]}J_{n+1}^{[2]} = \frac{2n-4}{12}\left(2^{n+1} + 1\right) > 0,$$

else

$$\left(J_n^{[2]}\right)^2 - J_{n-1}^{[2]}J_{n+1}^{[2]} = \frac{2n+2}{12}\left(2^{n+1} - 1\right) > 0.$$

Then $\left(J_n^{[2]}\right)_{n\geq 0}$ is log-concave. By induction hypothesis and Lemma 3.1 the sequence $\left(J_n^{[r]}\right)_{n\geq 0}$ $(r \geq 2)$ is log-concave.

One can verify that

$$j_n^{[3]} = \frac{1}{8}\left[j_{n+6} - 2(n+1)(n+9) + 31\right]. \tag{3.3}$$

It follows from (2.3), (2.5) and (3.3) that

$$\begin{aligned}
&\left(j_n^{[3]}\right)^2 - j_{n-1}^{[3]}j_{n+1}^{[3]} \\
&= \frac{1}{16}((j_{n+6} - 2(n+1)(n+9) + 31)^2 \\
&- (j_{n+5} - 2n(n+8) + 31) \\
&\times (j_{n+6} - 2(n+2)(n+10) + 31)) \\
&= \frac{1}{16}[-9(-1)^{n+3} 2^{n+3} + (2n(2n+4) - 1)j_{n+3} \\
&+ 3(4n+17)(-1)^{n+3} + (2n^2 + 20n + 5)].
\end{aligned} \tag{3.4}$$

For $n \geq 1$, there exist two cases. If n is even, we get

$$\left(j_n^{[3]}\right)^2 - j_{n-1}^{[3]}j_{n+1}^{[3]} = \frac{1}{16}\left[8 \times 2^{n+3} + 2n(n+4)2^{n+3}\right],$$

else

$$\begin{aligned}
&\left(j_n^{[3]}\right)^2 - j_{n-1}^{[3]}j_{n+1}^{[3]} \\
&= \frac{1}{16}\left[\left(2n^2 + 8n - 10\right)2^{n+3} + (4n^2 + 40n + 100)\right].
\end{aligned}$$

Hence $\left(j_n^{[3]}\right)_{n\geq 0}$ is log-concave. By induction hypothesis and Lemma 3.1 the sequence $\left(j_n^{[r]}\right)_{n\geq 0}$ $(r \geq 3)$ is log-concave. This completes the proof of Theorem 3.3.

Then we have the following corollary.

Corollary 3.4. The sequences $\left(\sum_{k=0}^{n}\binom{n}{k}J_k^{[r]}\right)_{n\geq 0}$ and $\left(\sum_{k=0}^{n}\binom{n}{k}j_k^{[r]}\right)_{n\geq 0}$ are log-concave for $r \geq 2$ and $r \geq 3$ respectively.

Proof. By Lemma 3.2.

Now we establish the log-concavity of order two of the sequences $\left(J_n^{[r]}\right)_{n\geq 0}$ and $\left(j_n^{[r]}\right)_{n\geq 0}$ for some special sub-sequences.

Theorem 3.5. Let $n \geq 1$

$$T_n := \left(J_n^{[2]}\right)^2 - J_{n-1}^{[2]}J_{n+1}^{[2]} \text{ and } R_n := \left(j_n^{[3]}\right)^2 - j_{n-1}^{[3]}j_{n+1}^{[3]}.$$

Then, the sub-sequences $(T_{2n})_{n\geq 1}$, $(T_{2n+1})_{n\geq 0}$, $(R_{2n})_{n\geq 1}$ and $(R_{2n+1})_{n\geq 0}$ are log-concave.

Proof. From (2.4), we get

$$J_{2n+1}^2 - J_{2n-1}J_{2n+3} = -2^{2n-1},\ n\geq 1. \tag{3.5}$$

$$J_{2n+2}^2 - J_{2n}J_{2n+4} = 2^{2n}, n\geq 1. \tag{3.6}$$

$$j_{2n+3}^2 - j_{2n+1}j_{2n+5} = 9\times 2^{2n+1}, n\geq 1. \tag{3.7}$$

It follows form (3.2) and (3.5) that

$$\begin{aligned}
&T_{2n}^2 - T_{2(n-1)}T_{2(n+1)} \\
&= \frac{1}{16}[-2^{2n+1} + (4n-1)J_{2n+1} + 3]^2 \\
&-\left[-2^{2n-1} + (4n-5)J_{2n-1} + 3\right] \\
&\times\left[-2^{2n-1} + (4n+3)J_{2n+3} + 3\right] \\
&= 2^{2n-1}\left[\frac{7}{9}2^{2n-1} - \frac{10}{9}\right] + \frac{1}{9} > 0.
\end{aligned}$$

Then $(T_{2n})_{n\geq 0}$ is log-concave.

It follows from (3.2) and (3.6)

$$\begin{aligned}
&T_{2n+1}^2 - T_{2n-1}T_{2n+3} \\
&= \frac{1}{16}\left[2^{2n+2} + (4n+1)J_{2n+2} - 1\right]^2 \\
&-\left[2^{2n} + (4n-3)J_{2n-1} - 1\right] \\
&\times\left[-2^{2n+4} + (4n+5)J_{2n+4} - 1\right] \\
&= \frac{1}{16}\left[\begin{array}{l}(4n+1)^2 2^{2n} + 16J_{2n}J_{2n+4} \\ +(2n-8)2^{2n} + (12n+23)\left(2^{2n}+2\right)\end{array}\right].
\end{aligned}$$

Then $(T_{2n+1})_{n\geq 0}$ is log-concave.

Similarly, by applying (3.3) and (3.7), we have

$$\begin{aligned}
&R_{2n}^2 - R_{2(n-1)}R_{2(n+1)} \\
&= \frac{2^{4n+6}}{256}\left[\left(8n^2+16n+8\right)^2 - 64n^2\left(4n^2+4n+4\right)^2\right] \\
&= 2^{4n+2}\left[2n^2+4n+1\right] > 0.
\end{aligned}$$

Then $(R_{2n})_{n\geq 1}$ is log-concave.

By same technic, we obtain

$$R_{2n+1}^2 - R_{2n-1}R_{2n+3} > 0.$$

Then $(R_{2n+1})_{n\geq 0}$ is log-concave. This completes the proof.

Then we have the following corollaries.

Corollary 3.6. The sequences $\left(\sum_{k=0}^{n}\binom{n}{k}T_{2k}^{[r]}\right)_{n\geq 1}$ and $\left(\sum_{k=0}^{n}\binom{n}{k}T_{2k+1}^{[r]}\right)_{n\geq 0}$ are log-concave.

Proof. By Lemma 3.2.

Corollary 3.7. The sequences $\left(\sum_{k=0}^{n}\binom{n}{k}R_{2k}^{[r]}\right)_{n\geq 1}$ and $\left(\sum_{k=0}^{n}\binom{n}{k}R_{2k+1}^{[r]}\right)_{n\geq 0}$ are log-concave.

Proof. By Lemma 3.2.

Now, we establish the q-log-concavity property as follows.

Theorem 3.8. Define, for $r \geq 1$, the polynomials

$$J_{n,r}(q) = \sum_{k=0}^{n} J_k^{[r]}q^k,\ \ j_{n,r}(q) = \sum_{k=0}^{n} j_k^{[r]}q^k.$$

The polynomials $J_{n,r}(q)$ and $j_{n,r}(q)$ are q-log-concave for ($r \geq 2$) and ($r \geq 3$) respectively.

Proof. When ≥ 1, $r \geq 2$,

$$\begin{aligned}
&\left[J_{n,r}(q)\right]^2 - J_{n-1,r}(q)J_{n+1,r}(q) \\
&= \left[\sum_{k=0}^{n} J_k^{[r]}q^k\right]^2 - \sum_{k=0}^{n-1} J_k^{[r]}q^k \times \sum_{k=0}^{n+1} J_k^{[r]}q^k \\
&= \sum_{k=1}^{n}\left[J_k^{[r]}J_n^{[r]} - J_{k-1}^{[r]}J_{n+1}^{[r]}\right]q^{k+n}.
\end{aligned}$$

When ≥ 1, $r \geq 3$, through computation, we get

$$\begin{aligned}
&\left[j_{n,r}(q)\right]^2 - j_{n-1,r}(q)j_{n+1,r}(q) \\
&= J_n^{[r]}q^n + \sum_{k=1}^{n}\left[j_k^{[r]}J_n^{[r]} - j_{k-1}^{[r]}j_{n+1}^{[r]}\right]q^{k+n}.
\end{aligned}$$

As $\left(J_n^{[r]}\right)_{n\geq 0}$ ($r \geq 2$) and $\left(j_n^{[r]}\right)_{n\geq 0}$ ($r \geq 3$) are log-concave, then the polynomials $J_{n,r}(q)$ and $j_{n,r}(q)$ are q-log-concave for ($r \geq 2$) and ($r \geq 3$) respectively.

4. Concluding remarks

We have discussed the log-concavity of hyperjacobsthal numbers and hyperjacobsthal-Lucas numbers. In addition, we estabilished the q-log-concavity of some polynomials related to the both numbers.

Acknowledgements

We would like to thank the referee for useful suggestions and several comments witch involve the quality of the paper.

References

[1] F. Brenti, Log-concave and unimodal sequence in algebra, combinatorics and geometry: an update. Elec. Contemp. Math. 178 (1994, 1997), 71-84.

[2] R. P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, Ann. New York Acad. Sci. 576 (1989), 500-534.

[3] Y. Wang, Y.-N. Yeh, Log-concavity and LC-positivity, J. Combin. Theory Ser. A, 114 (2007), 195-210.

[4] L. M. Butler, The q-log concavity of q-binomial coefficients, J. Combin. Theory Ser. A 54 (1990), 54-63.

[5] W. Y. C. Chen, L. X. W. Wang and A. L. B. Yang, Schur positivity and the q-log-convexity of the Narayana polynomials, J. Algebr. Comb. 32 (2010), 303-338.

[6] B.-X. Zhu, Log-convexity and strong q-log-convexity for some triangular arrays, Adv. in. Appl. Math. 50(4) (2013), 595-606.

[7] N-N. Cao, F-Z. Zhao, Some Properties of Hyperfibonacci and Hyperlucas Numbers, Journal of Integer Sequences, 13(8) (2010), Article 10.8.8.

[8] A. Dil, I. Mezö, A symmetric algorithm for hyperharmonic and Fibonacci numbers,Appl. Math. Comput. 206 (2008), 942-951.

[9] N. J. A. Sloane, On-line Encyclopedia of Integer Sequences, http://oeis.org, (2014).

[10] L.-N. Zheng, R. Liu, On the Log-Concavity of the Hyperfibonacci Numbers and the Hyperlucas Numbers, J. Integer Sequences, Vol. 17 (2014), Article 14.1.4.

[11] M. Ahmia, H. Belbachir, A. Belkhir, The log-concavity and log-convexity properties associated to hyperpell numbers and hyperpell-lucas numbers, Annales Mathematicae et Informaticae. 43 (2014), 3-12.

[12] A. F. Horadam. Jacobsthal Representation Numbers. Fibonacci Quarterly, 34 (1) (1996), 40-54.

[13] A. F. Horadam. Jacobsthal and Pell Curves. The Fibonacci Quarterly 26.1 (1988), 79-83.

[14] K. V. Menon. On the convolution of logarithmically concave sequences, Proc. Amer. Math. Soc, 23 (1969), 439-441.

[15] D. W. Walkup, Pólya sequences, binomial convolution and the union of random sets, J. Appl. Probab, 13 (1976), 76-85.

[16] M. Ahmia, H. Belbachir, Preserving log-concavity and generalized triangles. T. Komatsu (ed.), Diophantine analysis and related fields 2010. NY: American Institute of Physics (AIP). AIP Conference Proceedings 1264 (2010), 81-89.

[17] M. Ahmia, H. Belbachir, Preserving log-convexity for generalized Pascal triangles, Electron. J. Combin. 19(2) (2012), Paper 16, 6 pp.

[18] F. Brenti, Unimodal, log-concave and Pólya frequency sequences in combinatorics, Mem. Amer. Math. Soc. no. 413 (1989).

[19] H. Davenport, G. Pólya, On the product of two power series, Canadian J. Math. 1 (1949), 1-5.

[20] L. Liu, Y. Wang, On the log-convexity of combinatorial sequences, Advances in Applied Mathematics 39(4) (2007), 453-476.