

The Log-concavity Property Associated to Hyperjacobsthal and Hyperjacobsthal-Lucas Sequences

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Abstract In this paper, we show the log-concavity properties for the hyperjacobsthal, hyperjacobsthal-Lucas and associated sequences. Further, we investigate the q -log-concavity property.

Keywords: Hyperjacobsthal numbers, hyperjacobsthal-Lucas numbers, log-concavity, q -log-concavity

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1. Introduction

Let $(x_n)_n$ be a sequence of nonnegative numbers. If for all $n > 0$, $x_j^2 \geq x_{j-1}x_{j+1}$ (respectively $x_j^2 \leq x_{j-1}x_{j+1}$), the sequence is called log-concave (respectively log-convex), which is equivalent to $x_i x_j \geq x_{i-1} x_{j+1}$ (respectively $x_i x_j \leq x_{i-1} x_{j+1}$) for $j \geq i \geq 1$.

The log-concave and log-convex sequences arise often in combinatorics, algebra, geometry, analysis, probability and statistics and have been extensively investigated. We refer the reader to [1,2,3] for log-concavity.

Let $(f_n(q))_{n \geq 0}$ be a sequences of polynomials in q . If for each $n \geq 1$, $f_n^2(q) - f_{n-1}(q)f_{n+1}(q)$ has nonnegative coefficients as a polynomials in q ; we say that $(f_n(q))_{n \geq 0}$ is q -log-concave. The q -log-concavity of polynomials have been extensively studied; see for instance [4,5,6].

In [7,8], some properties of hyperfibonacci numbers $F_n^{[r]}$ and hyperlucas numbers $L_n^{[r]}$ are given. For Fibonacci numbers F_n and Lucas numbers L_n , it is well known that (F_{2n+1}) and (L_{2n}) are log-concave (see [9]).

L. Zheng and R. Liu [10] gave some properties of the hyperfibonacci numbers and hyperlucas numbers, and investigated the log-concavity and log-convexity of these numbers. Finally, they also studied the log-concavity (log-convexity) of generalized hyperfibonacci numbers and hyperlucas numbers. In [11], we established these properties for hyperpell numbers and hyperpell-Lucas numbers.

In section two, we give the definitions and some properties of hyperjacobsthal and hyperjacobsthal-Lucas sequences. In section three, we establish the generating functions of these sequences. In section 4, we discuss their log-concavity. In addition, we investigate the q -log-concavity of some polynomials related to hyperjacobsthal and hyperjacobsthal-Lucas numbers.

2. Definitions

Dil and Mezö [8] introduced the hyperfibonacci numbers $F_n^{[r]}$ and hyperlucas numbers $L_n^{[r]}$ to be

$$F_n^{[r]} = \sum_{k=0}^n F_k^{[r-1]}, \text{ with } F_n^{[0]} = F_n,$$

$$L_n^{[r]} = \sum_{k=0}^n L_k^{[r-1]}, \text{ with } L_n^{[0]} = L_n,$$

where r is a positive integer, and F_n and L_n are Fibonacci and Lucas numbers, respectively.

Definition 2.1. Let r be positive integer. The hyperjacobsthal numbers $J_n^{[r]}$ and hyperjacobsthal-Lucas numbers $j_n^{[r]}$ are defined as follows

$$J_n^{[r]} = \sum_{k=0}^n J_k^{[r-1]}, \quad j_n^{[r]} = \sum_{k=0}^n j_k^{[r-1]}.$$

where $J_n^{[0]} = J_n$, $j_n^{[0]} = j_n$, and $(J_n)_n$ and $(j_n)_n$ are Jacobsthal and Jacobsthal-Lucas sequences, respectively.

The initial values of $(J_n^{[r]})_n$ and $(j_n^{[r]})_n$ are as follows

n	0	1	2	3	4	5	6	7
$J_n^{[1]}$	0	1	2	5	10	21	42	85
$j_n^{[1]}$	2	3	8	15	32	63	128	255

Now we recall some formulas for Jacobsthal and Jacobsthal-Lucas numbers. It is well know that the Binet forms of J_n and j_n are

$$J_n = \frac{2^n - (-1)^n}{3}, \quad j_n = 2^n + (-1)^n. \quad (2.0)$$

See for instance [12].

The sequences $(J_n)_n$ and $(j_n)_n$ satisfy the following recurrences

$$W_n = W_{n-1} + 2W_{n-2}, n \geq 2. \tag{2.1}$$

For more details, see for instance [13].

It follows from (2.1) that the following formulas hold:

$$J_{n+2}^2 - J_{n+1}J_{n+3} = (-1)^{n+1} 2^{n+1}, \tag{2.2}$$

$$j_{n+2}^2 - j_{n+1}j_{n+3} = 9(-1)^n 2^{n+1}. \tag{2.3}$$

It is easy to see, for example by induction, that

$$J_{n+1} = 2J_n + (-1)^n, j_{n+1} = 2j_n - 3(-1)^n. \tag{2.4}$$

$$J_n \geq n-1 \text{ and } j_n \geq n \text{ for } n \geq 1. \tag{2.5}$$

The generating function of Jacobsthal numbers and Jacobsthal-Lucas numbers, denoted $G_J(t)$ and $G_j(t)$, are respectively

$$G_J(t) := \sum_{n=0}^{+\infty} J_n t^n = \frac{t}{1-t-2t^2}, \tag{2.6}$$

and

$$G_j(t) := \sum_{n=0}^{+\infty} j_n t^n = \frac{2-2t}{1-t-2t^2}. \tag{2.7}$$

So, we establish the generating function of hyperjacobsthal and hyperjacobsthal-Lucas numbers using respectively

$$J_n^{[r]} = J_{n-1}^{[r]} + J_n^{[r-1]} \text{ and } j_n^{[r]} = j_{n-1}^{[r]} + j_n^{[r-1]}. \tag{2.8}$$

The generating functions of hyperjacobsthal numbers and hyperjacobsthal-Lucas numbers are

$$G_J^{[r]}(t) := \sum_{n=0}^{+\infty} J_n^{[r]} t^n = \frac{t}{(1-t-2t^2)(1-t)^r}, \tag{2.9}$$

and

$$G_j^{[r]}(t) := \sum_{n=0}^{+\infty} j_n^{[r]} t^n = \frac{2-2t}{(1-t-2t^2)(1-t)^r}. \tag{2.10}$$

3. The Log-concavity Property

We start the section by some useful lemmas.

Lemma 3.1. [15] If the sequences $(x_n)_n$ and $(y_n)_n$ are log-concave, then so is their ordinary convolution $z_n = \sum_{k=0}^n x_k y_{n-k}$, $n = 0, 1, 2, \dots$

Lemma 3.2. [15] If the sequences $(x_n)_n$ is log-concave, then so is the binomial convolution $z_n = \sum_{k=0}^n \binom{n}{k} x_k$, $n = 0, 1, 2, \dots$

The following result deals with the log-concavity of hyperjacobsthal and hyperjacobsthal-Lucas sequences.

Theorem 3.3. The sequences $(J_n^{[r]})_{n \geq 0}$ and $(j_n^{[r]})_{n \geq 0}$ are log-concave for $r \geq 2$ and $r \geq 3$ respectively.

Proof. To prove the results, we use the following relations

$$J_n^{[2]} = \frac{1}{4}(J_{n+4} - 2n - 5) \text{ and } j_n^{[2]} = \frac{1}{4}(j_{n+4} - 2n - 9) \tag{3.1}$$

When $n = 1$, $(J_n^{[2]})^2 - J_{n-1}^{[2]} J_{n+1}^{[2]} = 1 > 0$. When $n \geq 2$ it follows from (2.2), (2.4), (2.5) and (3.1)

$$\begin{aligned} & (J_n^{[2]})^2 - J_{n-1}^{[2]} J_{n+1}^{[2]} \\ &= \frac{1}{16} ([J_{n+4} - (2n+5)]^2 \\ & - [J_{n+3} - (2n+3)] \times [J_{n+5} - (2n+7)]) \\ &= \frac{1}{4} [(-1)^{n+1} 2^{n+1} + (2n-1)J_{n+1} + 2(-1)^{n+1} + 1] \\ &\geq \frac{1}{4} \left[(-1)^{n+1} 2^{n+1} + (2n-1) \times \frac{2^n - (-1)^n}{3} - 1 \right]. \end{aligned} \tag{3.2}$$

There exist two cases. If n is even, then

$$(J_n^{[2]})^2 - J_{n-1}^{[2]} J_{n+1}^{[2]} = \frac{2n-4}{12} (2^{n+1} + 1) > 0,$$

else

$$(J_n^{[2]})^2 - J_{n-1}^{[2]} J_{n+1}^{[2]} = \frac{2n+2}{12} (2^{n+1} - 1) > 0.$$

Then $(J_n^{[2]})_{n \geq 0}$ is log-concave. By induction hypothesis and Lemma 3.1 the sequence $(J_n^{[r]})_{n \geq 0}$ ($r \geq 2$) is log-concave.

One can verify that

$$j_n^{[3]} = \frac{1}{8} [j_{n+6} - 2(n+1)(n+9) + 31]. \tag{3.3}$$

It follows from (2.3), (2.5) and (3.3) that

$$\begin{aligned} & (j_n^{[3]})^2 - j_{n-1}^{[3]} j_{n+1}^{[3]} \\ &= \frac{1}{16} ((j_{n+6} - 2(n+1)(n+9) + 31)^2 \\ & - (j_{n+5} - 2n(n+8) + 31) \\ & \times (j_{n+6} - 2(n+2)(n+10) + 31)) \\ &= \frac{1}{16} [-9(-1)^{n+3} 2^{n+3} + (2n(2n+4) - 1)j_{n+3} \\ & + 3(4n+17)(-1)^{n+3} + (2n^2 + 20n + 5)]. \end{aligned} \tag{3.4}$$

For $n \geq 1$, there exist two cases. If n is even, we get

$$(j_n^{[3]})^2 - j_{n-1}^{[3]} j_{n+1}^{[3]} = \frac{1}{16} [8 \times 2^{n+3} + 2n(n+4)2^{n+3}], \text{ ?}$$

else

$$\begin{aligned} & (j_n^{[3]})^2 - j_{n-1}^{[3]} j_{n+1}^{[3]} \\ &= \frac{1}{16} [(2n^2 + 8n - 10)2^{n+3} + (4n^2 + 40n + 100)]. \end{aligned}$$

Hence $(j_n^{[3]})_{n \geq 0}$ is log-concave. By induction hypothesis and Lemma 3.1 the sequence $(j_n^{[r]})_{n \geq 0}$ ($r \geq 3$) is log-concave. This completes the proof of Theorem 3.3.

Then we have the following corollary.

Corollary 3.4. The sequences $(\sum_{k=0}^n \binom{n}{k} J_k^{[r]})_{n \geq 0}$ and $(\sum_{k=0}^n \binom{n}{k} j_k^{[r]})_{n \geq 0}$ are log-concave for $r \geq 2$ and $r \geq 3$ respectively.

Proof. By Lemma 3.2.

Now we establish the log-concavity of order two of the sequences $(J_n^{[r]})_{n \geq 0}$ and $(j_n^{[r]})_{n \geq 0}$ for some special sub-sequences.

Theorem 3.5. Let $n \geq 1$

$$T_n := (J_n^{[2]})^2 - J_{n-1}^{[2]} J_{n+1}^{[2]} \text{ and } R_n := (j_n^{[3]})^2 - j_{n-1}^{[3]} j_{n+1}^{[3]}.$$

Then, the sub-sequences $(T_{2n})_{n \geq 1}$, $(T_{2n+1})_{n \geq 0}$, $(R_{2n})_{n \geq 1}$ and $(R_{2n+1})_{n \geq 0}$ are log-concave.

Proof. From (2.4), we get

$$J_{2n+1}^2 - J_{2n-1} J_{2n+3} = -2^{2n-1}, \quad n \geq 1. \tag{3.5}$$

$$J_{2n+2}^2 - J_{2n} J_{2n+4} = 2^{2n}, \quad n \geq 1. \tag{3.6}$$

$$j_{2n+3}^2 - j_{2n+1} j_{2n+5} = 9 \times 2^{2n+1}, \quad n \geq 1. \tag{3.7}$$

It follows from (3.2) and (3.5) that

$$\begin{aligned} & T_{2n}^2 - T_{2(n-1)} T_{2(n+1)} \\ &= \frac{1}{16} [-2^{2n+1} + (4n-1)J_{2n+1} + 3]^2 \\ & \quad - [-2^{2n-1} + (4n-5)J_{2n-1} + 3] \\ & \quad \times [-2^{2n-1} + (4n+3)J_{2n+3} + 3] \\ &= 2^{2n-1} \left[\frac{7}{9} 2^{2n-1} - \frac{10}{9} \right] + \frac{1}{9} > 0. \end{aligned}$$

Then $(T_{2n})_{n \geq 0}$ is log-concave.

It follows from (3.2) and (3.6)

$$\begin{aligned} & T_{2n+1}^2 - T_{2n-1} T_{2n+3} \\ &= \frac{1}{16} [2^{2n+2} + (4n+1)J_{2n+2} - 1]^2 \\ & \quad - [2^{2n} + (4n-3)J_{2n-1} - 1] \\ & \quad \times [-2^{2n+4} + (4n+5)J_{2n+4} - 1] \\ &= \frac{1}{16} \left[(4n+1)^2 2^{2n} + 16J_{2n} J_{2n+4} \right. \\ & \quad \left. + (2n-8)2^{2n} + (12n+23)(2^{2n} + 2) \right]. \end{aligned}$$

Then $(T_{2n+1})_{n \geq 0}$ is log-concave.

Similarly, by applying (3.3) and (3.7), we have

$$\begin{aligned} & R_{2n}^2 - R_{2(n-1)} R_{2(n+1)} \\ &= \frac{2^{4n+6}}{256} \left[(8n^2 + 16n + 8)^2 - 64n^2 (4n^2 + 4n + 4)^2 \right] \\ &= 2^{4n+2} [2n^2 + 4n + 1] > 0. \end{aligned}$$

Then $(R_{2n})_{n \geq 1}$ is log-concave.

By same technic, we obtain

$$R_{2n+1}^2 - R_{2n-1} R_{2n+3} > 0.$$

Then $(R_{2n+1})_{n \geq 0}$ is log-concave. This completes the proof.

Then we have the following corollaries.

Corollary 3.6. The sequences $(\sum_{k=0}^n \binom{n}{k} T_{2k}^{[r]})_{n \geq 1}$ and $(\sum_{k=0}^n \binom{n}{k} T_{2k+1}^{[r]})_{n \geq 0}$ are log-concave.

Proof. By Lemma 3.2.

Corollary 3.7. The sequences $(\sum_{k=0}^n \binom{n}{k} R_{2k}^{[r]})_{n \geq 1}$ and $(\sum_{k=0}^n \binom{n}{k} R_{2k+1}^{[r]})_{n \geq 0}$ are log-concave.

Proof. By Lemma 3.2.

Now, we establish the q -log-concavity property as follows.

Theorem 3.8. Define, for $r \geq 1$, the polynomials

$$J_{n,r}(q) = \sum_{k=0}^n J_k^{[r]} q^k, \quad j_{n,r}(q) = \sum_{k=0}^n j_k^{[r]} q^k.$$

The polynomials $J_{n,r}(q)$ and $j_{n,r}(q)$ are q -log-concave for $(r \geq 2)$ and $(r \geq 3)$ respectively.

Proof. When $n \geq 1, r \geq 2$,

$$\begin{aligned} & [J_{n,r}(q)]^2 - J_{n-1,r}(q) J_{n+1,r}(q) \\ &= \left[\sum_{k=0}^n J_k^{[r]} q^k \right]^2 - \sum_{k=0}^{n-1} J_k^{[r]} q^k \times \sum_{k=0}^{n+1} J_k^{[r]} q^k \\ &= \sum_{k=1}^n [J_k^{[r]} J_n^{[r]} - J_{k-1}^{[r]} J_{n+1}^{[r]}] q^{k+n}. \end{aligned}$$

When $n \geq 1, r \geq 3$, through computation, we get

$$\begin{aligned} & [j_{n,r}(q)]^2 - j_{n-1,r}(q) j_{n+1,r}(q) \\ &= J_n^{[r]} q^n + \sum_{k=1}^n [j_k^{[r]} j_n^{[r]} - j_{k-1}^{[r]} j_{n+1}^{[r]}] q^{k+n}. \end{aligned}$$

As $(J_n^{[r]})_{n \geq 0}$ ($r \geq 2$) and $(j_n^{[r]})_{n \geq 0}$ ($r \geq 3$) are log-concave, then the polynomials $J_{n,r}(q)$ and $j_{n,r}(q)$ are q -log-concave for $(r \geq 2)$ and $(r \geq 3)$ respectively.

4. Concluding remarks

We have discussed the log-concavity of hyperjacobsthal numbers and hyperjacobsthal-Lucas numbers. In addition, we established the q -log-concavity of some polynomials related to the both numbers.

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