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Symmetric Functions for *k*-Fibonacci Numbers and Orthogonal Polynomials

Khadidja Boubellouta[1,*], Ali Boussayoud[2], Mohamed Kerada[3]

[1]LMAM Laboratory and Department of Mathematic, University Center of Mila, Mila, Algeria
[2]LMAM Laboratory and Department of Mathematic, Mohamed Seddik Ben Yahia University, Jijel, Algeria
*Corresponding author: boubelouta.khadidja@gmail.com

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Abstract In this paper, we derive new generating functions for the products of *k*-Fibonacci numbers, *k*-Pell numbers, *k*-Jacobsthal numbers and the Chebychev polynomials of the second kind by making use of useful properties of the symmetric functions.

Keywords: *k-Fibonacci numbers, k-Pell numbers, k-Jacobsthal numbers, generating functions, symmetric functions*

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1. Introduction and Preliminaries

Fibonacci and Lucas numbers have been studied by many researchers for a long time to get intrinsic theory and applications of these numbers in many research areas as Physics, Engineering, Architecture, Nature and Art. For example, the ratio of two consecutive numbers converges to the Golden ratio $\alpha = \frac{1+\sqrt{5}}{2}$ which was thoroughly interested in [1]. We should recall that , for $k \in \mathbb{R}_+$, *k*-Fibonacci $\{F_{k,n}\}_{n\in\mathbb{N}}$ and *k*-Jacobsthal $\{J_{k,n}\}_{n\in\mathbb{N}}$ sequences have been defined by the recursive equations [2,3];

$$\begin{cases} F_{k,n+2} = kF_{k,n+1} + F_{k,n} \\ F_{k,0} = 0,\ F_{k,1} = 1 \end{cases}$$

and $\begin{cases} J_{k,n+2} = kJ_{k,n+1} + 2J_{k,n} \\ J_{k,0} = 0,\ J_{k,1} = 1 \end{cases}$

For the special case $k = 1$, it is clear that these two sequences are simplified to the well-known Fibonacci and Jacobsthal sequences, respectively. More recently, many papers are dedicated to Fibonacci sequence, such as the works of Caldwell *et al.* in [4], Marques in [5], Shattuck in [6] and Falcon *et al.* in [7].

The main purpose of this paper is to present some results involving the *k*-Fibonacci and *k*-Jacobsthal numbers using define a new useful operator denoted by $\delta^k_{b_1,b_2}$. By making use of this operator, we can derive new results based on our previous ones [8,9,10,11,12]. In order to determine generating functions of the product of *k*-Fibonacci and *k*-Jacobsthal numbers and Chebychev polynomials of second kind, we combine between our indicated past techniques and these presented polishing approaches.

Here, we recall some basic definitions and theorems that are needed in the sequel.

Definition 1. [2] *Let A and B be any two alphabets, then we give* $S_n(A-B)$ *by the following form*

$$\frac{\prod_{b\in B}(1-bt)}{\prod_{a\in A}(1-at)} = \sum_{n\geq 0} S_n(A-B)t^n \tag{1}$$

with the condition $S_n(A-B) = 0$ for $n < 0$.

Definition 2. [13] *Taking* $A = \{0,0,...,0\}$ *in* (1) *gives*

$$\prod_{b\in B}(1-bt) = \sum_{n\geq 0} S_n(-B)t^n. \tag{2}$$

Definition 3. [14] *Given a function* f *on* $\mathbb{R}^n$, *the divided difference operator is defined as follows*

$$\partial_{x_i,x_{i+1}}(f) = \frac{f(x_1,...,x_i,x_{i+1},...x_n) - f(x_1,...,x_{i+1},x_i,...x_n)}{x_i - x_{i+1}}.$$

Definition 4. [15] *The symmetrizing operator* $\delta^k_{b_1b_2}$ *is defined by*

$$\delta^k_{b_1,b_2}(g) = \frac{b_1^k g(b_1) - b_2^k g(b_2)}{b_1 - b_2},$$

for all $k \in \mathbb{N}$.

Remark 1. *If* $g(b_1) = b_1$, *we have*

$$\delta^k_{b_1b_2}(g) = S_k(b_1 + b_2).$$

2. Main Results

In this section, we combine all results obtained here in a unified way such that they can be considered as special cases of the following Theorems.

Theorem 1. *Given two alphabets* $A=\{a_1,a_2\}$ *and* $B=\{b_1,b_2\}$ *then*

$$\sum_{n=0}^{\infty} S_n(a_1+a_2)S_{n-1}(b_1+b_2)t^n = \frac{S_1(a_1+a_2)t - a_1a_2S_1(b_1+b_2)t^2}{\left(\sum_{n=0}^{\infty} S_n(-A_2)b_1^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-A_2)b_2^n t^n\right)} \quad (3)$$

Proof. Let $\sum_{n=0}^{\infty} S_n(A)t^n$ and $\sum_{n=0}^{\infty} S_n(-A)t^n$ be two sequences such that $\sum_{n=0}^{\infty} S_n(A)t^n = \frac{1}{\sum_{n=0}^{\infty} S_n(-A)t^n}$.

On one hand, since $f(b_1)=\sum_{n=0}^{\infty} S_n(A)b_1^n t^n$ and $f(b_2)=\sum_{n=0}^{\infty} S_n(A)b_2^n t^n$, we have

$$\partial_{b_1b_2} f(b_1) = \partial_{b_1b_2}\left(\sum_{n=0}^{\infty} S_n(a_1+a_2)b_1^n t^n\right) = \sum_{n=0}^{\infty} S_n(a_1+a_2)\frac{b_1^n-b_2^n}{b_1-b_2}t^n = \sum_{n=0}^{\infty} S_n(a_1+a_2)S_{n-1}(b_1+b_2)t^n$$

which is the left hand side of (3). On the other hand, since

$$f(b_1) = \frac{1}{\sum_{n=0}^{\infty} S_n(-A_2)b_1^n t^n}$$

we have that

$$\partial_{b_1b_2} f(b_1) = \frac{\frac{1}{\sum_{n=0}^{\infty} S_n(-A_2)b_1^n t^n} - \frac{1}{\sum_{n=0}^{\infty} S_n(-A_2)b_2^n t^n}}{b_1-b_2}$$

$$= \frac{\sum_{n=0}^{\infty} S_n(-A_2)b_2^n t^n - \sum_{n=0}^{\infty} S_n(-A_2)b_1^n t^n}{(b_1-b_2)\left(\sum_{n=0}^{\infty} S_n(-A_2)b_1^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-A_2)b_2^n t^n\right)}$$

$$= \frac{\prod_{a\in A_2}(1-ab_2t) - \prod_{a\in A_2}(1-ab_1t)}{(b_1-b_2)\left(\sum_{n=0}^{\infty} S_n(-A_2)b_1^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-A_2)b_2^n t^n\right)}$$

$$= \frac{S_1(a_1+a_2)t - a_1a_2S_1(b_1+b_2)t^2}{\left(\sum_{n=0}^{\infty} S_n(-A_2)b_1^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-A_2)b_2^n t^n\right)}.$$

So, this completes the proof.

Theorem 2. [16] *Given two alphabets* $A=\{a_1,a_2\}$ *and* $B=\{b_1,b_2\}$ *we have*:

$$\sum_{n=0}^{\infty} S_n(a_1+a_2)S_n(b_1+b_2)t^n = \frac{1-a_1a_2b_1b_2t^2}{\left(\sum_{n=0}^{\infty} S_n(-A_2)b_1^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-A_2)b_2^n t^n\right)} \quad (4)$$

Theorem 3. *Given two alphabets* $A=\{a_1,a_2\}$ *and* $B=\{b_1,b_2\}$ *we have*

$$\sum_{n=0}^{\infty} S_{n-1}(a_1+a_2)S_{n-1}(b_1+b_2)t^n = \frac{t-a_1a_2b_1b_2t^3}{\left(\sum_{n=0}^{\infty} S_n(-A_2)b_1^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-A_2)b_2^n t^n\right)} \quad (5)$$

3. On The Symmetric and Generating Functions

In this section, the new generating functions of the products of k-Fibonacci numbers, k-Pell numbers, k-Jacobsthal numbers and the Chebychev polynomials of the second kind are given by using the previous theorems.

Case 1: Replacing b_2 by $(-b_2)$ and a_2 by $(-a_2)$ in (5) yields

$$\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(b_1+[-b_2])t^n = \frac{t-a_1a_2b_1b_2t^3}{(1-a_1b_1t)(1+a_2b_1t)(1+a_1b_2t)(1-a_2b_2t)}. \quad (6)$$

This case consists of two related parts.

Firstly, the substitutions of

$$\begin{cases} a_1-a_2=k \\ a_1a_2=1 \end{cases} \text{and} \begin{cases} b_1-b_2=2 \\ b_1b_2=k \end{cases}.$$

In (6), we deduce the following theorem

Theorem 4. *We have the following a new generating function of the product of k-Fibonacci numbers and k-Pell numbers is given by*

$$\sum_{n=0}^{\infty} F_{k,n}P_{k,n}t^n = \frac{t-kt^3}{1-2kt-\left(k^3+2k+4\right)t^2-2k^2t^3+k^2t^4} \quad (7)$$

Corollary 1. *If* $k=1$ *in the relationship* (7) *we get*

$$\sum_{n=0}^{\infty} F_n P_n t^n = \frac{t-t^3}{1-2t-7t^2-2t^3+t^4}.$$

which represents a generating function of the product of Fibonacci numbers and Pell numbers [17].

Secondly, the substitution of

$$\begin{cases} a_1-a_2=2 \\ a_1a_2=k \end{cases} \text{and} \begin{cases} b_1-b_2=k \\ b_1b_2=2 \end{cases}$$

in (6) yields

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1+[-a_2]\right)S_{n-1}\left(b_1+[-b_2]\right)t^n = \frac{t-2kt^3}{1-2kt-(k^3+4k+8)-4k^2t^3+4k^2t^4}.$$

We deduce the following theorem.

Theorem 5. *For* $n\in\mathbb{N}$, *the new generating function of the product of k-Pell numbers and k-Jacobsthal numbers is given by*

$$\sum_{n=0}^{\infty} P_{k,n}J_{k,n}t^n = \frac{t-2kt^3}{1-2kt-\left(k^3+4k+8\right)t^2-4k^2t^3+4k^2t^4}. \quad (8)$$

Corollary 2. *If* $k=1$ in the *relationship* (8) *we get*

$$\sum_{n=0}^{\infty} P_n J_n t^n = \frac{t-2t^3}{1-2t-13t^2-4t^3+4t^4}.$$

which represents a generating function of the product of Pell numbers and Jacobsthal numbers [17].

Case 2. Replacing a_2 by $(-a_2)$ and b_2 by $(-b_2)$ in (4) yields

$$\sum_{n=0}^{\infty} S_n\left(a_1+[-a_2]\right)S_n\left(b_1+[-b_2]\right)t^n = \frac{1-a_1a_2b_1b_2t^2}{(1-a_1b_1t)(1+a_2b_1t)(1+a_1b_2t)(1-a_2b_2t)}. \quad (9)$$

The substitution of

$$\begin{cases} a_1-a_2=k \\ a_1a_2=1 \end{cases} \textit{and} \begin{cases} b_1-b_2=k \\ b_1b_2=2 \end{cases}$$

in (9), we deduce the following theorem.

Theorem 6. *For* $n\in\mathbb{N}$ *the new generating function of the produce of k-Fibonacci numbers and k-Jacobsthal numbers is given by*

$$\sum_{n=0}^{\infty} F_{k,n}J_{k,n}t^n = \frac{1-2t^2}{1-k^2t-(3k^2+4)t^2-2k^2t^3+4t^4}. \quad (10)$$

Corollary 3. *In the special case* $k=1$ *identity* (10) *gives*

$$\sum_{n=0}^{\infty} F_n J_n t^n = \frac{1-2t^2}{1-t-7t^2-2t^3+4t^4}.$$

which represents a generating function of the product of Fibonacci numbers and Jacobsthal numbers [17].

Case 3: Remplacing a_1 by $[2a_1]$, a_2 by $[-2a_2]$ and b_2 by $[-b_2]$ in (3) yields

$$\sum_{n=0}^{\infty} S_{n-1}\left(2a_1+[-2a_2]\right)S_{n-1}\left(b_1+[-b_2]\right)t^n = \frac{S_1\left(2a_1+[-2a_2]\right)t+4a_1a_2S_1\left(b_1+[-b_2]\right)t^2}{(1-2a_1b_1t)(1+2a_2b_1t)(1+2a_1b_2t)(1-2a_2b_2t)}. \quad (11)$$

This case consists of three related parts.

Firstly, the substitutions of $\begin{cases} b_1-b_2=k \\ b_1b_2=1 \\ 4a_1a_2=-1 \end{cases}$ in (11), we deduce the following theorem.

Theorem 7. [2] *We have a generating function of the product of k-Fibonacci numbers and Chebychev polynomial of the second kind*

$$\sum_{n=0}^{\infty} F_{k,n}U_n\left(a_1-a_2\right)t^n = \frac{2\left(a_1-a_2\right)t-kt^2}{\left[\begin{array}{c}1-2k\left(a_1-a_2\right)t-(4\left(a_1-a_2\right)^2-k^2-2)t^2 \\ +2k\left(a_1-a_2\right)t^3+t^4\end{array}\right]}. \quad (12)$$

Secondly, the substitution of $\begin{cases} b_1-b_2=2 \\ b_1b_2=k \\ 4a_1a_2=-1 \end{cases}$ in (11), we deduce the following theorem.

Theorem 8. *For* $n\in\mathbb{N}$, *the new generating function of the product of k-Pell numbers and Chebychev polynomial of the second kind is given by*

$$\sum_{n=0}^{\infty} P_{k,n}U_n\left(a_1-a_2\right)t^n = \frac{2\left(a_1-a_2\right)t-2t^2}{\left[\begin{array}{c}1-4\left(a_1-a_2\right)t-(4k\left(a_1-a_2\right)^2-2k-4)t^2 \\ +4k\left(a_1-a_2\right)t^3+k^2t^4\end{array}\right]}. \quad (13)$$

- if $k=1$ in the relationship (13) we get [2]

$$\sum_{n=0}^{\infty} P_nU_n\left(a_1-a_2\right)t^n = \frac{2\left(a_1-a_2\right)t-2t^2}{\left[\begin{array}{c}1-4\left(a_1-a_2\right)t-(4\left(a_1-a_2\right)^2-6)t^2 \\ +4\left(a_1-a_2\right)t^3+t^4\end{array}\right]}.$$

which represents a generating function of the product of Pell numbers and Chebychev polynomial of the second kind.

Finally, the substitution of $\begin{cases} b_1-b_2=k \\ b_1b_2=2 \\ 4a_1a_2=-1 \end{cases}$ in (11) gives

$$\sum_{n=0}^{\infty} S_n\left(2a_1+[-2a_2]\right)S_{n-1}\left(b_1+[-b_2]\right)t^n$$
$$=\frac{2\left(a_1-a_2\right)t-kt^2}{\left[\begin{array}{c}1-2k\left(a_1-a_2\right)t-(8\left(a_1-a_2\right)^2-k^2-4)t^2\\+4k\left(a_1-a_2\right)t^3+4t^4\end{array}\right]}.$$

We deduce the following theorem.

Theorem 9. *We obtain a new generating function of the product of k-Jacobsthal numbers and Chebychev polynomial of the second kind as*

$$\sum_{n=0}^{\infty} J_{k,n}U_n\left(a_1-a_2\right)t^n$$
$$=\frac{2\left(a_1-a_2\right)t-kt^2}{\left[\begin{array}{c}1-2k\left(a_1-a_2\right)t-(8\left(a_1-a_2\right)^2-k^2-4)t^2\\+4k\left(a_1-a_2\right)t^3+4t^4\end{array}\right]}. \quad (14)$$

Corollary 4. *If* $k=1$ *in the relationship* (14) *we get*

$$\sum_{n=0}^{\infty} J_nU_n\left(a_1-a_2\right)t^n$$
$$=\frac{2\left(a_1-a_2\right)t-t^2}{\left[\begin{array}{c}1-2\left(a_1-a_2\right)t-(8\left(a_1-a_2\right)^2-5)t^2\\+4\left(a_1-a_2\right)t^3+4t^4\end{array}\right]}$$

which represents a new generating function of the product of Jacobsthal numbers and Chebychev polynomial of the second kind.

Case 4. *Replacing* b_2 *by* $(-b_2)$ *and* a_2 *by* $(-a_2)$ *in (3) and (5) yields*

$$\sum_{n=0}^{\infty} S_n\left(a_1+[-a_2]\right)S_{n-1}\left(b_1+[-b_2]\right)t^n$$
$$=\frac{\left(a_1-a_2\right)t+a_1a_2\left(b_1-b_2\right)t^3}{\left(1-a_1b_1t\right)\left(1+a_2b_1t\right)\left(1+a_1b_2t\right)\left(1-a_2b_2t\right)}. \quad (15)$$

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1+[-a_2]\right)S_{n-1}\left(b_1+[-b_2]\right)t^n$$
$$=\frac{t-a_1a_2b_1b_2t^3}{\left(1-a_1b_1t\right)\left(1+a_2b_1t\right)\left(1+a_1b_2t\right)\left(1-a_2b_2t\right)}. \quad (16)$$

This case consists of three related parts.

Firstly, the substitution of

$$\begin{cases}a_1-a_2=k\\ a_1a_2=1\end{cases} \text{and} \begin{cases}b_1-b_2=k\\ b_1b_2=1\end{cases}$$

In (15) and (16), we obtain

$$\sum_{n=0}^{\infty} S_n\left(a_1+[-a_2]\right)S_{n-1}\left(b_1+[-b_2]\right)t^n$$
$$=\frac{kt-kt^2}{1-k^2t-\left(2k^2+2\right)t^2-k^2t^3+t^4}. \quad (17)$$

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1+[-a_2]\right)S_{n-1}\left(b_1+[-b_2]\right)t^n$$
$$=\frac{t-t^3}{1-k^2t-\left(2k^2+2\right)t^2-k^2t^3+t^4}. \quad (18)$$

Multiplying the equation (17) by 2 and added to (18) by -1, we obtain

$$\sum_{n=0}^{\infty} F_{k,n}L_{k,n}t^n=\frac{(2k-1)t+2kt^2+t^3}{1-k^2t-\left(2k^2+2\right)t^2-k^2t^3+t^4}. \quad (19)$$

which represents a new generating function of the product of k-Fibonacci numbers and k-Lucas numbers.

- For $k=1$ in (19) we obtain

$$\sum_{n=0}^{\infty} F_{2n}t^n=\frac{t+2t^2+t^3}{1-t-4t^2-t^3+t^4}=\frac{t}{1-3t+t^2}$$

which represents a generating function of even indices of Fibonacci numbers [17].

Secondly, the substitution of

$$\begin{cases}a_1-a_2=2\\ a_1a_2=k\end{cases} \text{and} \begin{cases}b_1-b_2=2\\ b_1b_2=k\end{cases}$$

in (15) and (16), we get

$$\sum_{n=0}^{\infty} S_n\left(a_1+[-a_2]\right)S_{n-1}\left(b_1+[-b_2]\right)t^n$$
$$=\frac{2t+2kt^2}{1-4t-\left(2k^2+8k\right)t^2-4k^2t^3+k^4t^4}. \quad (20)$$

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1+[-a_2]\right)S_{n-1}\left(b_1+[-b_2]\right)t^n$$
$$=\frac{t-k^2t^3}{1-4t-\left(2k^2+8k\right)t^2-4k^2t^3+k^4t^4}. \quad (21)$$

Multiplying the equation (20) by 2 and added to (21) by -2, we have

$$\sum_{n=0}^{\infty} P_{k,n}Q_{k,n}t^n$$
$$=\frac{2t+4kt^2+2k^2t^3}{1-4t-\left(2k^2+8k\right)t^2-4k^2t^3+k^4t^4}. \quad (22)$$

which represents a new generating function of the product of k-Pell numbers and k-Pell-Lucas numbers.

Corollary 5. *If* $k=1$ *in the relationship* (22) *we get*

$$\sum_{n=0}^{\infty} P_{2n}t^n=\frac{2t+4t^2+2t^3}{1-4t-10t^2-4t^3+t^4}$$
$$=\frac{2t}{1-6t+t^2}$$

which represents a generating function of even indice of Pell numbers [17].

Thirdly, the substitution of

$$\begin{cases} a_1 - a_2 = k \\ a_1 a_2 = 2 \end{cases} \text{and} \begin{cases} b_1 - b_2 = k \\ b_1 b_2 = 2 \end{cases}$$

in (15) and (16), we get

$$\sum_{n=0}^{\infty} S_n\left(a_1 + [-a_2]\right) S_{n-1}\left(b_1 + [-b_2]\right) t^n = \frac{kt + 2kt^2}{1 - k^2 t - \left(4k^2 + 8\right) t^2 - 4k^2 t^3 + 16t^4} \quad (23)$$

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1 + [-a_2]\right) S_{n-1}\left(b_1 + [-b_2]\right) t^n = \frac{t - 4t^3}{1 - k^2 t - \left(4k^2 + 8\right) t^2 - 4k^2 t^3 + 16t^4}. \quad (24)$$

Multiplying the equation (23) by 2 added to (24) by (-1), we obtain

$$\sum_{n=0}^{\infty} J_{k,n} j_{k,n} t^n = \frac{(2k-1)t + 4kt^2 + 4t^3}{1 - k^2 t - \left(4k^2 + 8\right) t^2 - 4k^2 t^3 + 16t^4} \quad (25)$$

which represents a new generating function of the product of k-Jacobsthal numbers and k-Jacobsthal-Lucas numbers.

Corollary 6. *If* $k = 1$ *in the relationship* (25) *we have*

$$\sum_{n=0}^{\infty} J_{2n} t^n = \frac{t + 4t^2 + 4t^3}{1 - t - 12t^2 - 4t^3 + 16t^4} = \frac{t}{1 - 5t + 4t^2}$$

which represents a generating function of even indices for Jacobsthal numbers [17]

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