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On Multiple Zeta Function and Associated Properties

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Abstract The purpose of this paper is to introduce and investigate a new class of multiple zeta functions of n variables. We study its properties, integral representations, differential relation, series expansion and discuss the link with known results.

Keywords: *multi-variable zeta functions, Hurwitz-Lerch Zeta function, Hyper-geometric function, Lauricella functions, Summation formula, expansion formulas*

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1. Introduction

The generalized (Hurwitz's) zeta function is defined by [1,2]

$$\zeta(s,a) = \sum_{n=0}^{\infty}(a+n)^{-s} (a \neq 0,-1,-2,...;\Re(s)>1), \quad (1.1)$$

so that when $a=1$, we have

$$\zeta(s,1) = \sum_{n=0}^{\infty} n^{-s} = \zeta(s), \quad (1.2)$$

where $\zeta(s)$ is the Riemann zeta function. The function $\Phi(x,s,a)$ extends (1.1) further, and this generalized Hurwitz-Lerch zeta function [[1], p. 316], is defined by

$$\Phi(x,s,a) = \sum_{n=0}^{\infty} \frac{x^n}{(a+n)^s}, \quad (1.3)$$

$(a \neq 0,-1,-2,...,|x|<1;\Re(s)>1)$, when $|x|=1)$?

A generalization of (1.3) is the Zeta function Φ_μ^* which is defined by [[3], p.100, (1.5)]:

$$\Phi_\mu^*(x,s,a) = \sum_{n=0}^{\infty} \frac{(\mu)_n x^n}{(a+n)^s n!}, \quad (1.4)$$

$\mu \in \mathbb{C}; a \in \mathbb{C} \setminus \mathbb{Z}_0^-; |x|<1$, where $(\mu)_n = \frac{\Gamma(\mu+n)}{\Gamma(\mu)}$, for $n=0,1,2,\ldots$, denotes the Pochhammer's symbol and Γ denotes the Gamma function. Evidently, we have

$$\Phi_1^*(x,s,a) = \Phi(x,s,a), \quad (1.5)$$

$$\Phi(1,s,1) = \zeta(s), \quad (1.6)$$

$$\Phi(1,s,a) = \zeta(s,a). \quad (1.7)$$

The zeta functions in (1.3) and (1.4) have since been extended and generalized by a number of workers (see e.g. [3-14]. The present sequel to these earlier papers is motivated largely by the aforementioned works of Matsumoto and Kamano [15,16] in which the zeta function $\zeta(s,a)$ in (1.1) was generalized to the following multiple Hurwitz zeta function

$$\zeta_n(s_1,\ldots,s_n;a) = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{1}{(a+m_1)^{s_1}\ldots(a+m_n)^{s_n}}, \quad (1.8)$$

$\left(a>0;(m_1,\ldots,m_n)\in\mathbb{Z}^n, 0\le m_1<\ldots<m_n;(s_1,\ldots,s_n)\in\mathbb{C}^n\right)$.

In the present paper we introduce a new class of zeta functions $\zeta_n^{\mu,s_1,\ldots,s_n}(x_1,\ldots,x_n;a_1,\ldots,a_n)$ which is defined by

$$\zeta_n^{\mu,s_1,\ldots,s_n}(x_1,\ldots,x_n;a_1,\ldots,a_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\mu)_{m_1+\ldots+m_n}}{(a_1+m_1)^{s_1}\ldots(a_n+m_n)^{s_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}, \quad (1.9)$$

$(\{s_1,\ldots,s_n\}\in C; |x_i|<1, a_i>0, (i=1,2,\ldots,n);$
$\mu \in C\setminus\mathbb{Z}_0^-; 0\le m_1<\ldots<m_n)$.

Clearly, we have the following relationship

$$\zeta_1^{1,s}(x;a) = \Phi(x,s,a),$$

$$\text{and } \zeta_1^{\mu,s}(x;a) = \Phi_\mu^*(x,s,a).$$

In the case when $s=1, (\forall i=1,2,\ldots,n)$, we have simply

$$\zeta_n^{\mu,1,\ldots,1}(x_1,\ldots,x_n;a_1,\ldots,a_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\mu)_{m_1+\ldots+m_n}}{(a_1+m_1)\ldots(a_n+m_n)} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}$$

which implies the next result.

Corollary 1.1. Let $|x_1|+\ldots+|x_n|<1, \mathrm{Re}(a_i)>0$, $(\forall i=1,2,\ldots,n)$. Then

$$\zeta_n^{\mu,1,\ldots,1}(x_1,\ldots,x_n;a_1,\ldots,a_n) = \left(a_1^{-1}\ldots a_n^{-1}\right) F_A^{(n)}\left[\begin{matrix}\mu, a_1,\ldots,a_n; \\ a_1+1,\ldots,a_n+1; \\ x_1,\ldots,x_n\end{matrix}\right] \tag{1.10}$$

where $F_A^{(n)}$ is the Lauricella function of n variables defined by the series (see e.g. [17] and [18]),

$$F_A^{(n)}[a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_s] = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c_1)_{m_1}\ldots(c_n)_{m_1}} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}. \tag{1.11}$$

By using the Hankel's contour integral for Gamma function (for details see e.g. [[19], Section 12.12])

$$\frac{1}{(a)_m} = \frac{\Gamma(a)}{2i\pi}\int t^{-a-m}e^{-t}dt, \tag{1.12}$$

we can derive the following interesting formula.

Corollary 1.2. Let $s_1=\ldots=s_n=1, |x_j|<1, (j=1,2,..,n)$. Then

$$\frac{\Gamma(\mu)}{2i\pi}\int e^t t^{-\mu}\ \zeta_n^{\mu,1,\ldots,1}\left(x_1t^{-1},\ldots,x_nt^{-1};a_1,\ldots,a_n\right)dt = \left(a_1^{-1}\ldots a_n^{-1}\right)\prod_{j=1}^{n}\left\{{}_1F_1\left[\begin{matrix}a_j & ; \\ a_j+1; & x_j\end{matrix}\right]\right\}. \tag{1.13}$$

Proof. The result follow directly from definition (1.9) and the integral representation (1.11).

2. Integral Representations

First, by using Eulerian integral formula of the second kind (see e.g. [2]):

$$(a+n)^{-s} = \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}e^{-(a+n)t}dt, \quad \left(\min\{\mathrm{Re}(s),\mathrm{Re}(a+n)\}>0\right), \tag{2.1}$$

and the formula(see, e.g., [[20], p.67, Eq (22)]) :

$$(1-x)^{-\mu} = \sum_{m=0}^{\infty}\frac{(\mu)_m}{m!}x^m, \tag{2.2}$$

we should proof the following result.

Theorem 2.1. Let $\mathrm{Re}(\mu)>0, \mathrm{Re}(a_i)>0$ and either

$$|x_i|<1, \mathrm{Re}(s_i)>0 \text{ or } x_i=1, \mathrm{Re}(s_i)>1 (\forall i=1,2,\ldots,n, n\in\mathbb{Z}^+).$$

Then

$$\zeta_n^{\mu,s_1,\ldots,s_n}(x_1,\ldots,x_n;a_1,\ldots,a_n) = \frac{1}{\Gamma(s_1)}\ldots\frac{1}{\Gamma(s_n)} \times\int_0^{\infty}\ldots\int_0^{\infty} t_1^{s_1-1}\ldots t_n^{s_n-1}e^{-(a_1t_1+\ldots+a_nt_n)} \times\left(1-x_1e^{-t_1}-\ldots-x_ne^{-t_n}\right)^{-\mu}dt_1\ldots dt_n. \tag{2.3}$$

Proof. Denote, for convenience, the right-hand side of equation (2.3) by *I*. Then in view of (2.2), it is easily seen that

$$I = \sum_{m_1,\ldots,m_n=0}^{\infty}(\mu)_{m_1+\ldots+m_n}\frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}\frac{1}{\Gamma(s_1)}\ldots\frac{1}{\Gamma(s_n)} \times\int_0^{\infty}\ldots\int_0^{\infty} t_1^{s_1-1}e^{-(a_1+m_1)t_1}\ldots t_n^{s_n-1}e^{-(a_n+m_n)t_n}dt_1\ldots dt_n.$$

Upon using (2.1) and in view of the definition (1.9), we led finally to the left-hand side of the formula (2.3). Next, by using the contour integral formula [[2], p.14 (4)]:

$$2i\sin(\pi s)\Gamma(s) = -\int_{\infty}^{(0+)}(-t)^{s-1}e^{-t}dt, \quad |\arg(-t)|\le\pi, \tag{2.4}$$

one can derive the following contour integral representation.

Theorem 2.2. Let $\mathrm{Re}(\mu)>0, \mathrm{Re}(a_i)>0$ and

$$|\arg(-t_i)|\le\pi, (i=1,2,\ldots,n).$$

Then

$$\zeta_n^{\mu,s_1,\ldots,s_n}(x_1,\ldots,x_n;a_1,\ldots,a_n) = \frac{-\Gamma(1-s_1)}{2\pi i}\ldots\frac{-\Gamma(1-s_n)}{2\pi i} \times\int_{\infty}^{(0+)}\ldots\int_{\infty}^{(0+)}(-t_1)^{s_1-1}\ldots(-t_n)^{s_n-1}e^{-(a_1t_1+\ldots+a_nt_n)} \times\left(1-x_1e^{-t_1}-\ldots-x_ne^{-t_n}\right)^{-\mu}dt_1\ldots dt_n. \tag{2.5}$$

Proof. Denote, for convenience, the right-hand side of equation (2.5) by *I*. Then in view of (2.2), it is easily seen that

$$I = \sum_{m_1,\ldots,m_n=0}^{\infty}(\mu)_{m_1+\ldots+m_n}\frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}\frac{-\Gamma(1-s_1)}{2\pi i}\ldots\frac{-\Gamma(1-s_n)}{2\pi i} \times\int_{\infty}^{(0+)}\ldots\int_{\infty}^{(0+)}(-t_1)^{s_1-1}e^{-(a_1+m_1)t_1}\ldots(-t_n)^{s_n-1}e^{-(a_n+m_n)t_n}dt_1\ldots dt_n.$$

Upon using (2.4) and in view of the definition (1.9) with the reflection formula

$$\Gamma(1-s)\Gamma(s)=\frac{\pi}{\sin(\pi s)}$$

for Gamma function, we led finally to the left-hand side of the formula (2.5).

Further, we evaluate some definite integrals involving the function $\zeta_n^{\mu,s_1,\ldots,s_n}(x_1,\ldots,x_n;a_1,\ldots,a_n)$. First, we recall the Eulerian integral formula of first kind (cf. e.g [12]):

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad (2.6)$$
$$\mathrm{Re}(x)>0, \mathrm{Re}(y)>0.$$

From the term-by-term integration, we can derive the following formula.

Theorem 2.3. Let $\mathrm{Re}(\gamma-\mu)>0$ and $\mathrm{Re}(\mu)>0$. Then

$$\zeta_n^{\mu,s_1,\ldots,s_n}(x_1,\ldots,x_n;a_1,\ldots,a_n) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\mu)\Gamma(\mu)}\int_0^1 t^{\mu-1}(1-t)^{\gamma-\mu-1} \zeta_n^{\gamma,s_1,\ldots,s_n}(x_1t,\ldots,x_nt;a_1,\ldots,a_n)\,dt. \qquad (2.7)$$

Proof. Denote for convenience the left-hand side of equation (2.7) by *I*.

Then in view of definition (1.9), it is easily seen that

$$I=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\gamma)_{m_1+\ldots+m_n}}{(a_1+m_1)^{s_1}\ldots(a_n+m_n)^{s_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_s^{m_n}}{m_n!} \times\frac{\Gamma(\gamma)}{\Gamma(\gamma-\mu)\Gamma(\mu)}\int_0^1 t^{\mu+m_1+\ldots+m_n-1}(1-t)^{\gamma-\mu-1}\,dt.$$

Upon using (2.6) and the relation

$$\Gamma(\mu+m_1+..+m_n)=(\mu)_{m_1+\ldots+m_n}\Gamma(\mu),$$

we are finally led to the right-hand of the relation (2.7).

Finally, we prove the following result.

Theorem 2.4. Let $Re(\gamma)>Re(\mu)>0$. Then

$$\zeta^{\mu,s_1,\ldots,s_n}(x_1,\ldots,x_n;a_1,\ldots,a_n) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\mu)\Gamma(\mu)}\int_0^\infty \frac{w^{\mu-1}}{(1+w)^{\gamma}} \zeta_n^{\gamma,s_1,\ldots,s_n}\left(\frac{x_1w}{1+w},\ldots,\frac{x_nw}{1+w};a_1,\ldots,a_n\right)dw. \qquad (2.8)$$

Proof. Setting $a=\mu+m_1+\ldots+m_n$ and

$$b=\gamma+m_1+\ldots+m_n; n\in\mathbb{Z}^+$$

in the Eulerian Beta function formula (see, e.g., [[20], p.8, Eq. (45)]):

$$B(a,b-a)=\frac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)}=\int_0^\infty\frac{t^{a-1}}{(1+t)^b}dt,$$
$$(Re(b)>Re(a)>0),$$

we find that

$$\frac{(\mu)_{m_1+\ldots+m_n}}{(\gamma)_{m_1+\ldots+m_n}}=\frac{\Gamma(\gamma)}{\Gamma(\gamma-\mu)\Gamma(\mu)}\int_0^\infty\frac{w^{\mu+m_1+\ldots+m_n-1}}{(1+w)^{\gamma+m_1+\ldots+m_n}}dw,$$
$$(Re(\gamma)>Re(\mu)>0; m_i\in\mathbb{N}_0, (i=1,2,\ldots,n)),$$

which, by appealing to the definition (1.9), immediately yields the assertion (2.8).

3. Operational Relations

The multiple Zeta function $\zeta_n^{\mu,s_1,\ldots,s_n}$ satisfies some operational relations. Fortunately these properties of $\zeta_n^{\mu,s_1,\ldots,s_n}$ be developed directly from the definition (1.9). First, by recalling the familiar derivative formula from calculus in terms of the gamma function [8]

$$D_x^m x^n=\frac{\Gamma(n+1)}{\Gamma(n-m+1)}x^{n-m}, n-m\ge 0, D_x=\frac{d}{dx}, \qquad (3.1)$$

where $m\in\mathbb{N}$, we aim now to derive the following differential relations for $\zeta_n^{\mu,s_1,\ldots,s_n}$.

Theorem 3.1. Let $k_i\in\mathbb{N}, (i=1,2,\ldots,n)$. Then

$$\prod_{i=}^{n}\left\{D_{x_i}^{k_i}\right\}\zeta_n^{\mu,s_1,\ldots,s_n}(x_1,\ldots,x_n;a_1,\ldots,a_n) =(\mu)_{k_1+\ldots+k_n}\zeta_n^{\mu+k_1+\ldots+k_n,s_1,\ldots,s_n}\begin{pmatrix}x_1,\ldots,x_n;\\ a_1+k_1,\ldots,a_n+k_n\end{pmatrix}. \qquad (3.2)$$

Proof. By starting from the left-hand side of (3.2) and in view of (1.9) and by using the relation (3.1), we get:

$$\prod_{i=}^{n}\left\{D_{x_i}^{k_i}\right\}\zeta_n^{\mu,s_1,\ldots,s_n}(x_1,\ldots,x_n;a_1,\ldots,a_n) =\sum_{m_1,\ldots,m_n=0}^{\infty}\left(\begin{array}{c}\dfrac{(\mu)_{m_1+\ldots+m_n}}{(a_1+m_1)^{s_1}\ldots(a_n+m_n)^{s_n}}\\ \dfrac{x_1^{m_1-k_1}}{(m_1-k_1)!}\cdots\dfrac{x_n^{m_n-k_n}}{(m_n-k_n)!}\end{array}\right). \qquad (3.3)$$

Now, letting $m_i\to m_i+k_i, (i=1,2,\ldots,n)$ in (3.3) using the formula $(\mu)_{m+k}=(\mu)_k(\mu+k)_m$ and considering the definition (1.9), we get the right-hand side of formula (3.2).

Theorem 3.2. Let

$$k_i\in\mathbb{N},\ s_i\ne 0,-1,-2,\ldots\ldots,(i=1,2,\ldots,n).$$

Then

$$\prod_{i=1}^{n}\left\{\frac{\partial^{k_i}}{\partial a_i^{k_i}}\right\}\zeta_n^{\mu,s_1,\ldots,s_n}(x_1,\ldots,x_n;a_1,\ldots,a_n) =\prod_{i=1}^{n}\left\{(-1)^{k_i}(s_i)_{k_i}\right\} \times\zeta_n^{\mu,s_1+k_1,\ldots,s_n+k_n}(x_1,\ldots,x_n;a_1,\ldots,a_n). \qquad (3.4)$$

Proof. By starting from the left-hand side of (3.4) and in view of (1.9) and by using the relation

$$\frac{\partial^k}{\partial a^k}(a+m)^{-s} = (-1)^k (s)_k (a+m)^{-s-k},$$

we get:

$$\prod_{i=1}^{n}\left\{D_{a_i}^{k_i}\right\}\zeta_n^{\mu,s_1,\ldots,s_n}\left(x_1,\ldots,x_n;a_1,\ldots,a_n\right)$$

$$=\prod_{i=1}^{n}\left\{(-1)^{k_i}(s_i)_{k_i}\right\}$$

$$\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\mu)_{m_1+\ldots+m_n}}{(a_1+m_1)^{s_1+k_1}\ldots(a_n+m_n)^{s_n+k_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}.$$

Now, considering the definition (1.9), we get the right-hand side of formula (3.4).

Closely associated with the derivative of the gamma function is the digamma function defined by [[21], p.74(2.51)]:

$$\psi(x)=\frac{d}{dx}\ln\Gamma(x)=\frac{\Gamma'(x)}{\Gamma(x)},\ (x)\neq 0,-1,-2,\ldots. \qquad (3.5)$$

Now, we wish to establish the derivative of the function $\zeta_n^{\mu,s_1,\ldots,s_n}$ with respect to the parameters μ.

Theorem 3.3. Let $\operatorname{Re}(\mu)\in\mathbb{C}\setminus\mathbb{Z}_0^-$. Then

$$\frac{\partial}{\partial\mu}\zeta_n^{\mu,s_1,\ldots,s_n}\left(x_1,\ldots,x_n;a_1,\ldots,a_n\right)$$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\mu)_{m_1+\ldots+m_n}}{(a_1+m_1)^{s_1}\ldots(a_n+m_n)^{s_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_s^{m_n}}{m_n!} \qquad (3.6)$$

$$\times\left[\psi(\mu+m_1+\ldots+m_n)-\psi(\mu)\right].$$

Proof. By starting from the left-hand side of formula (3.6) and by using the relation (3.5), then according to the result by using:

$$\frac{d}{d\mu}(\mu)_n=\frac{d}{d\mu}\left[\frac{\Gamma(\mu+n)}{\Gamma(\mu)}\right]=(\mu)_n\left[\psi(\mu+n)-\psi(\mu)\right], \qquad (3.7)$$

we obtain the right-hand side of formula (3.6).

Next, let us recall the definition of the Weyl fractional derivative of exponential function $e^{-at}, a>0$ of order v in the form (see [[22], p.248(7.4)]):

$$D_t^{v}e^{-at}=a^{v}e^{-at}, \qquad (3.8)$$

(v not restricted to be positive integer).

We now proceed to find the fractional derivative of the function $\zeta_n^{\mu,s_1,\ldots,s_n}$ with respect to s.

Theorem 3.4. Let $v_i>0, (i=1,2,\ldots,n)$. Then

$$\prod_{i=1}^{n}\left\{D_{s_i}^{v_i}\right\}\zeta_n^{\mu,s_1,\ldots,s_n}\left(x_1,\ldots,x_n;a_1,\ldots,a_n\right)$$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\mu)_{m_1+\ldots+m_n}}{(a_1+m_1)^{s_1}\ldots(a_n+m_n)^{s_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \qquad (3.9)$$

$$\times\prod_{i=1}^{n}\left\{\left[\log(a_i+m_i)\right]^{v_i}\right\}.$$

Proof. Since $(a+m)^{(-s)}=e^{(-s\log(a+m))}$, we have

$$\zeta_n^{\mu,s_1,\ldots,s_n}\left(x_1,\ldots,x_n;a_1,\ldots,a_n\right)=\sum_{m_1,\ldots,m_n=0}^{\infty}(\mu)_{m_1+\ldots+m_n}$$

$$\times\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}e^{-s_1\log(a_1+m_1)}\ldots e^{-s_n\log(a_n+m_n)}$$

The desired result now follows by applying the formula (3.8) to the above identity.

4. Series Expansions

First we derive the following basic sums of series

Theorem 4.1. Let $s_i\neq 0,1,2,\ldots\ldots,(\forall i=1,2,\ldots,n)$. Then

$$\sum_{k_1,\ldots,k_n=0}^{\infty}\zeta_n^{\mu,s_1-k_1,\ldots,s_n-k_n}\left(\begin{matrix}x_1,\ldots,x_n;\\ a_1,\ldots,a_n\end{matrix}\right)\frac{y_1^{k_1}}{k_1!}\cdots\frac{y_n^{k_n}}{k_n!}$$

$$=\exp\left(\sum_{i=1}^{n}a_iy_i\right)\zeta_n^{\mu,s_1,\ldots,s_n}\left(x_1e^{y_1},\ldots,x_ne^{y_n};a_1,\ldots,a_n\right). \qquad (4.1)$$

Proof. If in formula (1.9), we replace s_i by s_i-k_i, $(i=1,2,\ldots,n)$, multiply throughout by $\frac{y_1^{k_1}}{k_1!}\cdots\frac{y_n^{k_n}}{k_n!}$ and then sums up, we get (4.1).

Further, from definition (1.9) we easily have the following interesting series relation.

Theorem 4.2. Let $|x_i|<1,\{|y_i|<|a_i|\}$, $s_i\neq 0,-1,-2,\ldots$, $(\forall i=1,2,\ldots,n)$. Then

$$\sum_{k_1,\ldots,k_n=0}^{\infty}\zeta_n^{\mu,s_1+k_1,\ldots,s_n+k_n}\left(x_1,\ldots,x_n;a_1,\ldots,a_n\right)\prod_{i=1}^{n}\left\{(s_i)_{k_i}\frac{y_i^{k_i}}{k_i!}\right\} \qquad (4.2)$$

$$=\zeta_n^{\mu,s_1,\ldots,s_n}\left(x_1,\ldots,x_n;a_1-y_1,\ldots,a_n-y_n\right),$$

$$=\exp\left(-\sum_{i=1}^{n}\boldsymbol{y}_i\frac{\partial}{\partial\boldsymbol{a}_i}\right)\zeta_n^{\mu,s_1,\ldots,s_n}\left(x_1,\ldots,x_n;a_1,\ldots,a_n\right). \qquad (4.3)$$

Proof. Since

$$(a+m-y)^{-s}=(a+m)^{-s}\left(1-\frac{y}{a+m}\right)^{-s}$$

$$=\sum_{k=0}^{\infty}\frac{(s)_k}{(a+m)^{s+k}}\frac{y^k}{k!},$$

it is easily seen that :

$$\zeta_n^{\mu,s_1,\ldots,s_n}\left(x_1,\ldots,x_n;a_1-y_1,\ldots,a_n-y_n\right)$$

$$=\sum_{k_1,\ldots,k_n=0}^{\infty}\prod_{i=1}^{n}\left\{(s_i)_{k_i}\frac{y_i^{k_i}}{k_i!}\right\}$$

$$\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\mu)_{m_1+\ldots+m_n}}{(a_1+m_1)^{s_1+k_1}\ldots(a_n+m_n)^{s_n+k_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}.$$

The desired result (4.2) now follows by using definition (1.9). Also, by right-hand formula (4.3) and using (3.4) we get to left-hand side of formula (4.3).

Theorem 4.3. Let $s_i \neq 0,-1,-2,\ldots\ldots, \quad (\forall i=1,2,\ldots,n)$. Then

$$\sum_{k_1,\ldots,k_n=0}^{\infty}(\alpha)_{k_1+\ldots+k_n}\zeta_n^{\mu,s_1+k_1,\ldots,s_n+k_n}(x_1,\ldots,x_n;a_1,\ldots,a_n)\times\prod_{i=1}^{n}\left\{(\beta_i)_{k_i}\frac{y_i^{k_i}}{k_i!}\right\}=F_A^{(n)}\left[\alpha,\beta_1,\ldots,\beta_n;s_1,\ldots,s_n;-y_1\frac{\partial}{\partial a_1},\ldots,-y_n\frac{\partial}{\partial a_n}\right]\times\zeta_n^{\mu,s_1,\ldots,s_n}(x_1,\ldots,x_n;a_1,\ldots,a_n), \tag{4.4}$$

where $\left|-y_1\frac{\partial}{\partial a_1}\right|+\ldots+\left|-y_n\frac{\partial}{\partial a_n}\right|<1;$

$$\sum_{k_1,\ldots,k_n=0}^{\infty}\frac{1}{(\gamma)_{k_1+\ldots+k_n}}\zeta_n^{\mu,s_1+k_1,\ldots,s_n+k_n}(x_1,\ldots,x_n;a_1,\ldots,a_n)\times\prod_{i=1}^{n}\left\{(\alpha_i)_{k_i}\frac{y_i^{k_i}}{k_i!}\right\}=F_B^{(n)}\left[\alpha_1,\ldots,\alpha_n,s_1,\ldots,s_n;\gamma;-y_1\frac{\partial}{\partial a_1},\ldots,-y_n\frac{\partial}{\partial a_n}\right]\times\zeta_n^{\mu,s_1,\ldots,s_n}(x_1,\ldots,x_n;a_1,\ldots,a_n), \tag{4.5}$$

where $max\left\{\left|-y_1\frac{\partial}{\partial a_1}\right|+\ldots+\left|-y_n\frac{\partial}{\partial a_n}\right|\right\}<1;$

$$\sum_{k_1,\ldots,k_n=0}^{\infty}(\alpha)_{k_1+\ldots+k_n}(\beta)_{k_1+\ldots+k_n}\times\zeta_n^{\mu,s_1+k_1,\ldots,s_n+k_n}(x_1,\ldots,x_n;a_1,\ldots,a_n)\prod_{i=1}^{n}\left\{\frac{y_i^{k_i}}{k_i!}\right\}=F_C^{(n)}\left[\alpha,\beta;s_1,\ldots,s_n;-y_1\frac{\partial}{\partial a_1},\ldots,-y_n\frac{\partial}{\partial a_n}\right]\times\zeta_n^{\mu,s_1,\ldots,s_n}(x_1,\ldots,x_n;a_1,\ldots,a_n),? \tag{4.6}$$

where $\sqrt{\left|-y_1\frac{\partial}{\partial a_1}\right|}+\ldots+\sqrt{\left|-y_n\frac{\partial}{\partial a_n}\right|}<1;$ and

$$\sum_{k_1,\ldots,k_n=0}^{\infty}\frac{(\alpha)_{k_1+\ldots+k_n}}{(\gamma)_{k_1+\ldots+k_n}}\zeta_n^{\mu,s_1+k_1,\ldots,s_n+k_n}(x_1,\ldots,x_n;a_1,\ldots,a_n)\times\prod_{i=1}^{n}\left\{\frac{y_i^{k_i}}{k_i!}\right\}=F_D^{(n)}\left[\alpha,s_1,\ldots,s_n;\gamma;-y_1\frac{\partial}{\partial a_1},\ldots,-y_n\frac{\partial}{\partial a_n}\right]\times\zeta_n^{\mu,s_1,\ldots,s_n}(x_1,\ldots,x_n;a_1,\ldots,a_n). \tag{4.7}$$

where $F_A^{(n)},F_B^{(n)},F_C^{(n)}$ and $F_D^{(s)}$ are the Lauricella's hypergeometric functions of n-variables (see [[18], p.60, Eq (1), (2), (3), (4)]).

Proof. We refer to the proof of Theorem 4.2.

Theorem 4.4. Let $|y_i|<1,(i=1,2,\ldots,n)$. Then

$$\sum_{k_1,\ldots,k_n=0}^{\infty}(\mu)_{k_1+\ldots+k_n}\zeta_n^{\mu+k_1+\ldots+k_n,s_1,\ldots,s_n}(x_1,\ldots,x_n;a_1+k_1,\ldots,a_n+k_n)\prod_{i=1}^{n}\left\{\frac{y_i^{k_i}}{k_i!}\right\}=\zeta_n^{\mu,s_1,\ldots,s_n}(x_1+y_1,\ldots,x_n+y_n;a_1,\ldots,a_n).? \tag{4.8}$$

Proof. By starting from the right-hand side of (4.8) and in view of (1.9) and by using the relation

$$\sum_{m=0}^{\infty}\frac{(\mu)_m}{(a+m)^s}\frac{(x+y)^m}{m!}=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\mu)_{m+k}}{(a+m+k)^s}\frac{x^m}{m!}\frac{y^k}{k!},$$

we get the left-hand side of formula (4.8).

Theorem 4.5. Let $|x_i|<1,|y_i|<|a_i|$ and $|t_i|<|a_i|$; $(i=1,2,\ldots,n)$. Then

$$\sum_{k_1,\ldots,k_n=0}^{\infty}\zeta_n^{\mu,s_1+k_1,\ldots,s_n+k_n}(x_1,\ldots,x_n;a_1-t_1,\ldots,a_n-t_n)\times\prod_{i=1}^{n}\left\{(b_i)_{k_i}\frac{y_i^{k_i}}{k_i!}\right\}=\sum_{m_1,\ldots,m_n=0}^{\infty}(\mu)_{m_1+\ldots+m_n}\times\prod_{i=1}^{n}\left\{\frac{x_i^{m_i}}{(a_i+m_i)^{s_i}m_i!}F_2\left[s_i,b_i,1;s_i,1;\frac{y_i}{a_i+m_i},\frac{t_i}{a_i+m_i}\right]\right\}, \tag{4.9}$$

where F_2 is the Appell's function of two variables defined by the series [18]

$$F_2\left[a,b,b';c,c';x,y\right]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n}\frac{x^m}{m!}\frac{y^n}{n!}.$$

Proof. By starting from the right-hand side of (4.9) and in view of (1.9), we get

$$\sum_{k_1,\ldots,k_n=0}^{\infty}\zeta_n^{\mu,s_1+k_1,\ldots,s_n+k_n}(x_1,\ldots,x_n;a_1-t_1,\ldots,a_n-t_n)\prod_{i=1}^{n}\left\{(b_i)_{k_i}\frac{y_i^{k_i}}{k_i!}\right\}=\sum_{k_1,\ldots,k_n=0}^{\infty}\sum_{m_1,\ldots,m_n=0}^{\infty}(\mu)_{m_1+\ldots+m_n}\prod_{i=1}^{n}\left\{\frac{(b_i)_{k_i}}{(a_i+m_i-t_i)^{s_i+k_i}}\frac{x_i^{m_i}}{m_i!}\frac{y_i^{k_i}}{k_i!}\right\}.$$

Since

$$\sum_{k=0}^{\infty}(b)_k\frac{y^k}{k!}(a+m-t)^{-(s+k)}=\sum_{k=0}^{\infty}(b)_k\frac{y^k}{k!}(a+m)^{-(s+k)}\left(1-\frac{t}{a+m}\right)^{-(s+k)}$$
$$=\sum_{k=0}^{\infty}(b)_k\frac{\left(y/a+m\right)^k}{k!}\sum_{r=0}^{\infty}\frac{(s+k)_r}{(a+m)^s}\frac{\left(t/a+m\right)^r}{r!},?$$
$$=(a+m)^s\sum_{k,r=0}^{\infty}\frac{(s)_{k+r}(b)_k(1)_r}{(1)_r(s)_k}\frac{\left(t/a+m\right)^r}{r!}\frac{\left(y/a+m\right)^k}{k!}.$$

Hence, the right-hand side of formula (4.9) follows.

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