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Fixed Point Theorems for Expansive Type Mappings in Multiplicative Metric Spaces

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Abstract In this paper, we prove some common fixed point theorems for various types of compatible mappings in the setting of complete multiplicative metric spaces.

Keywords: *multiplicative metric space, compatible mapping, expansive mappings*

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1. Introduction and Preliminaries

The study of expansive mappings is an interesting research area in the fixed point theory. It was grown in 1984 from the paper of Wang *et al.* [7] by introducing the concept of expanding mappings in the complete metric space. Daffer and Kaneko [10] used two self mappings to generalize the result of Wang in a complete metric space.

Definition 1.1. *Let* X *be a nonempty set. A multiplicative metric is a mapping* $d: X \times X \to \mathbb{R}_+$ *satisfying the following conditions*:

i) $d(x,y) \geq 1\ \forall\, x,y \in X$ and $d(x,y)=1$ iff $x=y$;
ii) $d(x,y)=d(y,x)\ \forall\, x,y \in X$;
iii) $d(x,y) \leq d(x,z).d(z,y)\ \forall\, x,y \in X$ (multiplicative triangle inequality).

Then mapping d together with X, i.e. (X,d) is a multiplicative metric space.

Example 1.2. ([9]) Let $d:\mathbb{R}\times\mathbb{R} \to [1,\infty)$ be defined as $d(x,y)=a^{|x-y|}$, where $x,y\in\mathbb{R}$ and $a>1$. Then d is a multiplicative metric and $(\mathbb{R},d)$ is a multiplicative metric space. We may call it usual multiplicative metric spaces.

Example 1.3. ([9]) Let (X,d) be a metric space. Define a mapping d_a on X by

$$d_a(x,y) = a^{d(x,y)} = \begin{cases} 1 & \text{if } x=y, \\ a & \text{if } x \neq y. \end{cases}$$

where $x,y\in X$ and $a>1$. Then d_a is a multiplicative metric and (X,d_a) is known as the discrete multiplicative metric space.

In 2012, Ozavsar and Cevikel [5] gave the concept of multiplicative contraction mappings and proved some fixed point theorems of such mappings in a multiplicative metric space.

Also they proved the Banach Contraction Principle in the setting of multiplicative metric spaces as follows:

"Let f be a multiplicative contraction mapping of a complete multiplicative metric space (X,d) into itself. Then f has a unique fixed point."

In 1984, Wang et al. [7] and in 1993, Rhoades [8] proved some fixed point theorems for expansion mappings, which corresponds to some contractive mappings in metric spaces.

Definition 1.6. Let f be a mapping of a multiplicative metric space (X,d) into itself. Then f is said to be an *expansive mapping* if there exists a constant $\alpha>1$ such that

$$d(fx,fy) \geq d^{\alpha}(x,y), \forall\, x,y\in X.$$

Recently Kang et al. [6] introduced the notion of compatible mappings and its variants as follows:

Definition 1.7. Let f and g be two mappings of a multiplicative metric space (X,d) into itself. Then f and g are called

(1) *Compatible* if

$$\lim_{n\to\infty} d(fgx_n, gfx_n)=1,$$

where $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

(2) *Compatible of type (A)* if

$$\lim_{n\to\infty} d(fgx_n, ggx_n) = 1 \text{ and } \lim_{n\to\infty} d(gfx_n, ffx_n) = 1,$$

where $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$..

(3) *Compatible of type (B)* if

$$\lim_{n\to\infty} d(fgx_n, ggx_n) \le [\lim_{n\to\infty} (fgx_n, ft). \lim_{n\to\infty} d(ft, ffx_n)]^{1/2}$$

and

$$\lim_{n\to\infty} d(gfx_n, ffx_n) \le [\lim_{n\to\infty} d(gfx_n, gt). \lim_{n\to\infty} d(gt, ggx_n)]^{1/2},$$

where $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

(4) *Compatible of type (C)* if

$$\lim_{n\to\infty} d(fgx_n, ggx_n) \le \left[\lim_{n\to\infty} d(fgx_n, ft). \lim_{n\to\infty} d(ft, ffx_n). \lim_{n\to\infty} d(ft, ggx_n)\right]^{1/3}$$

and

$$\lim_{n\to\infty} d(gfx_n, ffx_n) \le \left[\lim_{n\to\infty} d(gfx_n, gt). \lim_{n\to\infty} d(gt, ggx_n). \lim_{n\to\infty} d(gt, ffx_n)\right]^{1/3}$$

where $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

(5) *Compatible of type (P)* if

$$\lim_{n\to\infty} d(ffx_n, ggx_n) = 1,$$

where $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

Next is a result which is useful for our main results.

Proposition 1.8. *Let f and g be compatible mappings of a multiplicative metric space (X,d) into itself. Suppose that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. Then $\lim_{n\to\infty} gfx_n = ft$ iff f is continuous at t.*

2. Main Results

In 1993, Rhoades [8] proved the following fixed point theorem for expansive mappings in metric spaces as follows:

Definition 2.1. *Let f and g be compatible mappings of a complete metric space into itself satisfying the condition*

$$d(fx, fy) \ge q d(gx, gy)$$

$\forall\ x, y \in X$, where $q > 1$ and assume that $g(X) \subset f(X)$ and f is continuous. Then f and g have a unique common fixed point.

Theorem 2.2. Let f be a mapping of a multiplicative metric space (X,d) into itself. Then f is said to be *multiplicative contraction* if $\exists$ a real constant $\lambda \in (0,1]$ such that

$$d(fx, fy) \le d^{\lambda}(x, y) \ \ \forall\, x, y \in X.$$

Now we generalized Theorem 2.2 in the string of setting of multiplicative metric spaces in the following way:

Theorem 2.3. *Let f and g be compatible mappings of a complete multiplicative metric space into itself satisfying the condition*

$$\begin{aligned} &[d(fx, fy).d(fx, gy).d(fy, gx).d(fy, gy)] \\ &\ge d^{2q}(gx, gy) \end{aligned} \tag{2.1}$$

$\forall\, x, y \in X$, where $q > 2$ and assume that $g(X) \subset f(X)$ and f is continuous. Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. Since $g(X) \subset f(X)$, $\exists$ $x_1 \in X$ such that $fx_1 = gx_0$. In general, $\exists$ $x_{n+1} \in X$ such that

$$y_n = fx_{n+1} = gx_n. \tag{2.2}$$

From (2.1), consider

$$\begin{aligned} d(y_n, y_{n+1}) &= d(gx_n, gx_{n+1}) \\ &\le \begin{bmatrix} d(fx_n, fx_{n+1}).d(fx_n, gx_{n+1}) \\ .d(fx_{n+1}, gx_n).d(fx_{n+1}, gx_{n+1}) \end{bmatrix}^{1/2q} \\ &= \begin{bmatrix} d(gx_{n-1}, gx_n).d(gx_{n-1}, gx_{n+1}) \\ .d(gx_n, gx_n).d(gx_n, gx_{n+1}) \end{bmatrix}^{1/2q} \\ &= [d(gx_{n-1}, gx_n).d(gx_n, gx_{n+1})]^{1/q} \\ &= [d(y_{n-1}, y_n).d(y_n, y_{n+1})]^{1/q} \end{aligned}$$

$$d(y_n, y_{n+1}) \ \le \ d^{\frac{1}{q}-1}(y_{n-1}, y_n). \tag{2.3}$$

Again,

$$d(y_{n-1}, y_n) = d(gx_{n-1}, gx_n)$$

$$\leq \left[\begin{matrix} d(fx_{n-1},fx_n).d(fx_{n-1},gx_n). \\ d(fx_n,gx_{n-1}).d(fx_n,gx_n) \end{matrix}\right]^{1/2q}$$

$$=\left[\begin{matrix} d(gx_{n-2},gx_{n-1}).d(gx_{n-2},gx_n). \\ d(gx_{n-1},gx_{n-1}).d(gx_{n-1},gx_n) \end{matrix}\right]^{1/2q}$$

$$=\left[d^2(gx_{n-2},gx_{n-1}).d^2(gx_{n-1},gx_n)\right]^{1/2q}$$

$$d(gx_{n-1},gx_n)\leq\left[d(gx_{n-2},gx_{n-1}).d(gx_{n-1},gx_n)\right]^{1/q}$$

$$d(y_{n-1},y_n)\leq\left[d(y_{n-2},y_{n-1}).d(y_{n-1},y_n)\right]^{1/q}$$

$$d(y_{n-1},y_n)\leq d^{1/q}(y_{n-2},y_{n-1}).d^{1/q}(y_{n-1},y_n)$$

$$d(y_{n-1},y_n)\leq d^{1/q-1}(y_{n-2},y_{n-1}) \qquad (2.4)$$

Therefore using (2.3) and (2.4), we have

$$d(y_n,y_{n+1})\leq d^{1/q-1}(y_{n-1},y_n)$$
$$\leq d^{1/(q-1)^2}(y_{n-2},y_{n-1})$$

and so on.

In general, we have

$$d(y_n,y_{n+1})\leq d^{k^n}(y_0,y_1) \qquad (2.5)$$

where $k=\frac{1}{q-1}<1$ or $q>2$.

Now for $m,n\in N$ with $n<m$, consider

$$d(y_n,y_m)$$
$$\leq d(y_n,y_{n+1}).d(y_{n+1},y_{n+2})....d(y_{m-1},y_m)$$
$$\leq d^{k^n}(y_0,y_1).d^{k^{n+1}}(y_0,y_1)....d^{k^{m-1}}(y_0,y_1)$$
[using (2.5)]
$$\leq d^{\frac{k^n}{1-k}}(y_0,y_1)$$
$$\to 1$$

As $n\to\infty$. It follows that the sequence $\{y_n\}$ is a multiplicative Cauchy sequence. Since (X,d) is complete, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z. \qquad (2.6)$$

Since f and g are compatible and f is continuous, by Proposition 1.15,

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fg_n = \lim_{n\to\infty} gfx_n = fz. \qquad (2.7)$$

Consider

$$d(gfx_n,gz)\leq\left[\begin{matrix} d(ffx_n,fz).d(ffx_n,gz). \\ d(fz,gfx_n).d(fz,gz) \end{matrix}\right]^{1/2q} \text{[using (2.1)]}$$

Letting $n\to\infty$, we get

$$d(fz,gz)\leq\left[\begin{matrix} d(fz,fz).d(fz,gz) \\ .d(fz,fz).d(fz,gz) \end{matrix}\right]^{1/2q} \text{[using (2.7)]}$$
$$=[d^2(fz,gz)]^{1/2q}$$

$$d(fz,gz)\leq[d(fz,gz)]^{1/q}$$

$$\Rightarrow fz=gz, \text{ since } q>2. \qquad (2.8)$$

Now we show that z is fixed point of f and g. Again considering

$$d(gz,gx_n)=d(gx_n,gz)$$
$$\leq\left[\begin{matrix} d(fx_n,fz).d(fx_n,gz). \\ d(fz,gx_n).d(fz,gz) \end{matrix}\right]^{1/2q} \text{[using (2.1)]}$$

Letting $n\to\infty$, we get

$$d(gz,z)\leq\left[\begin{matrix} d(z,fz).d(z,gz). \\ d(fz,z).d(fz,gz) \end{matrix}\right]^{1/2q} \text{[using (2.6)]}$$

$$=\left[\begin{matrix} d(z,gz).d(z,gz). \\ d(gz,z).d(gz,gz) \end{matrix}\right]^{1/2q} \text{[using (2.8)]}$$

$$d(gz,z)\leq[d(z,gz)]^{3/2q} \text{ ,since q>2}$$
$$\Rightarrow gz=z.$$

Hence z is fixed point of f and g.

Uniqueness follows easily from (2.1). This completes the proof.

Next we prove a common fixed point theorem for compatible mappings of type (B) as follows:

Theorem 2.3. *Let f and g be compatible mappings of type (B) of a complete multiplicative metric space into itself satisfying the condition (2.1) and assume that $g(X)\subset f(X)$ and f is continuous. Then f and g have a unique common fixed point.*

***Proof*.** From the proof of Theorem 2.2, $\{gx_n\}$ is a multiplicative Cauchy sequence. Since (X,d) is complete, there exists a point $z\in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z.$$

Since f is continuous, we have

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = fz.$$

Since f and g are compatible of type (B), so

$$\lim_{n\to\infty} d(fgx_n, ggx_n)$$

$$\leq \left[\lim_{n\to\infty} d(fgx_n, fz).\lim_{n\to\infty} d(fz, ffx_n)\right]^{1/2}.$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} d(fz, ggx_n) \leq \left[d(fz, fz).d(fz, fz)\right]^{1/2},$$

$$\Rightarrow \lim_{n\to\infty} ggx_n = fz.$$

Consider

$$\lim_{n\to\infty} d(ggx_n, gfx_n) = \lim_{n\to\infty} d(ggx_n, ffx_n)$$

$$= \lim_{n\to\infty} \begin{bmatrix} d(fgx_n, ffx_n).d(fgx_n, gfx_n) \\ .d(ffx_n, ggx_n).d(ffx_n, gfx_n) \end{bmatrix}^{1/2q} \text{[using (2.1)]}$$

$$d(fz, gfx_n) = \begin{bmatrix} d(fz, fz).d(fz, gfx_n) \\ .d(fz, fz).d(fz, gfx_n) \end{bmatrix}^{1/2q}$$

$$= \left[d^2(fz, gfx_n)\right]^{1/2q} = \left[d(fz, gfx_n)\right]^{1/q}$$

$$\Rightarrow \lim_{n\to\infty} ggx_n = \lim_{n\to\infty} gfx_n = fz$$

Again consider

$$d(gfx_n, gz)$$

$$\leq [d(ffx_n, fz).d(ffx_n, gz).d(fz, gfx_n).d(fz, gz)]^{1/2q}$$

[using (2.1)]

Letting $n \to \infty$, we have

$$d(fz, gz) \leq [d(fz, fz).d(fz, gz).d(fz, fz).d(fz, gz)]^{1/2q}$$

$$d(fz, gz) \leq d^{1/q}(fz, gz)$$

Since $q > 2$, $\Rightarrow fz = gz$.

Next we have to show that z is a fixed point of f and g, that easily follows from the proof of Theorem 2.2.

Uniqueness follows from (2.1). This completes the proof.

Now we prove a common fixed point theorem for compatible mappings of type (C) as follows:

Theorem 2.4. *Let f and g be compatible mappings of type (C) of a complete multiplicative metric space into itself satisfying the conditions (2.1) and assume that $g(X) \subset f(X)$ and f is continuous. Then f and g have a unique common fixed point.*

Proof. From the proof of Theorem 2.2, $\{gx_n\}$ is a multiplicative Cauchy sequence. Since (X, d) is complete, there exists a point $z \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z.$$

Since f is continuous, we have

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = fz.$$

Since f and g are compatible of type (C), so

$$\lim_{n\to\infty} d(fgx_n, ggx_n)$$

$$\leq \left[\lim_{n\to\infty} d(fgx_n, fz).\lim_{n\to\infty} d(fz, ffx_n).\lim_{n\to\infty} d(fz, ggx_n)\right]^{1/3}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} d(fz, ggx_n)$$

$$\leq \left[d(fz, fz).d(fz, fz).\lim_{n\to\infty} d(fz, ggx_n)\right]^{1/3},$$

$$\Rightarrow \lim_{n\to\infty} ggx_n = fz.$$

Consider

$$\lim_{n\to\infty} d(ggx_n, gfx_n)$$

$$\leq \lim_{n\to\infty} \begin{bmatrix} d(fgx_n, ffx_n).d(fgx_n, gfx_n) \\ .d(ffx_n, ggx_n).d(ffx_n, gfx_n) \end{bmatrix}^{1/2q} \text{[using (2.1)]}$$

$$d(fz, gfx_n) \leq \begin{bmatrix} d(fz, fz).d(fz, gfx_n) \\ .d(fz, fz).d(fz, gfx_n) \end{bmatrix}^{1/2q}$$

$$= \left[d^2(fz, gfx_n)\right]^{1/2q}$$

$$d(fz, gfx_n) \leq \left[d(fz, gfx_n)\right]^{1/q}$$

$$\Rightarrow \lim_{n\to\infty} ggx_n = \lim_{n\to\infty} gfx_n = fz.$$

Again consider

$$d(gfx_n, gz) \leq \begin{bmatrix} d(ffx_n, fz).d(ffx_n, gz) \\ .d(fz, gfx_n).d(fz, gz) \end{bmatrix}^{1/2q}$$

Letting $n \to \infty$, we have

$$d(fz, gz) \leq \left[d(fz, fz).d(fz, gz).d(fz, fz).d(fz, gz)\right]^{1/2q}$$

$$= \left[d^2(fz, gz)\right]^{1/2q}$$

$$d(fz, gz) \leq \left[d(fz, gz)\right]^{1/q}$$

$$\Rightarrow fz = gz.$$

The rest of the proof follows easily from Theorem 2.2.

Now we prove a common fixed point theorem for compatible mappings of type (P) as follows:

Theorem 2.5. *Let f and g be compatible mappings of type (P) of a complete metric space into itself satisfying the condition (2.1) and assume that $g(X) \subset f(X)$ and f is continuous. Then f and g have a unique common fixed point.*

Proof. From the proof of Theorem 2.2, $\{gx_n\}$ is a multiplicative Cauchy sequence. Since (X, d) is complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z.$$

Since f and g are compatible of type (P) and f is continuous, we have

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} ggx_n = \lim_{n\to\infty} fgx_n = fz.$$

Consider

$$\lim_{n\to\infty} d(ggx_n, gz)$$

$$\leq \lim_{n\to\infty} \begin{bmatrix} d(fgx_n, fz).d(fgx_n, gz) \\ .d(fz, ggx_n).d(fz, gz) \end{bmatrix}^{1/2q} \text{ [using (2.1)]}$$

$$d(fz, gz) \leq \left[d(fz, fz).d(fz, gz).d(fz, fz).d(fz, gz)\right]^{1/2q}$$

$$= \left[d^2(fz, gz)\right]^{1/2q}$$

$$d(fz, gz) \leq \left[d(fz, gz)\right]^{1/q} \Rightarrow gz = fz$$

The rest of the proof follows easily from Theorem 2.2.

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