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A Shortened Recurrence Relation for Bernoulli Numbers

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Abstract In this note, starting with a little-known result of Kuo, I derive a recurrence relation for the Bernoulli numbers B_{2n}, n being a positive integer. This formula is shown to be advantageous in comparison to other known formulae for the *exact* symbolic computation of B_{2n}. Interestingly, it is suitable for large values of n since it allows the computation of both B_{4n} and B_{4n+2} from only B_0, B_2, ..., B_{2n}.

Keywords: *Bernoulli numbers, recurrence relations, Riemann zeta function*

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1. Introduction

The Bernoulli numbers B_n, n being a nonnegative integer, can be defined by the generating function

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \qquad (1)$$

which holds for all complex z with $|z| < 2\pi$. The first few values are $B_0 = 1$, $B_1 = -1/2$, and $B_2 = 1/6$. It is well-known that $B_n = 0$ for *odd* values of n, $n>1$. For *even* values of n, $n>1$, the numbers B_n form a subsequence of non-null rational numbers such that $(-1)^{m+1} B_{2m}>0$ for all positive integer m. In other words, the entire subsequence of non-null Bernoulli numbers up to any B_n, $n>1$, consists of B_0 and B_1, given above, and the preceding numbers B_{2m}, $m = 1, ..., \lfloor n/2 \rfloor$. The basic properties of these numbers can be found, e.g., in Sec. 9.61 of [1].

The numbers B_n appear in many instances in pure and applied mathematics, most notably in number theory, finite differences calculations, and asymptotic analysis. They also appear as coefficients in the Euler-Maclaurin formula, thus being important for accurate numerical evaluation of special functions. Therefore, numerical computations of B_n are of great mathematical interest. To this end, recurrence formulae were soon recognized as the most efficient tool [2]. One of the simplest such formulas is found by multiplying both sides of Eq. (1) by e^x-1, using the Cauchy product with the Maclaurin series for e^x-1, and equating the coefficients, which yields

$$\sum_{j=0}^{n}\binom{n+1}{j} B_j = 0, \;\; n \ge 1. \qquad (2)$$

This kind of recurrence has the disadvantage of demanding the previous knowledge of *every* B_0, B_1, ...,B_{n-1} for the computation of B_n. On searching for more efficient formulae in literature, one finds shortened recurrence relations of two different types. The first type consists of *lacunary* recurrences, in which B_n is determined only from every second, or every third, etc., preceding B_n (see, e.g., the lacunary formula by Ramanujan in [3]). The second type demands the knowledge of the *second-half* of Bernoulli numbers up to B_{n-1} in order to compute B_n [4]. For an extensive study of these and other recurrence relations involving the numbers B_n, see [5].

Here in this note, I apply the Euler's formula relating the even zeta value $\zeta(2n)$ to B_{2n} to a recurrence formula for $\zeta(2n)$ obtained by Kuo in [6] in order to convert it into a new recurrence formula for B_{2n}. This reveals a third type of recurrence for the Bernoulli numbers, in the sense that it allows us to compute both B_{4n} and B_{4n+2} from only the *first-half* of the preceding numbers, i.e. B_0, B_2, ..., B_{2n}.

2. Recurrence Relation for Bernoulli Numbers

For complex values of s with Re(s)>1, the Riemann zeta function is defined as $\zeta(s) := \sum_{n=1}^{\infty} 1/n^s$. In this domain, the convergence of this series is guaranteed by the integral test. For s=1, one has the well-known *harmonic serie* $\sum_{n=1}^{\infty} 1/n$, which diverges to infinity. For positive even values of s, one has a celebrated formula by Euler [7]:

$$\zeta(2n) = (-1)^{n-1}\frac{2^{2n-1} B_{2n}}{(2n)!}\pi^{2n}. \qquad (1)$$

Since $(-1)^{n-1} B_{2n}$ is a positive rational for all integer $n>0$, then $\zeta(2n)$ is a positive rational multiple of π^{2n}. The function $\zeta(s)$, as defined above, can be extended to the entire complex plane (except for the only simple pole at s=1) by analytic continuation, which yields $\zeta(0) = -1/2$ [8]. Alternatively, we can use a globally convergent series representation due to Hasse (1930) [9]:

$$\frac{1}{1-2^{1-s}}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{(k+1)^s}, \tag{4}$$

valid for all complex $s \neq 1 + (2\pi m)i/\ln(2)$, m being an integer. At s=0, it reduces to $\zeta(0) = -1/2$, which shows that Eq. (3) is also valid for n=0.

These are the ingredients we need to state our first lemma, which comes from a little-known recurrence formula by Kuo (1949) [6], written here in terms of even zeta values.

Lemma 1 (Kuo's recurrence formula). For any positive integer n, one has

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{4(n-1)!^2(2n-1)} + \frac{1}{(n-1)!} \times \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{\zeta(2k)(2\pi)^{2n-2k}}{(n-2k)!(2n-2k)} + \frac{1}{\pi}\sum_{k=0}^{\lfloor n/2\rfloor}\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^{k+j}\zeta(2k)\zeta(2j) \times \frac{(2\pi)^{2n-2k-2j+1}}{(n-2k)!(n-2j)!(2n-2k-2j+1)}.$$

This formula is proved in [6] by developing successive integrations (from 0 to x) of the Fourier series $\sum_{n=1}^{\infty}\sin(nx)/n$, which converges to $(\pi-x)/2$ for all $x \in (0,2\pi)$, and then applying Parseval's theorem.

We are now in a position to state and prove our main result. In what follows, $(\lambda)_n := \lambda(\lambda+1)\cdots(\lambda+n-1)$ is the Pochhammer symbol.

Theorem 1 (Recurrence for B_{2n}). For any integer $n>0$, the following formula holds:

$$B_{2n} = (-1)^{n-1} n\,(n)_n\left[a_n - \sum_{k=0}^{\lfloor n/2\rfloor}\frac{B_{2k}}{(2k)!(n-2k)!(n-k)} + 2(n-1)!\sum_{k=0}^{\lfloor n/2\rfloor}\frac{B_{2k}}{(2k)!(n-2k)!} \times \sum_{j=0}^{\lfloor n/2\rfloor}\frac{B_{2j}}{(2j)!(n-2j)!}\frac{1}{2n-2k-2j+1}\right],$$

where $a_n := \dfrac{1}{2(2n-1)(n-1)!}$.

Proof. On dividing both sides of Kuo's formula, our Lemma 1, by $(2\pi)^{2n}$, one finds

$$\frac{\zeta(2n)}{(2\pi)^{2n}} = \frac{2^{-1}}{4(n-1)!^2(2n-1)} + \frac{1}{(n-1)!}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{\zeta(2k)}{\pi^{2k}}\frac{2^{-2k}}{(n-2k)!(2n-2k)} + \sum_{k=0}^{\lfloor n/2\rfloor}\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^{k+j}\frac{\zeta(2k)}{\pi^{2k}}\frac{\zeta(2j)}{\pi^{2j}} \times \frac{2^{-2k-2j+1}}{(n-2k)!(n-2j)!(2n-2k-2j+1)}. \tag{2}$$

From Eq. (3), we know that $\dfrac{\zeta(2m)}{(2\pi)^{2m}} = (-1)^{m-1}\dfrac{B_{2m}}{2(2m)!}$, which is valid for all integer $m \geq 0$. By substituting this in Eq. (5), one finds

$$\frac{B_{2n}}{(2n)!} = \frac{(-1)^{n-1}}{4(n-1)!^2(2n-1)} - \frac{(-1)^{n-1}}{(n-1)!}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{B_{2k}}{(2k)!}\frac{1}{(n-2k)!(2n-2k)} + (-1)^{n-1}\sum_{k=0}^{\lfloor n/2\rfloor}\sum_{j=0}^{\lfloor n/2\rfloor}\frac{B_{2k}}{(2k)!}\frac{B_{2j}}{(2j)!} \times \frac{1}{(n-2k)!(n-2j)!(2n-2k-2j+1)}. \tag{3}$$

Now, by factoring $(-1)^{n-1}$ one finds

$$B_{2n} = (-1)^{n-1}\left[\frac{n(2n-2)!}{2(n-1)!^2} - \frac{(2n)!}{2(n-1)!} \times \sum_{k=0}^{\lfloor n/2\rfloor}\frac{B_{2k}}{(2k)!(n-2k)!(n-k)} + (2n)!\sum_{k=0}^{\lfloor n/2\rfloor}\frac{B_{2k}}{(2k)!(n-2k)!} \times \sum_{j=0}^{\lfloor n/2\rfloor}\frac{B_{2j}}{(2j)!(n-2j)!}\frac{1}{2n-2k-2j+1}\right]. \tag{4}$$

On writing the first term in the right-hand side as $(2n)!/[4(2n-1)(n-1)!^2]$ and putting $(2n)!/(n-1)!$ in evidence (out of the square brackets), the desired result follows.

For instance, when n=1, since $(1)_1 = 1$ and B_0=1, one readily finds

$$B_2 = a_1 - B_0 + 2\frac{B_0^{\,2}}{3} = \frac{1}{2} - 1 + \frac{2}{3} = \frac{1}{6}. \tag{5}$$

For larger values of n, a more efficient, fast computation is achieved when one avoids the repeated computation of the factorials in our Theorem 1. For this, it is advantageous to make use of the algorithm given below. A short *Maple* code implementation of this algorithm is freely available in the journal website (file B4mB4mp2.mws).

Of course, more efficient numerical routines for floating-point computation of B_{2n} can be found in some recent works (see, e.g., [10,11,12]), but we are interested here only in *exact symbolic computations* of the rational numbers B_{2n}, which are important for implementation in modern mathematical software such as *Maple* and *Mathematica*. Admittedly, most algorithms in [12] can, *a priori*, be implemented in a manner to yield exact symbolic results.

2.1. Algorithm

For completion, the algorithm corresponding to an efficient implementation of the recurrence formula given in our Theorem 1 is presented below.

Algorithm: Algorithmic form of Theorem 1. Computes B_{2N} and B_{2N+2} at once.

```
Input: N (a positive integer greater than 1)
Output: B_2N and B_2N+2
Constant: B_0=1, B_2=1/6
Var: c=1, max=0
BEGIN
N ← 2N
for n from 2 by 2 to N do
   if n > 2 then
         c = c (n-2) (n-1)
   a = 1/(c (4n-2))
   max = max+1
   S = 0
   for k from 0 to max do
         d_k = B_2k/((2k)! (n-2k)!)
         S = S+d_k/(n-k)
         fator = n (n+1) ··· (2n-1) (2n)
         sum_k = 0
         for k from 0 to max do
            sum_j = 0
            for j from 0 to max do
                  sum_j = sum_j+d_j/(2n-2k-2j+1)
            sum_k = sum_k+d_k sum_j
          B_2n = -fator (a -S +2 c sum_k)/2
 REM # Updating the parameters to compute B_2N+2
   a = a (2n-1)/(n (2n+1))
   fator = fator (2n+1) (2n+2)/n
   S = 0
   for k from 0 to max do
          d_k = d_k/(n+1-2k)
          S = S+d_k/(n+1-k)
 REM # Go to label '1' above, substitute n by n+1, and
 recalculate sum_k
 B_2n+2 = fator (a –S +2 c n sum_k)/2
```

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