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Generalized Dynamic Process for Generalized Multivalued F-contraction of Hardy Rogers Type in b-metric Spaces

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Abstract The aim of this paper is to establish common fixed point results for multivalued mappings satisfying generalized F-contractive conditions of Hardy Rogers type with respect to generalized dynamic process in *b*-metric space. Our results improve and generalize several well known results in the existing literature.

Keywords: *fixed point, generalized F-contraction, b-metric space, generalized dynamic process, Hausdorff metric*

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1. Introduction and Preliminaries

Let X be a non empty set, $f : X \to X$ be a mapping. A point $x \in X$ is called a fixed point of f if $x = fx$. Fixed points results of mappings, which satisfies some specific contractive conditions on some space have been very useful in research activity (see [1-30]).

Recently, Wardowski [30] introduced a new concept of contraction named F-contraction and proved a fixed point theorem which generalizes Banach contraction principle. Klim et al. [22] further established fixed point result for F-contractive mapping in dynamic process. Cosentino et al. [16] further generalized this concept as F-Contractive Mappings of Hardy-Rogers-Type. Arshad et al. [4] proved fixed point result in $\alpha - GF -$ contraction of Hardy-Rogers-type. Following this direction of research, in this paper, we will present some fixed point results of Hardy-Rogers-type for multivalued mappings in b-metric space with generalized dynamic process. This paper contain common fixed point results for two mappings. Throughout our paper R_+, N and $CB(X)$ represent set of real numbers, set of natural numbers and family of non-empty closed bounded subsets X respectively.

Definition 1 [10] *Let X be a non-empty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to R_+$ is called a b-metric provided that, for all $x, y, z \in X$*

1) $d(x,y) = 0$ iff $x = y$
2) $d(x,y) = d(y,x)$
3) $d(x,z) \leq s[d(x,y) + d(y,z)]$. The pair (X,d) is called a b-metric space.

Definition 2 [15] *Let (X,d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if for all $\varepsilon > 0$ there exist $n(\varepsilon) \in N$ such that for each $n, m > n(\varepsilon)$, we have $d(x_n, x_m) < \varepsilon$.*

Definition 3 [30] *Let $F : R_+ \to R$ be a mapping satisfying:*

(F1) F is strictly increasing.

(F2) for each sequence $\{a_n\} \subset R_+$ of positive numbers $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$

(F3) there exists $k \in (0,1)$ such that $\lim_{a \to 0^+} a^k F(a) = 0.$

We denote with $\mathcal{F}$ the family of all functions F that satisfy the conditions (F1)-(F3).

Let (X,d) be a metric space. A self-mapping T on X is called an F-contraction if there exist $\tau \in R_+$ such that

$$\tau + F(d(Tx,Ty)) \leq F(d(x,y)),$$
$$\text{for all } x, y \in X \text{ with } d(Tx,Ty) > 0.$$

Theorem 4 [30] *Let (X,d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ a sequence $\{T^n x_0\}_{n \in N}$ is convergent to x^*.*

Definition 5 [16] *Let (X,d) be a metric space. A self-mapping T on X is called a generalized F-contraction of Hardy-Rogers-type if there exist $F \in \mathcal{F}$ and $\tau \in R_+$ such that*

$$\tau + F(d(Tx,Ty)) \leq F(\alpha d(x,y) + \beta d(x,Tx)$$
$$+\gamma d(y,Ty) + \delta d(x,Ty) + L d(y,Tx)),$$

for all $x, y \in X$ *with* $d(Tx, Ty) > 0$, *where* $\alpha, \beta, \delta \in [0,1)$, $\alpha + \beta + \gamma + 2\delta = 1, \gamma \neq 1$ *and* $L \geq 0$.

Theorem 6 [16] *Let* (X, d) *be a complete metric space and let* T *be a self-mapping on* X. *Assume that there exist* $F \in \mathcal{F}$ *and* $\tau \in R_+$ *such that* T *is an F-contraction of Hardy-Rogers-type, that is,*

$$\tau + F(d(Tx,Ty)) \leq F(\alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \delta d(x,Ty) + Ld(y,Tx)),$$

for all $x, y \in X$, $Tx \neq Ty$, *where* $\alpha, \beta, \delta \in [0,1)$, $\alpha + \beta + \gamma + 2\delta = 1$, $\gamma \neq 1$ *and* $L \geq 0$. *Then* T *has a fixed point. Moreover, if* $\alpha + \delta + L \leq 1$, *then the fixed point of* T *is unique.*

Definition 7 [20] *Let* (X, d) *be a metric space. For* $A, B \in CB(X)$, *the Hausdorff metric* H *on* $CB(X)$ *induced by metric* d *is given as*:

$$H(A,B) = \max\left\{\sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A)\right\}$$

where $d(x,B) = \inf\{d(x,y) : y \in B\}$ *and* $H : CB(X) \times CB(X) \to R_+$ *is the Hausdorff metric induced by* d.

Definition 8 [22] *Let* $x_0 \in X$ *and let* $F : R_+ \to R$ *satisfies (F1)-(F3). The mapping* $T : X \to C(X)$ *is called a set-valued F-contraction with respect to a dynamic process* $\{x_n\} \in D(T, x_0)$ *if there exists a function* $\tau : R_+ \to R_+$ *such that*

$$\text{for all } n \in N \ (d(x_n, x_{n+1}) > 0 \Rightarrow \tau(d(x_{n-1}, x_n)) + F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n))).$$

In the above discussion $C(X)$ denotes the collection of all non empty closed subsets of X and $D(T, x_0)$ is called a dynamic process of T starting at x_0. The dynamic process $(x_n)_{n \in N \cup \{0\}}$ will be written simply as (x_n) where

$$D(T, x_0) = \{(x_n)_{n \in N \cup \{0\}} \subset X : x_n \in Tx_{n-1}, \text{ for all n} \in \mathrm{N}\}.$$

Definition 9 [3] *Let* $f : X \to X$, $T : X \to CB(X)$ *and* x_0 *be an arbitrary but fixed element in* X. *The set* $D(f, T, x_0) = \{(x_n)_{n \in N \cup \{0\}} : x_{n+1} = fx_n \in Tx_{n-1}$ *for all* $n \in N\}$ *is called a generalized dynamic process of* f *and* T *starting at* x_0. *The generalized dynamic process* $D(f, T, x_0)$ *will simply be written as* (fx_n). *The sequence* (x_n) *for which* (fx_n) *is a generalized dynamic process is called* f *iterative sequence of* T *starting at* x_0.

Lemma 10 [23] *Suppose* (M, d) *be a b-metric space and* $\{x_n\}$ *be a sequence in* M *such that*

$$d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1}), n = 0, 1, 2, \ldots$$

where $0 \leq \lambda < 1$. Then the sequence $\{x_n\}$ is a Cauchy sequence in M provided that $s.\lambda < 1$.

2. Main Result

Definition 11 Let $F_1 : R_+ \to R$ be a mapping satisfying:

(F1) F_1 is strictly increasing.

(F2) for each sequence $\{a_n\} \subset R_+$ of positive numbers $\lim_{n \to \infty} a_n = 0$ if and only if $\lim_{n \to \infty} F_1(a_n) = -\infty$.

We denote with $\mathcal{F}_1$ the family of all functions F_1 that satisfy the conditions (F1) and (F2).

Let (X, d) be a b-metric space. A self-mapping T on X is called an F_1-contraction if there exist $\tau \in R_+$ such that

$$\tau + F_1(d(Tx,Ty)) \leq F_1(d(x,y)), \text{ for all } x, y \in X \text{ with } d(Tx,Ty) > 0. \quad (1)$$

Definition 12 *Let* (X, d) *be a b-metric space,* $f : X \to X$ *is continuous,* $T : X \to CB(X)$ *and* x_0 *be an arbitrary point in* X. *A mapping* f *is called a generalized multivalued* F_1 *-contraction of Hardy-Rogers-type with respect to a generalized dynamic process* $D(f, T, x_0)$ *if there exist* $F_1 \in \mathcal{F}_1$ *and* $\tau : R_+ \to R_+$ *is non decreasing such that*

$$\tau(M(x_{n-1}, x_n)) + F_1(d(fx_n, fx_{n+1})) \leq F_1(M(x_{n-1}, x_n))$$

where

$$M(x_{n-1}, x_n) = \alpha d(fx_{n-1}, fx_n) + \beta d(fx_{n-1}, Tx_{n-1}) + \gamma d(fx_n, Tx_n) + \delta d(fx_{n-1}, Tx_n) + Ld(fx_n, Tx_{n-1})$$

for all $x, y \in D(f, T, x_0)$ *with* $d(fx, fy) > 0$, *where* $\alpha + \beta + \gamma + 2s\delta < 1$, $\gamma \neq 1$, $L \geq 0$, $\frac{(\alpha + \beta + s\delta)}{(1 - \gamma - s\delta)} = \lambda$ *and* $s.\lambda < 1$.

Now we state and prove our main result.

Theorem 13 *Let* (X, d) *be a complete b-metric space and* $T : X \to CB(X)$. *If* f *is a generalized multivalued* F_1 *-contraction of Hardy-Rogers-type with respect to a generalized dynamic process* $D(f, T, x_0)$. *Then* f *and* T *have a common fixed point.*

Proof: Let $x_0 \in X$ be an arbitrary point, by definition of generalized multivalued F_1 -contraction of Hardy-Rogers-type with respect to a generalized dynamic process $D(f, T, x_0)$, we have

$$\begin{aligned} F_1(d_{n+1}) &= F_1(d(x_{n+1}, x_{n+2})) = F_1(d(fx_n, fx_{n+1})) \\ &\leq F_1[(\alpha d(fx_{n-1}, fx_n) + \beta d(fx_{n-1}, Tx_{n-1}) \\ &\quad + \gamma d(fx_n, Tx_n) + \delta d(fx_{n-1}, Tx_n) \\ &\quad + Ld(fx_n, Tx_{n-1}))] - \tau[(\alpha d(fx_{n-1}, fx_n) \\ &\quad + \beta d(fx_{n-1}, Tx_{n-1}) + \gamma d(fx_n, Tx_n) \\ &\quad + \delta d(fx_{n-1}, Tx_n) + Ld(fx_n, Tx_{n-1}))] \end{aligned}$$

$$\begin{aligned}
&\le F_1[(\alpha d(fx_{n-1},fx_n)+\beta d(fx_{n-1},fx_n)\\
&\quad+\gamma d(fx_n,fx_{n+1})+\delta d(fx_{n-1},fx_{n+1})\\
&\quad+Ld(fx_n,fx_n))]-\tau[(\alpha d(fx_{n-1},fx_n)\\
&\quad+\beta d(fx_{n-1},Tx_{n-1})+\gamma d(fx_n,Tx_n)\\
&\quad+\delta d(fx_{n-1},Tx_n)+Ld(fx_n,Tx_{n-1}))]\\
&=F_1[(\alpha d(x_n,x_{n+1})+\beta d(x_n,x_{n+1})\\
&\quad+\gamma d(x_{n+1},x_{n+2})+\delta d(x_n,x_{n+2})\\
&\quad+Ld(x_{n+1},x_{n+1})]-\tau[(\alpha d(fx_{n-1},fx_n)\\
&\quad+\beta d(fx_{n-1},Tx_{n-1})+\gamma d(fx_n,Tx_n)\\
&\quad+\delta d(fx_{n-1},Tx_n)+Ld(fx_n,Tx_{n-1}))]\\
&\le F_1[(\alpha d(x_n,x_{n+1})+\beta d(x_n,x_{n+1})+\gamma d(x_{n+1},x_{n+2})\\
&\quad+\delta s[d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})]\\
&\quad+Ld(x_{n+1},x_{n+1})]-\tau[(\alpha d(fx_{n-1},fx_n)\\
&\quad+\beta d(fx_{n-1},Tx_{n-1})+\gamma d(fx_n,Tx_n)\\
&\quad+\delta d(fx_{n-1},Tx_n)+Ld(fx_n,Tx_{n-1}))]\\
&=F_1[(\alpha+\beta+s\delta)d_n+(\gamma+s\delta)d_{n+1}]\\
&\quad-\tau[(\alpha d(fx_{n-1},fx_n)+\beta d(fx_{n-1},Tx_{n-1})+\gamma d(fx_n,Tx_n)\\
&\quad+\delta d(fx_{n-1},Tx_n)+Ld(fx_n,Tx_{n-1}))]
\end{aligned}$$

$$F_1(d_{n+1})<F_1[(\alpha+\beta+s\delta)d_n+(\gamma+s\delta)d_{n+1}].$$

As F_1 is strictly increasing, therefore

$$d_{n+1}<(\alpha+\beta+s\delta)d_n+(\gamma+s\delta)d_{n+1},$$

$$(1-\gamma-s\delta)d_{n+1}<(\alpha+\beta+s\delta)d_n.$$

As $\alpha+\beta+\gamma+2s\delta<1$ and $\gamma\ne 1$, we deduce that $1-\gamma-s\delta>0$ and so

$$d_{n+1}<[(\alpha+\beta+s\delta)/(1-\gamma-s\delta)]d_n=d_n.$$

This implies that

$$d_{n+1}<\lambda d_n.$$

Continuing this process, we can easily say that

$$d_{n+1}<\lambda^n d_0.$$

Now, to show that $\{x_n\}$ is a Cauchy sequence in X. Let $m,n>0$ with $m>n$

$$\begin{aligned}
&d(x_n,x_m)\\
&\le s[d(x_n,x_{n+1})+d(x_{n+1},x_m)]\\
&\le sd(x_n,x_{n+1})+s^2d(x_{n+1},x_{n+2})+s^3d(x_{n+2},x_{n+3})+...\\
&\le s\lambda^n d(x_0,x_1)+s^2\lambda^{n+1}d(x_0,x_1)+s^3\lambda^{n+3}d(x_0,x_1)+...\\
&=s\lambda^n d(x_0,x_1)[1+s\lambda+(s\lambda)^2+(s\lambda)^3...]=\frac{s\lambda}{1-s\lambda}d(x_0,x_1).
\end{aligned}$$

Using Lemma 10, and taking limit $n\to\infty$, we get

$$\lim_{n\to\infty} d(x_n,x_m)=0.$$

which proves that $\{x_n\}$ is a Cauchy, so there exist some $x^*\in X$ such that $\lim_{n\to\infty} x_n=x^*$. Now we prove that $x^*=fx^*\in Tx^*$. For this, we have

$$\begin{aligned}
&\lim_{n\to\infty}[\alpha d(fx_{n-1},fx_n)+\beta d(fx_{n-1},Tx_{n-1})\\
&\quad+\gamma d(fx_n,Tx_n)+\delta d(fx_{n-1},Tx_n)+Ld(fx_n,Tx_{n-1}))]\\
&\le\lim_{n\to\infty}[\alpha d(fx_{n-1},fx_n)+\beta d(fx_{n-1},fx_n)\\
&\quad+\gamma d(fx_n,fx_{n+1})+\delta[d(fx_{n-1},fx_n)+d(fx_n,fx_{n+1})]\\
&=\alpha d(fx^*,fx^*)+\beta d(fx^*,fx^*)+\gamma d(fx^*,fx^*)\\
&\quad+\delta[d(fx^*,fx^*)+d(fx^*,fx^*)=0.
\end{aligned}$$

By (F2)

$$\lim_{n\to\infty}F_1[\alpha d(fx_{n-1},fx_n)+\beta d(fx_{n-1},Tx_{n-1})+\gamma d(fx_n,Tx_n)$$
$$+\delta d(fx_{n-1},Tx_n)+Ld(fx_n,Tx_{n-1}))]=-\infty.$$

Therefore

$$\lim_{n\to\infty}F_1(M(fx_n,fx_{n+1}))=-\infty.$$

From above we can write

$$\lim_{n\to\infty}F_1(d(x_{n+1},fx_{n+1})=\lim_{n\to\infty}F_1(d(fx_n,fx_{n+1}))=-\infty.$$

Again by (F2) $\lim_{n\to\infty}d(x_{n+1},fx_{n+1})=0.$ Therefore $d(x^*,fx^*)=0.$ So $x^*=fx^*$. Moreover

$$\lim_{n\to\infty}d(x_{n+1},Tx_n)\le\lim_{n\to\infty}d(fx_n,fx_{n+1})=d(fx^*,fx^*)=0.$$

Hence, $d(x^*,Tx^*)=0\Rightarrow x^*\in Tx^*$, so $x^*=fx^*\in Tx^*$, that is x^* is the common fixed point of f and T.

Putting $\alpha=\delta=L=0$, we obtain a generalize form of Kannan's result in dynamic process.

Corollary 14 *Let (X,d) be a complete b-metric space and $T: X\to CB(X)$. Assume that there exist $\tau\in R_+$ and $F_1\in\mathcal{F}_1$ such that F_1 is continuous satisfying*:

$$\tau(\beta d(fx_{n-1},Tx_{n-1})+\gamma d(fx_n,Tx_n))+F_1(d(fx_n,fx_{n+1}))$$
$$\le F_1(\beta d(fx_{n-1},Tx_{n-1})+\gamma d(fx_n,Tx_n))$$

for all $x,y\in D(f,T,x_0)$ with $d(fx,fy)>0$, where $\beta+\gamma<1$, $\gamma\ne 1$, $L\ge 0$, $\frac{\beta}{1-\gamma}=\lambda$ and $s.\lambda<1$. Then f and T have a common fixed point.

Choosing $\delta=L=0$, we obtain a generalize version of Reich's result.

Corollary 15 *Let (X,d) be a complete b-metric space and $T: X\to CB(X)$. Assume that there exist $\tau\in R_+$ and $F_1\in\mathcal{F}_1$ such that F_1 is continuous satisfying*:

$$\begin{aligned}
&\tau[(\alpha d(fx_{n-1},fx_n)+\beta d(fx_{n-1},Tx_{n-1})\\
&\quad+\gamma d(fx_n,Tx_n)]+F_1(d(fx_n,fx_{n+1}))\\
&\le F_1[(\alpha d(fx_{n-1},fx_n)+\beta d(fx_{n-1},Tx_{n-1})+\gamma d(fx_n,Tx_n)]
\end{aligned}$$

for all $x,y\in D(T,x_0)$ with $d(fx,fy)>0$, where $\alpha+\beta+\gamma<1$, $\gamma\ne 1$, $\frac{(\alpha+\beta)}{(1-\gamma)}=\lambda$ and $s.\lambda<1$. Then f and T have a common fixed point.

Theorem 16 Let (X,d) *be a complete b-metric space,* $f : X \to X$ *is continuous and* $T : X \to CB(X)$. *Assume that there exist* $\tau \in R_+$ *and* $F_1 \in \mathcal{F}_1$ *such that* F_1 *is continuous from the right. Now if* $D(f,T,x_0)$ *is a generalized dynamic process such that*

$$2\tau(M(x,y)) + F_1(H(Tx,Ty)) \leq F_1(M(x,y)) \quad (2)$$

where

$$M(x,y) = \alpha d(fx,fy) + \beta d(fx,Tx) + \gamma d(fy,Ty) + \delta d(fx,Ty) + Ld(fy,Tx)$$

for all $x, y \in D(f,T,x_0)$ *with* $d(fx,fy) > 0$, *where* $\alpha + \beta + \gamma + 2s\delta < 1$, $\gamma \neq 1$, $L \geq 0$, $\frac{(\alpha+\beta+s\delta)}{(1-\gamma-s\delta)} = \lambda$ *and* $s.\lambda < 1$.. *Then* f *and* T *have a common fixed point.*

Proof: Let $x_0 \in X$ be an arbitrary point of X. By definition of generalized dynamic process $fx_1 \in Tx_0$. Since F_1 is continuous from the right, there exists a real number $h > 1$ such that

$$F_1(hH(Tx_0,Tx_1)) < F_1(H(Tx_0,Tx_1)) + \tau(M(x_0,x_1)).$$

Now, from $d(fx_1,Tx_1) < hH(Tx_0,Tx_1)$, we deduce that there exists $fx_2 \in Tx_1$ such that $d(fx_1,fx_2) \leq hH(Tx_0,Tx_1)$. Consequently, we get

$$F_1(d(fx_1,fx_2)) \leq F_1(hH(Tx_0,Tx_1)) < F_1(H(Tx_0,Tx_1)) + \tau(M(x_0,x_1)),$$

which implies

$$\begin{aligned}
&2\tau(M(x_0,x_1)) + F_1(d(x_2,x_3)) \\
&= 2\tau(M(x_0,x_1)) + F_1(d(fx_1,fx_2)) \\
&\leq 2\tau(M(x_0,x_1)) + F_1(H(Tx_0,Tx_1)) + \tau(M(x_0,x_1)) \\
&= F_1(\alpha d(x_1,x_2) + \beta d(x_1,Tx_0) + \gamma d(x_2,Tx_1) \\
&\quad + \delta d(x_1,Tx_1) + Ld(x_2,Tx_0)) + \tau(M(x_0,x_1)) \\
&\leq F_1(\alpha d(x_1,x_2) + \beta d(x_1,fx_1) + \gamma d(x_2,fx_2) + \delta d(x_1,fx_2) \\
&\quad + Ld(x_2,fx_1)) + \tau(M(x_0,x_1)) \\
&= F_1(\alpha d(x_1,x_2) + \beta d(x_1,x_2) \\
&\quad + \gamma d(x_2,x_3) + \delta d(x_1,x_3) \\
&\quad + Ld(x_2,x_2)) + \tau(M(x_0,x_1)) \\
&\leq F_1[\alpha d(x_1,x_2) + \beta d(x_1,x_2) \\
&\quad + \gamma d(x_2,x_3) + s[\delta(d(x_1,x_2) + d(x_2,x_3))] \\
&\quad + \tau(M(x_0,x_1)) \\
&= F_1[(\alpha+\beta+s\delta)d_1 + (\gamma+s\delta)d_2] + \tau(M(x_0,x_1))
\end{aligned}$$

$$\tau(M(x_0,x_1)) + F_1((1-\gamma-s\delta)d_2) \leq F_1[(\alpha+\beta+s\delta)d_1].$$

As F_1 is strictly increasing we deduce

$$(1-\gamma-s\delta)d_2 \leq (\alpha+\beta+s\delta)d_1.$$

As $\alpha+\beta+\gamma+2s\delta < 1$ and $\gamma \neq 1$ hence we deduce that $1-\gamma-s\delta > 0$ and so $d_2 \leq \frac{(\alpha+\beta+s\delta)}{(1-\gamma-s\delta)} d_1 = \lambda d_1$, consequently,

$$d_2 \leq \lambda d_1.$$

Continuing this way we get

$$d_{n+1} < \lambda d_n,$$

and hence

$$d_{n+1} < \lambda^n d_0.$$

Proceeding this as in Theorem 13, we obtain that $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists some $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Now we prove that $x^* = fx^* \in Tx^*$. For this, since

$$\begin{aligned}
&\lim_{n\to\infty}[\alpha d(fx_{n-1},fx_n) + \beta d(fx_{n-1},Tx_{n-1}) + \gamma d(fx_n,Tx_n) \\
&+ \delta d(fx_{n-1},Tx_n) + Ld(fx_n,Tx_{n-1}))] \\
&\leq \lim_{n\to\infty}[\alpha d(fx_{n-1},fx_n) + \beta d(fx_{n-1},fx_n) \\
&+ \gamma d(fx_n,fx_{n+1}) + \delta[d(fx_{n-1},fx_n) + d(fx_n,fx_{n+1})] \\
&= \alpha d(fx^*,fx^*) + \beta d(fx^*,fx^*) + \gamma d(fx^*,fx^*) \\
&+ \delta[d(fx^*,fx^*) + d(fx^*,fx^*)] = 0.
\end{aligned}$$

By (F2)

$$\begin{aligned}
&\lim_{n\to\infty} F_1[\alpha d(fx_{n-1},fx_n) + \beta d(fx_{n-1},Tx_{n-1}) \\
&+ \gamma d(fx_n,Tx_n) + \delta d(fx_{n-1},Tx_n) + Ld(fx_n,Tx_{n-1}))] = -\infty.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} F_1(H(Tx_{n-1},Tx_n)) = -\infty$. Hence

$$\begin{aligned}
&\lim_{n\to\infty} F_1(d(x_{n+1},fx_{n+1})) \\
&\leq \lim_{n\to\infty} F_1(hH(Tx_{n-1},Tx_n)) \\
&< \lim_{n\to\infty}[F_1(H(Tx_{n-1},Tx_n)) + \tau(M(x_{n-1},x_n))] = -\infty.
\end{aligned}$$

By using (F2) $\lim_{n\to\infty} d(x_{n+1},fx_{n+1}) = 0$ which implies $x^* = fx^*$. Also

$$\begin{aligned}
&\lim_{n\to\infty} F_1(d(fx_n,Tx_n)) \leq \lim_{n\to\infty} F_1(hH(Tx_{n-1},Tx_n)) \\
&< \lim_{n\to\infty}[F_1(H(Tx_{n-1},Tx_n)) + \tau(M(x_0,x_1))] = -\infty.
\end{aligned}$$

Therefore $\lim_{n\to\infty} d(fx_n,Tx_n) = 0$, hence $fx^* = Tx^*$, that is $x^* = fx^* \in Tx^*$.

Replacing b-metric space by metric space in the above result we get the following corollary:

Corollary 17 *Assume that* (X,d) *is a complete metric space,* $f : X \to X$ *is continuous and* $T : X \to CB(X)$. *Assume that there exist* $\tau \in R_+$ *and* $F_1 \in \mathcal{F}_1$ *such that* F_1 *is continuous from the right. Now if* $D(f,T,x_0)$ *is a generalized dynamic process in such a way that*

$$2\tau(M(x,y)) + F_1(H(Tx,Ty)) \leq F_1(M(x,y))$$

where

$$M(x,y) = \alpha d(fx,fy) + \beta d(fx,Tx) + \gamma d(fy,Ty) + \delta d(fx,Ty) + Ld(fy,Tx)$$

for each $x, y \in (fx_n)$ with $d(fx, fy) > 0$, where $\alpha+\beta+\gamma+2\delta=1$, $\gamma \neq 1$ and $L \geq 0$. Then f and T possess a common fixed point.

Example 18 *Assume that $X=[0,\infty)$ and d is defined by*

$$d(x,y)=\begin{cases}0 & \text{if } x=y\\ 80h & \text{if } x\neq y\end{cases}$$

where $h>1$. Define $f: X \to X$, $T: X \to CB(X)$ and $F: R_+ \to R$ by $fx=\frac{x}{2}$, $Tx=[0,\frac{x}{2}]$ and $F(x)=\ln(x)$. Define a sequence $\{x_n\}$ by for all $n \in N$, $x_n = x_0 r^{n-1}$ with $x_0=2$ and $r=\frac{1}{2}$ Then

$$x_2 = 1 = f(x_1) \in T(x_0) = [0,1]$$
$$x_3 = \frac{1}{2} = f(x_2) \in T(x_1) = [0,1]$$
$$x_4 = \frac{1}{4} = f(x_3) \in T(x_2) = [0,\frac{1}{2}]$$

and so on. Here

$$D(f,T,2) = \{1, \frac{1}{2}, \frac{1}{4}, ...\}$$

Fix $\alpha=\beta=\gamma=\frac{1}{20}$, $\delta=\frac{1}{80}$, $L=1$ and $s=\frac{3}{2}$, clearly $s.\lambda<1$. We can check that d is a b-metric. Now checking for all $x,y \in D(f,T,2)$ we can find some $\tau : R_+ \to R_+$ that satisfy the inequality (2) in such a way that $H(Tx,Ty)$, $M(x,y)>0$ for each $x,y \in D(f,T,2)$. Moreover $0=f(0)\in T(0)$ is found to be the common fixed point of f and T.

Putting $\beta=\gamma=\delta=L=0$ and $\alpha=1$ in the above result we get the following corollary:

Corollary 19 *Assume that (X,d) is a complete b-metric space, $f: X\to X$ is continuous, $T: X \to CB(X)$, there exist a function $\tau: R_+ \to R_+$ and let $F_1 : R^+ \to R$ satisfy (F1)-(F2). Now if (fx_n) is a generalized dynamic process in such a way that*

$$\text{for all } x,y\in(fx_n) \, (d(Tx,Ty)>0$$
$$\Rightarrow 2\tau(d(fx,fy)) + F_1(H(Tx,Ty)) \leq F_1(d(fx,fy))),$$

then there is a common fixed point of f and T i.e. $x = fx \in Tx$.

Putting f as an identity function in Corollary 19 we get:

Corollary 20 *Assume that (X,d) is a complete metric space, $f: X \to X$ is continuous $T: X\to CB(X)$, there exist a function $\tau: R_+ \to R_+$ and suppose $F_1 : R^+ \to R$ satisfy (F1)-(F2). Now if (x_n) is a dynamic process in such a way that*:

$$d(Tx,Ty)>0 \Rightarrow 2\tau(d(x,y)) + F_1(H(Tx,Ty)) \leq F_1(d(x,y)),$$

then there exists a fixed point of T.

Interests

The authors declare that they have no competing interests.

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