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Notes on Several Families of Differential Equations Related to the Generating Function for the Bernoulli Numbers of the Second Kind

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Abstract In the paper, the author notes several families of ordinary differential equations related to the generating function for the Bernoulli numbers of the second kind.

Keywords: *ordinary differential equation, generating function, Bernoulli number of the second kind*

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It is well known [1,2,3,4] that the Bernoulli numbers of the second kind b_n for $n\ge 0$ can be generated by

$$\frac{t}{\ln(1+t)}=\sum_{n=0}^{\infty}b_n t^n.$$

This means that the reciprocal $\frac{1}{\ln(1+t)}$ of the logarithmic function $\ln(1+t)$ is related to the Bernoulli numbers of the second kind b_n for $n\ge 0$.

In [[5], Theorem 2.1], it was obtained inductively and recursively that the nonlinear differential equations

$$n!F^{n+1}(t)=(-1)^n\sum_{i=1}^{n}a_i(n)(1+t)^i F^{(i)}(t), n\in\mathbb{N} \tag{1}$$

have a solution $F(t)=\frac{1}{\ln(1+t)}$, where $a_1(n)=1$ and

$$a_i(n)=\sum_{k_{i-1}=0}^{n-i}\sum_{k_{i-2}=0}^{n-i-k_{i-1}}\cdots\sum_{k_1=0}^{n-i-k_{i-1}-\cdots-k_2} i^{k_{i-1}}(i-1)^{k_{i-2}}\dots 2^{k_1} \tag{2}$$

for $2\le i\le n$.

In [[6], Lemma 2.1], it was obtained also inductively and recursively that the family of differential equations

$$n!G^{n+1}(t)=(-1)^n\sum_{k=1}^{n}S(n,k)\lambda^{n-k}(1+\lambda t)^k G^k(t) \tag{3}$$

for $n\in\mathbb{N}$ has a solution $G(t)=\frac{\lambda}{\ln(1+\lambda t)}$, where $S(n,k)$ denotes the Stirling numbers of the second kind. In [[6], Corollary 2.2], it was claimed that

$$a_k(n)=S(n,k). \tag{4}$$

In [[7], Theorem 1], it was established still inductively and recursively that the family of differential equations

$$(-1)^n(r)_n H(t)=[\ln(1+t)]^n\sum_{i=1}^{n}b_i(n)(1+t)^i H^{(i)}(t)$$

for $n\in\mathbb{N}$ has a solution

$$H(t)=H(t,r)=\left[\frac{1}{\ln(1+t)}\right]^r, r\in\mathbb{N},$$

where $b_1(n)=1$ and

$$b_i(n)=\sum_{k_{i-1}=0}^{n-i}\sum_{k_{i-2}=0}^{n-i-k_{i-1}}\cdots\sum_{k_1=0}^{n-i-k_{i-1}-\cdots-k_2}\prod_{\ell=2}^{i}\ell^{k_{\ell-1}} \tag{5}$$

for $2\le i\le n$, and

$$(x)_n=\prod_{\ell=0}^{n-1}(x+\ell)=\begin{cases}x(x+1)(x+2)\cdots(x+n-1), & n\ge 1\\ 1, & n=0\end{cases}$$

is called [8] the rising factorial.

The proofs of [[7], Theorem 1], [[5], Theorem 2.1], and [[6], Lemma 2.1] are almost the same and have the same shortcoming: they are rather long and tedious. More importantly, the expressions (2) and (5) are difficult to be understood, to be remembered, and to be computed.

By virtue of the Faà di Bruno formula

$$\frac{d^n}{dt^n} f \circ h(t) = \sum_{k=0}^{n} f^{(k)}(h(t)) B_{n,k}\left(h'(t), h''(t), \dots, h^{(n-k+1)}(t)\right)$$

for $n \ge 0$, where the Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ for $n \ge k \ge 0$ are defined [[9], p. 134, Theorem A] and [[9], p. 139, Theorem C] by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \le i \le n, \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^{n} i\ell_i = n \\ \sum_{i=1}^{n} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i},$$

in view of two identities

$$B_{n,k}\left(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\right) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$$

and

$$B_{n,k}(0!, 1!, 2!, \dots, (n-k)!) = (-1)^{n-k} s(n,k)$$

in [[9], p. 135], where a and b are any complex numbers and $s(n,k)$ stands for the Stirling numbers of the first kind, and in light of an inversion theorem [[10], p. 171, Theorem 12.1] which reads that

$$a_n = \sum_{\alpha=0}^{n} S(n,\alpha) b_\alpha \text{ if and only if } b_n = \sum_{k=0}^{n} s(n,k) a_k, \tag{6}$$

two families of differential equations

$$H^{(n)}(t) = \left(\frac{1}{1+t}\right)^n \left[\sum_{k=0}^{n} s(n,k) \frac{\langle -r \rangle_k}{[\ln(1+t)]^k}\right] H(t) \tag{7}$$

and

$$\sum_{k=0}^{n} s(n,k)(1+t)^k H^k(t) = \frac{\langle -r \rangle_n}{[\ln(1+t)]^n} H(t) \tag{8}$$

for $n \ge 0$ and $r \in \mathbb{R}$ were standardly established in [[11], Theorem 1], where the falling factorial $\langle x \rangle_n$ of $x \in \mathbb{R}$ for $0 \in \{0\} \cup \mathbb{N}$ is defined [8] by

$$\langle x \rangle_n = \prod_{\ell=0}^{n-1} (x-\ell) = \begin{cases} x(x-1)(x-2)\cdots(x-n+1), & n \ge 1; \\ 1, & n = 0. \end{cases}$$

Since $H(t,1) = F(t)$, taking $r = 1$ in (8) derives

$$\sum_{k=0}^{n} S(n,k)(1+t)^k F^{(k)}(t) = \langle -1 \rangle_n F^{n+1}(t) = (-1)^n n! F^{n+1}(t). \tag{9}$$

Comparing (9) with (1) results in the equality (4). Replacing t by λt in (9) yields the differential equation (3).

Letting $r = 1$ in (7) arrives at

$$F^{(n)}(t) = \left(\frac{1}{1+t}\right)^n \sum_{k=1}^{n} (-1)^k k! s(n,k) F^{k+1}(t) \tag{10}$$

for $n \in \mathbb{N}$, which is an inversion formula of (1). This can also be deduced from applying (6) to (1).

Replacing t by λt in (10) gives

$$G^{(n)}(t) = \frac{\lambda^n}{(1+\lambda t)^n} \sum_{k=1}^{n} (-1)^k S(n,k) \frac{k!}{\lambda^k} G^{k+1}(t)$$

for $n \in \mathbb{N}$, which is an inversion formula of (3). This can also be acquired by applying (6) to (3).

Remark. The motivations in the papers [4,11-36] are same as the one in this paper. This paper is a slightly modified version of the preprint [37].

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