

Rational Inequalities Using Compatibility, Weak Compatibility and Common Properties in G-Metric Space

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Abstract We explain the popularized concept of G-metric space and some current common properties of functions like compatibility and others. Particularly two self functions are specified and some most achieved fixed point theorems are generalized using some rational inequalities.

Keywords: G-metric space, rational inequality, compatibility of maps, property (E.A) and CLR_g

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1. Introduction

Fixed point theory is a theory with enormous applications in multi branches of mathematics. Pioneering research work of this standard theory is published in 1922 by Stefan Banach [1], and iterative method was used to fix the unique fixed points. This theory is very interesting to understand and is applicable in almost all fields of mathematics especially in ordinary differential equations. In 2004, Mustafa and Sims [6] introduced the concept of generalized metric space which is generalization of the ordinary metric space. It was sturdy generalization of metric spaces. The following concept of G-metric space and some basic definitions of G-metric space is given by some mathematicians (see, for example, [2,3,5,6,7,8]).

Definition 1.1. Let $X \neq \emptyset$ be a set and $G: X \times X \times X \rightarrow [0, \infty)$ be a function satisfying the following axioms:

- (G1) $G(x, y, z) = 0$ if $x = y = z$,
- (G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$
- (G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$
- (G4) $G(x, y, z) = G(x, z, y) = G(y, z, x)$
(symmetry in all three variables),
- (G5) $G(x, y, z) = G(x, a, a) + G(a, y, z)$
for all $x, y, z, a \in X$, (rectangle inequality)

then the function G is called a generalized metric, or specifically a G-metric on X and the pair (X, G) is called a G-metric Space.

Definition 1.2. Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points in X , a point x in X is said to be

the limit of the sequence $\{x_n\}$ if $G(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is G-convergent to x . Thus, if $x_n \rightarrow x$ or $x_n = x$ as $n \rightarrow \infty$, in a G-metric space (X, G) , then for each $\varepsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \varepsilon$ for all $m, n \in N$.

Proposition 1.1. For a G-metric space (X, G) , the following are equivalent:

- i. $\{x_n\}$ is G-convergent to x ,
- ii. $G(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$,
- iii. $G(x_n, x, x) \rightarrow 0$ as $n \rightarrow \infty$,
- iv. $G(x_m, x_n, x) \rightarrow 0$ as $m, n \rightarrow \infty$.

Definition 1.3. Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if, for each $\varepsilon > 0$, there exists a positive integer N such that

- 1. $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \in N$, i.e., if
- 2. $G(x_n, x_m, x_l) \rightarrow 0$ as $n, m, l \rightarrow \infty$.

Definition 1.4. If (X, G) and (X', G') be two G-metric space and let $f: (X, G) \rightarrow (X', G')$ be a function, then f is said to be G-continuous at a point $x_0 \in X$ if for given $\varepsilon > 0$, there exists $\delta > 0$ such that for $x, y \in X$ and $G(x_0, x, y) < \delta$ implies $G(f(x_0), f(x), f(y)) < \varepsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $x_0 \in X$ or function f is said to be G-continuous at a point $x_0 \in X$ if and only if it is G-sequentially continuous at x_0 , that is, whenever $\{x_n\}$ is G-convergent to x_0 , $\{f(x_n)\}$ is G-convergent to $f(x_0)$.

Definition 1.5. A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in X .

Proposition 1.2. A G-metric space (\mathcal{X}, G) is said to be G-complete if and only if (\mathcal{X}, d_G) is a complete metric space.

Proposition 1.3. Let (\mathcal{X}, G) be a G-metric space. Then, for any x, y, z and a in \mathcal{X} , it follows that:

- (i) if $G(x, y, z) = 0$, then $x = y = z$,
- (ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
- (iii) $G(x, y, y) \leq 2G(y, x, x)$,
- (iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
- (v) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,
- (vi) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

Definition 1.6. Let f and g be two self mappings on a metric space (\mathcal{X}, G) . The mappings f and g are said to be compatible if there exist a sequence $\{x_n\}$ such that $\lim_{n \rightarrow \infty} (fgx_n, gfx_n, gfx_n) = 0$, or $G(fgx_n, gfx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in \mathcal{X} such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t, \text{ for some } t \in \mathcal{X}.$$

Definition 1.7. Let f and g be two self mappings on a metric space (\mathcal{X}, G) . The mappings f and g are said to satisfy the property (E.A), if there exist a sequence $\{x_n\}$ in \mathcal{X} such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = x$$

for some $x \in \mathcal{X}$.

Definition 1.8. Let f and g be two self mappings on a metric space (\mathcal{X}, G) . The mappings f and g are said to satisfy the common limit range (CLR_g) property if there exist a sequence $\{x_n\}$ in \mathcal{X} such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = gx$ for some $x \in \mathcal{X}$.

Definition 1.9. Two maps are said to be weakly compatible if they commute at coincidence points.

2. Main Result

This is the result for unique fixed point for pair of weak compatible maps.

Theorem 2.1. Let (\mathcal{X}, G) be a complete G-metric space and $f, g : \mathcal{X} \rightarrow \mathcal{X}$ be the self mapping on (\mathcal{X}, G) satisfying the following conditions:

- (1.1) $f(\mathcal{X}) \subseteq G(\mathcal{X})$
- (1.2) f or g is continuous,
- (1.3)

$$G(fx, fy, fz) \leq \Omega \max \left(\begin{aligned} & \left(\frac{G(gx, gy, gy) + G(fx, fy, gy)}{G(gx, gy, gz) + G(fx, gz, fz)} \right) \cdot (G(gx, gy, gz)), \\ & \left(\frac{G(gx, gz, fz) + G(gz, fz, fz)}{G(gx, gz, fy) + G(gy, fy, fy)} \right) \cdot (G(fx, fy, fz)), \\ & \frac{G^2(fy, fz, fx) \cdot G(gy, gy, gz)}{G(fx, fy, fz) + G(fz, fy, fx)}, \\ & \left(\frac{G(fx, fy, gz) + G(gy, fy, fz)}{G(fx, fz, gy) + G(gz, fz, fz)} \right) \cdot (G(fy, gx, gx)) \end{aligned} \right)$$

for all $x, y, z \in \mathcal{X}$, where $0 \leq \Omega < \frac{1}{8}$. Including the fact

that f and g are compatible maps then f and g have a unique common fixed point in \mathcal{X} .

Proof. Let $x_0 \in \mathcal{X}$ be a random selection, then by (1.1), there may exist one point or one may choose some point $x_1 \in \mathcal{X}$ such that $fx_0 = gx_1$. In general one can choose $x_{n+1} \in \mathcal{X}$ such that

$$y_n = fx_n = gx_{n+1}, n=0, 1, 2, 3.$$

From (1.3), we have

$$G(y_n, y_{n+1}, y_{n+1}) = G(fx_n, fx_{n+1}, fx_{n+1})$$

$$= \Omega \max \left(\begin{aligned} & \left(\frac{G(fx_{n-1}, fx_n, fx_n) + G(fx_n, fx_{n+1}, fx_n)}{G(fx_{n-1}, fx_n, fx_n) + G(fx_n, fx_n, fx_{n+1})} \right) \cdot \\ & \quad (G(fx_{n-1}, fx_n, fx_n)), \\ & \left(\frac{G(fx_{n-1}, fx_n, fx_n) + G(fx_n, fx_{n+1}, fx_{n+1})}{G(fx_{n-1}, fx_n, fx_{n+1}) + G(fx_n, fx_{n+1}, fx_{n+1})} \right) \cdot \\ & \quad (G(fx_n, fx_{n+1}, fx_{n+1})), \\ & \frac{G^2(fx_{n+1}, fx_{n+1}, fx_n) \cdot G(fx_n, fx_n, fx_n)}{G(fx_n, fx_{n+1}, fx_{n+1}) + G(fx_{n+1}, fx_{n+1}, fx_n)}, \\ & \left(\frac{G(fx_n, fx_{n+1}, fx_n) + G(fx_n, fx_{n+1}, fx_{n+1})}{G(fx_n, fx_{n+1}, fx_n) + G(fx_n, fx_{n+1}, fx_{n+1})} \right) \cdot \\ & \quad (G(fx_{n+1}, fx_{n-1}, fx_{n-1})) \end{aligned} \right)$$

$$= \Omega \max \left(\begin{aligned} & \left(\frac{G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_n)}{G(y_{n-1}, y_n, y_n) + G(y_n, y_{n-1}, y_n)} \right) \cdot \\ & \quad G(y_{n-1}, y_n, y_n), \\ & \left(\frac{G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1})}{G(y_{n-1}, y_n, y_{n+1}) + G(y_n, y_{n+1}, y_{n+1})} \right) \cdot \\ & \quad (G(y_n, y_{n+1}, y_{n+1})), \\ & \frac{G^2(y_{n+1}, y_{n+1}, y_n) \cdot G(y_n, y_n, y_n)}{G(y_n, y_{n+1}, y_{n+1}) + G(y_{n-1}, y_{n+1}, y_n)}, \\ & \left(\frac{G(y_n, y_{n+1}, y_n) + G(y_n, y_{n+1}, y_{n+1})}{G(y_n, y_{n+1}, y_n) + G(y_n, y_{n+1}, y_{n+1})} \right) \cdot \\ & \quad (G(y_{n+1}, y_{n-1}, y_{n-1})) \end{aligned} \right)$$

$$= \Omega \max \left(G(y_{n-1}, y_n, y_n), G(y_n, y_{n+1}, y_{n+1}), 0 \right),$$

Using the preposition $G(x, y, y) \leq 2G(y, x, x)$, we have

$$\leq \Omega \max \left(G(y_{n-1}, y_n, y_n), G(y_n, y_{n+1}, y_{n+1}), 0, 2G(y_{n-1}, y_{n+1}, y_{n+1}) \right)$$

Case 1.

If the max is taken as,

$$\max \left(G(y_{n-1}, y_n, y_n), G(y_n, y_{n+1}, y_{n+1}), 0, 2G(y_{n-1}, y_{n+1}, y_{n+1}) \right) = G(y_{n-1}, y_n, y_n)$$

Then using (1.3), we Get

$$G(y_n, y_{n+1}, y_{n+1}) \leq \Omega G(y_{n-1}, y_n, y_n).$$

Repetitively we have

$$G(y_n, y_{n+1}, y_{n+1}) \leq \Omega^n G(y_0, y_1, y_1).$$

Hence for every natural n and m where $n < m$, we have

$$\begin{aligned} &G(y_n, y_m, y_m) \\ &\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) \\ &\quad + \dots + G(y_{m-1}, y_m, y_m) \\ &\leq (\Omega^n + \Omega^{n+1} + \Omega^{n+2} \dots + \Omega^{m-1}) G(y_0, y_1, y_1) \\ &\leq \frac{k^n}{1-k} G(y_0, y_1, y_1). \end{aligned}$$

For limiting values of n and m as infinity, we have $\lim_{n \rightarrow \infty} G(y_n, y_m, y_m) = 0$. Therefore $\{y_n\}$ is a G-Cauchy sequence.

Case 2.

If

$$\max \left(G(y_{n-1}, y_n, y_n), G(y_n, y_{n+1}, y_{n+1}), 0, 2G(y_{n-1}, y_{n+1}, y_{n+1}) \right) = G(y_n, y_{n+1}, y_{n+1})$$

Then (1.3) becomes,

$$G(y_n, y_{n+1}, y_{n+1}) \leq \Omega G(y_{n-1}, y_{n+1}, y_{n+1}),$$

Which is a contradiction because $0 \leq \Omega < \frac{1}{8}$.

Hence the sequence is again a G-Cauchy sequence.

Case 3.

If the max is taken as,

$$\max \left(G(y_{n-1}, y_n, y_n), G(y_n, y_{n+1}, y_{n+1}), 0, 2G(y_{n-1}, y_{n+1}, y_{n+1}) \right) = 2G(y_{n-1}, y_{n+1}, y_{n+1})$$

Then using (1.3), we Get

$$G(y_n, y_{n+1}, y_{n+1}) \leq 2\Omega G(y_{n-1}, y_n, y_n).$$

$$G(y_n, y_{n+1}, y_{n+1}) \leq 2\Omega(G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1}))$$

$$G(y_n, y_{n+1}, y_{n+1}) \leq 2\Omega(G(y_{n-1}, y_n, y_n))$$

$$G(y_n, y_{n+1}, y_{n+1}) = \frac{2\Omega}{1 + 2\Omega} (G(y_{n-1}, y_n, y_n))$$

$$G(y_n, y_{n+1}, y_{n+1}) = p (G(y_{n-1}, y_n, y_n)),$$

where $p = \frac{2\Omega}{1 + 2\Omega}$, it is obvious that $p < 1$. Also

$$0 \leq \Omega < \frac{1}{8}.$$

It is clear from case 1 that it is again a G-Cauchy sequence in G-metric space.

For all the cases we have G-Cauchy sequence, also it is mentioned that the given G-metric space is complete.

Therefore the Cauchy sequence is convergent and the point of convergence belongs to the given metric space. i.e.

$$\lim_{n \rightarrow \infty} y_n = \mu,$$

We have

$$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_{n+1} = \mu.$$

Because of the continuity of one of the two maps f or g , let us assume that the map g is continuous and hence we have,

$$\lim_{n \rightarrow \infty} g f x_n = \lim_{n \rightarrow \infty} g g x_{n+1} = g\mu.$$

Additionally, f and g are compatible, hence

$$\lim_{n \rightarrow \infty} G(g f x_n, f g x_n, f g x_n) = 0 \text{ gives}$$

$$\lim_{n \rightarrow \infty} f g x_n = g\mu.$$

Now, from (2.1), we have

$$G(f g x_n, f x_n, f x_n)$$

$$\leq \Omega \max \left(\begin{aligned} &\left(\frac{G(g g x_n, g x_n, g x_n) + G(f g x_n, f x_n, g x_n)}{G(g g x_n, g x_n, g x_n) + G(f g x_n, g x_n, f x_n)} \right), \\ &G(g g x_n, g x_n, g x_n), \\ &\left(\frac{G(g g x_n, g x_n, f x_n) + G(g x_n, f x_n, f x_n)}{G(g g x_n, g x_n, f x_n) + G(g x_n, f x_n, f x_n)} \right), \\ &G(f g x_n, f x_n, f x_n), \\ &\frac{G^2(f x_n, f x_n, f g x_n) * G(g x_n, g x_n, g x_n)}{G(f g x_n, f x_n, f x_n) + G(f x_n, f x_n, f g x_n)}, \\ &\left(\frac{G(f g x_n, f x_n, g x_n) + G(g x_n, f x_n, f x_n)}{G(f g x_n, f x_n, g x_n) + G(g x_n, f x_n, f x_n)} \right), \\ &G(f x_n, g g x_n, g g x_n) \end{aligned} \right).$$

taking n approaches to infinity,

$$G(g\mu, u, u) \leq \Omega \max \left(\begin{array}{l} \frac{G(g\mu, \mu, \mu) + G(g\mu, \mu, \mu)}{G(g\mu, \mu, \mu) + G(g\mu, \mu, \mu)} \cdot G(g\mu, \mu, \mu), \\ \frac{G(g\mu, \mu, \mu) + G(\mu, \mu, \mu)}{G(g\mu, \mu, \mu) + G(g\mu, \mu, \mu)} \cdot G(g\mu, \mu, \mu), \\ \frac{G^2(\mu, \mu, g\mu) * G(\mu, \mu, \mu)}{G(g\mu, \mu, \mu) + G(\mu, \mu, g\mu)}, \\ \frac{G(g\mu, \mu, \mu) + G(\mu, \mu, \mu)}{G(g\mu, \mu, \mu) + G(\mu, \mu, \mu)} \cdot G(\mu, g\mu, g\mu) \end{array} \right)$$

$$G(gu, fx_n, fx_n) \leq \Omega \max \left(\begin{array}{l} G(g\mu, \mu, \mu), G(g\mu, \mu, \mu), 0 \\ G(\mu, g\mu, g\mu) \end{array} \right)$$

Case 1

$$\Omega \max \left(\begin{array}{l} G(g\mu, \mu, \mu), G(g\mu, \mu, \mu) \\ \frac{G^2(\mu, \mu, g\mu)}{2} \\ G(\mu, g\mu, g\mu) \end{array} \right) = G(g\mu, \mu, \mu)$$

$$G(gu, u, u) \leq \Omega G(g\mu, \mu, \mu)$$

It is a contradiction since $\Omega < \frac{1}{8}$

Case 2

$$\Omega \max \left(\begin{array}{l} G(g\mu, \mu, \mu), G(g\mu, \mu, \mu), 0 \\ 2G(g\mu, \mu, \mu) \end{array} \right) = 2G(g\mu, \mu, \mu)$$

$$G(gu, u, u) \leq \Omega G(g\mu, \mu, \mu)$$

which is a contradiction since $\Omega < \frac{1}{8}$.

Hence $g\mu = \mu$, next we are to show that $g\mu = f\mu = \mu$, to prove this substitute $x = x_m, y = z = \mu$ in (1.3)

$$G(fx_n, f\mu, f\mu) \leq \Omega \max \left(\begin{array}{l} \frac{G(gx_n, g\mu, g\mu) + G(f\mu, g\mu, f\mu)}{G(gx_n, g\mu, g\mu) + G(fx_n, gx_n, g\mu)} \cdot G(gx_n, g\mu, g\mu), \\ \frac{G(gx_n, g\mu, f\mu) + G(g\mu, f\mu, f\mu)}{G(gx_n, g\mu, f\mu) + G(g\mu, f\mu, f\mu)} \cdot G(fx_n, f\mu, f\mu), \\ \frac{G^2(f\mu, f\mu, fx_n) \cdot G(g\mu, g\mu, g\mu)}{G(fx_n, f\mu, f\mu) + G(f\mu, f\mu, fx_n)}, \\ \frac{G(fx_n, f\mu, g\mu) + G(g\mu, f\mu, f\mu)}{G(fx_n, f\mu, g\mu) + G(g\mu, f\mu, f\mu)} \cdot G(f\mu, gx_n, gx_n) \end{array} \right)$$

If n approaches to infinity,

$$G(\mu, f\mu, f\mu) \leq \Omega \max \left(\begin{array}{l} \left(\frac{G(\mu, g\mu, g\mu) + G(\mu, f\mu, \mu)}{G(\mu, \mu, \mu) + G(\mu, \mu, f\mu)} \right) \cdot (G(\mu, \mu, \mu)), \\ \left(\frac{G(\mu, \mu, f\mu) + G(\mu, f\mu, f\mu)}{G(\mu, \mu, f\mu) + G(\mu, f\mu, f\mu)} \right) \cdot (G(\mu, f\mu, f\mu)), \\ \frac{G^2(f\mu, f\mu, \mu) \cdot G(\mu, \mu, \mu)}{G(\mu, f\mu, f\mu) + G(f\mu, f\mu, \mu)}, \\ \left(\frac{G(\mu, f\mu, \mu) + G(\mu, f\mu, f\mu)}{G(\mu, f\mu, \mu) + G(\mu, f\mu, f\mu)} \right) \cdot (G(f\mu, \mu, \mu)) \end{array} \right)$$

$$G(\mu, f\mu, f\mu)$$

$$\leq \Omega \max \left(\begin{array}{l} 0, G(\mu, f\mu, f\mu) \\ 0, 2G(\mu, f\mu, f\mu) \end{array} \right)$$

$$G(\mu, f\mu, f\mu) = \Omega 2G(\mu, f\mu, f\mu), \text{ a contradiction as}$$

$$0 \leq \Omega < \frac{1}{8}.$$

Hence $f\mu = g\mu = \mu$. Thus μ is a fixed point of f and g . by the compatibility of f and g , it is obvious that $fg\mu = gf\mu = \mu$. Uniqueness

It is to be assume that that are two distinct fixed points μ and w of f and g then from (1.3). we have

$$G(\mu, w, w) = G(f\mu, fw, fw) \leq \Omega \max \left(\begin{array}{l} G(\mu, w, w), G(\mu, w, w) \\ 0, 2G(\mu, \mu, w) \end{array} \right)$$

hence $\mu = w$ since $\Omega < \frac{1}{8}$.

Theorem 2.2. Let (X, G) be a complete G -metric space and $f, g : X \rightarrow X$ be weakly compatible self mapping on (X, G) satisfying (1.1) and (1.3) also (1.4) one of the subspace $f(X)$ or $g(X)$ is complete then f and g have unique common fixed point.

proof

it is clear from theorem (2.1) that $\{y_n\}$ is G -Cauchy sequence, without loss of generality it can be assumed that $g(X)$ is complete. Then the subsequence is going to have a limit in $g(X)$. let it be τ . let $\mu = g^{-1}\tau$ then $g\mu = \tau$. Since $\{y_n\}$ is G -Cauchy sequence, containing a convergent subsequence therefore the sequence y_n is also convergent. It is to prove that $f\mu = \tau$.

Let $x=x_n, y= \mu$ and $z= \mu$

From (1.3), we have

$$G(fx_n, f\mu, f\mu) \leq \Omega \max \left(\begin{array}{l} \left(\frac{G(gx_n, g\mu, g\mu) + G(fx_n, f\mu, g\mu)}{G(gx_n, g\mu, g\mu) + G(fx_n, g\mu, f\mu)} \right) \cdot (G(gx_n, g\mu, g\mu)), \\ \left(\frac{G(gx_n, g\mu, f\mu) + G(g\mu, f\mu, f\mu)}{G(gx_n, g\mu, f\mu) + G(g\mu, f\mu, f\mu)} \right) \cdot (G(fx_n, f\mu, f\mu)), \\ \frac{G^2(f\mu, f\mu, fx_n) \cdot G(g\mu, g\mu, g\mu)}{G(fx_n, f\mu, f\mu) + G(f\mu, f\mu, fx_n)}, \\ \left(\frac{G(fx_n, f\mu, g\mu) + G(g\mu, f\mu, f\mu)}{G(fx_n, f\mu, g\mu) + G(g\mu, f\mu, f\mu)} \right) \cdot (G(f\mu, gx_n, gx_n)) \end{array} \right)$$

Taking n approaches to infinity, we have

$$G(\tau, f\mu, f\mu) \leq \Omega \max \left(\begin{array}{l} \left(\frac{G(\tau, \tau, \tau) + G(\tau, f\mu, f\mu)}{G(\tau, \tau, \tau) + G(\tau, \tau, f\mu)} \right) \cdot (G(\tau, \tau, \tau)), \\ \left(\frac{G(\tau, \tau, f\mu) + G(\tau, f\mu, f\mu)}{G(\tau, \tau, f\mu) + G(\tau, f\mu, f\mu)} \right) \cdot (G(\tau, f\mu, f\mu)), \\ \frac{G^2(f\mu, f\mu, \tau) \cdot G(\tau, \tau, \tau)}{G(\tau, f\mu, f\mu) + G(f\mu, f\mu, \tau)}, \\ \left(\frac{G(\tau, f\mu, \tau) + G(\tau, f\mu, f\mu)}{G(\tau, f\mu, \tau) + G(\tau, f\mu, f\mu)} \right) \cdot (G(f\mu, \tau, \tau)) \end{array} \right)$$

$$G(\tau, f\mu, f\mu) \leq \Omega \max(0, G(\tau, f\mu, f\mu), 0, 2G(\tau, f\mu, f\mu))$$

$\tau = \mu$ as $\Omega < \frac{1}{8}$,

i.e., $f\tau = g\tau$ now. It is to be shown that $f\tau = \tau$. Suppose that $f\tau \neq \tau$ now put on setting $x= \tau, y = z = \mu$ in (1.3), we have

$$G(f\tau, f\mu, f\mu) \leq \Omega \max \left(\begin{array}{l} \left(\frac{G(g\tau, g\mu, g\mu) + G(f\tau, f\mu, g\mu)}{G(g\tau, g\mu, g\mu) + G(f\tau, g\mu, f\mu)} \right) \cdot (G(g\tau, g\mu, g\mu)), \\ \left(\frac{G(g\tau, g\mu, f\mu) + G(g\mu, f\mu, f\mu)}{G(g\tau, g\mu, f\mu) + G(g\mu, f\mu, f\mu)} \right) \cdot (G(f\tau, f\mu, f\mu)), \\ \frac{G^2(f\mu, f\mu, f\tau) \cdot G(g\mu, g\mu, g\mu)}{G(f\tau, f\mu, f\mu) + G(f\mu, f\mu, f\tau)}, \\ \left(\frac{G(f\tau, f\mu, f\mu) + G(g\mu, f\mu, f\mu)}{G(f\tau, f\mu, g\mu) + G(g\mu, f\mu, f\mu)} \right) \cdot (G(f\mu, g\tau, g\tau)) \end{array} \right)$$

Using the prepositions (1.3) and using the fact that $f\tau = g\tau$, we have

$$G(f\tau, \tau, \tau) = \Omega \max(G(f\tau, \tau, \tau), G(f\tau, \tau, \tau), 0, 2G(f\tau, \tau, \tau))$$

Clearly we have, $f\tau = \tau$

It proves that $f\tau = g\tau = \tau$, i.e., τ is a common fixed point. Uniqueness is confirmed by the uniqueness of theorem 2.1.

Theorem 2.3. Let (X, G) be a complete G -metric space and $f, g : X \rightarrow X$ be the self mapping on (X, G) satisfying the conditions (1.3) and the following conditions
 (1.6) f and g satisfy property (E.A)
 (1.5) $g(X)$ is complete subspace of X
 Then f and G have a unique common fixed point in X provided f and G are compatible maps.

Proof. Since e property (E.A) is satisfied by f and g , therefore, there exists one sequence $\{x_n\}$ in X . Such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = \mu$$

for some $\mu \in X$. By the closeness of $g(X)$ in X . $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = g\gamma = \mu$ for some $\gamma \in X$

implies $\mu = g\gamma \in X$. We declare that

$$f\gamma = \mu = g\gamma.$$

Using the equation (1.3)

$$\begin{aligned} & \Omega(fx_n, f\gamma, f\gamma) \\ & \left(\begin{array}{c} \left(\frac{G(gx_n, g\gamma, g\gamma) + G(fx_n, f\gamma, g\gamma)}{G(gx_n, g\gamma, g\gamma) + G(fx_n, g\gamma, f\gamma)} \right) \\ G(gx_n, g\gamma, g\gamma), \\ \left(\frac{G(gx_n, g\gamma, f\gamma) + G(g\gamma, f\gamma, f\gamma)}{G(gx_n, g\gamma, f\gamma) + G(g\gamma, f\gamma, f\gamma)} \right) \\ G(fx_n, f\gamma, f\gamma), \\ \frac{G^2(f\gamma, f\gamma, fx_n) \cdot G(g\gamma, g\gamma, g\gamma)}{G(fx_n, f\gamma, f\gamma) + G(f\gamma, f\gamma, fx_n)}, \\ \left(\frac{G(fx_n, f\gamma, g\gamma) + G(g\gamma, f\gamma, f\gamma)}{G(fx_n, f\gamma, g\gamma) + G(g\gamma, f\gamma, f\gamma)} \right) \\ G(f\gamma, gx_n, gx_n) \end{array} \right) \\ & = \Omega \max \\ & G(\mu, f\gamma, f\gamma) \\ & \left(\begin{array}{c} \left(\frac{G(\mu, \mu, \mu) + G(\mu, \mu, \mu)}{G(\mu, \mu, \mu) + G(\mu, \mu, f\gamma)} \right) \cdot G(\mu, \mu, \mu), \\ \left(\frac{G(\mu, \mu, f\gamma) + G(\mu, f\gamma, f\gamma)}{G(\mu, \mu, f\gamma) + G(\mu, f\gamma, f\gamma)} \right) \\ G(\mu, f\gamma, f\gamma), \\ \frac{G^2(f\gamma, f\gamma, \mu) \cdot G(\mu, \mu, \mu)}{G(\mu, f\gamma, f\gamma) + G(f\gamma, f\gamma, \mu)}, \\ \left(\frac{G(\mu, f\gamma, \mu) + G(\mu, f\gamma, f\gamma)}{G(\mu, f\gamma, \mu) + G(\mu, f\gamma, f\gamma)} \right) \\ G(f\gamma, \mu, \mu) \end{array} \right) \\ & \leq \Omega \max \\ & \leq \Omega \max(0, G(\mu, f\gamma, f\gamma), 0, G(f\gamma, \mu, \mu)) \\ & \leq \Omega \max(0, G(\mu, f\gamma, f\gamma), 0, 2G(\mu, f\gamma, f\gamma)), \end{aligned}$$

which is a contradiction because of range of Ω hence

$$f\gamma = \mu = g\gamma$$

Hence γ is a fixed point of f and g .

Theorem 2.4. Let (X, G) be a complete G-metric space and $f, g: X \rightarrow X$ be the self mapping on (X, G) satisfying the following conditions (1.2) and (1.3) and the condition (1.4) any one of the subspaces $f(X)$ or $g(X)$ is complete. Then f and g have a unique common fixed point in X provided f and g are compatible maps.

Proof. Since the self maps f and g satisfies the common limit range property of type g i.e. (CLR_g) , therefore there exists a sequence $\{x_n\}$ in X in such a way

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = g\gamma = \mu,$$

for some $\gamma \in X$.

We declare that $f\gamma = g\gamma = \mu$, using the equation (1.3) and taking limiting value of n as infinity, we have

$$\begin{aligned} & G(fx_n, f\gamma, f\gamma) \\ & \left(\begin{array}{c} \left(\frac{G(\mu, g\gamma, g\gamma) + G(fx_n, f\gamma, g\gamma)}{G(gx_n, g\gamma, g\gamma) + G(fx_n, g\gamma, f\gamma)} \right) \\ G(gx_n, g\gamma, g\gamma), \\ \left(\frac{G(gx_n, g\gamma, f\gamma) + G(g\gamma, f\gamma, f\gamma)}{G(gx_n, g\gamma, f\gamma) + G(g\gamma, f\gamma, f\gamma)} \right) \\ G(fx_n, f\gamma, f\gamma), \\ \frac{G^2(f\gamma, f\gamma, fx_n) \cdot G(g\gamma, g\gamma, g\gamma)}{G(fx_n, f\gamma, f\gamma) + G(f\gamma, f\gamma, fx_n)}, \\ \left(\frac{G(fx_n, f\gamma, g\gamma) + G(g\gamma, f\gamma, f\gamma)}{G(fx_n, f\gamma, g\gamma) + G(g\gamma, f\gamma, f\gamma)} \right) \\ G(f\gamma, gx_n, gx_n) \end{array} \right) \\ & \leq \Omega \max \\ & G(\mu, f\gamma, f\gamma) \leq \Omega \max(0, G(\mu, f\gamma, f\gamma), 0, G(f\gamma, \mu, \mu)) \\ & = \Omega \max(0, G(\mu, f\gamma, f\gamma), 0, 2G(\mu, f\gamma, f\gamma)). \end{aligned}$$

This equation is a contradiction because of the values of Ω , hence

$$f\gamma = \mu = g\gamma.$$

Thus μ is a common fixed point of f and g .

References

- [1] Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales Fund. Math. 3 (1922), 133-181
- [2] B.C Dhage, "Generalized Metric Spaces and Mappings with Fixed points," Bulletin Of Calcutta mathematical Society, vol. 84. 1992, pp.329-336.
- [3] H. Aydi, W. Shatanawi, "On Generalized weakly G-contraction mapping in G-metric Spaces," Comput. Math. Appl. 62 (2011) 4222-4229.
- [4] G. Jungck, "Commuting mappings and fixed point," Amer. Math. Monthly, 83 (1976), 261-263.
- [5] G. Jungck, "Compatible Mappings and common Fixed Points theorems," International Journal of Mathematics and Mathematical Sciences, Vol. 9, No. 4, 1986, pp. 771-779.
- [6] Z. Mustafa and B. Sims, A new approach to Generalized metric spaces, Journal of Nonlinear Convex Analysis, 7 (2006), 289-297.
- [7] Z. Mustafa, H. Obiedat and F. Awawdeh, "Some fixed point theorem for mapping on complete G-metric spaces," Fixed Point Theory and Applications, Volume 2008, Article ID 189870.
- [8] Z. Mustafa and B. Sims, "Fixed point theorems for contractive mappings in complete G-metric spaces, Fixed Point Theory and Applications, Volume 2009, Article ID 917175.
- [9] H.K. Pathak, R. Rodrigue-Lopez and R.K. Vverma, "A Common Fixed Point Theorem Using Implicit Relation and E.A Property in Metric Spaces," Filomat, Vol.21, No.2, 2007, pp. 211-234.