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Some New Ostrowski Type Inequalities Concerning Differentiable Generalized Relative Semi-(r; m, p, q, h_1, h_2)-Preinvex Mappings

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Abstract In this article, we first presented a new integral identity concerning differentiable mappings defined on m-invex sets. By using the notion of generalized relative semi-(r; m, p, q, h_1, h_2)-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Ostrowski type inequalities are established. It is pointed out that some new special cases can be deduced from main results of the article.

Keywords: *Ostrowski type inequality, Hölder's inequality, Minkowski inequality, power mean inequality, m-invex.*

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1. Introduction

The subsequent inequality is known as Ostrowski inequality which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a,b]$.

Theorem 1.1. *Let $f: I \to \mathbb{R}$ be a mapping differentiable on I° and let $a,b \in I^\circ$ with $a<b$. If $|f'(x)| \le M$ for all $x \in [a,b]$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right], \forall x \in [a,b]. \tag{1.1}$$

Ostrowski inequality is playing a very important role in all the fields of mathematics, especially in the theory of approximations. Thus such inequalities were studied extensively by many researches and numerous generalizations, extensions and variants of them for various kind of functions like bounded variation, synchronous, Lipschitzian, monotonic, absolutely, continuous and n-times differentiable mappings etc. appeared in a number of papers [1-3,5-10,12,13,15-24,26,29,30,31,36,37,38,40,43,45]. In numerical analysis many quadrature rules have been established to approximate the definite integrals [14,25,27,28,32,35,39,41]. Ostrowski inequality provides the bounds for many numerical quadrature rules.

Let us recall some special functions and evoke some basic definitions as follows.

Definition 1.2. The incomplete beta function is defined for $a,b>0$ as

$$\beta(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt, \; 0 < x \le 1.$$

Definition 1.3. [44] A set $M_\varphi \subseteq \mathbb{R}^n$ is said to be a relative convex (φ-convex) set, if and only if, there exists a function $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ such that,

$$t\varphi(x) + (1-t)\varphi(y) \in M_\varphi, \quad \forall x,y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi, t \in [0,1]. \tag{1.2}$$

Definition 1.4. [44] A function f is said to be a relative convex (φ-convex) function on a relative convex (φ-convex) set M_φ, if and only if, there exists a function $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ such that,

$$f(t\varphi(x) + (1-t)\varphi(y)) \le tf(\varphi(x)) + (1-t)f(\varphi(y)), \tag{1.3}$$

$\forall x,y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi, t \in [0,1]$.

Definition 1.5. [4] A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta: K \times K \to \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x,y \in K$ and $t \in [0,1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y,x)=y-x$, but the converse is not necessarily true [4,42].

Definition 1.6. [34] The function f defined on the invex set $K\subseteq\mathbb{R}^n$ is said to be preinvex with respect η, if for every $x,y\in K$ and $t\in[0,1]$, we have that

$$f\left(x+t\eta(y,x)\right)\le(1-t)f(x)+tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x)=y-x$, but the converse is not true.

Definition 1.7. [25] Let $h:[0,1]\to\mathbb{R}$ be a non-negative function and $h\neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η, if

$$f\left(x+t\eta(y,x)\right)\le h(1-t)f(x)+h(t)f(y) \tag{1.4}$$

for each $x,y\in K$ and $t\in[0,1]$ where $f(.)>0$.

Definition 1.8. [41] Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$ be a positive function, $h\not\equiv 0$. We say that $f:J\subseteq\mathbb{R}\to\mathbb{R}$ is h-convex, if f is non-negative and for all $x,y\in I$ and $t\in(0,1)$, one has

$$f\left(tx+(1-t)y\right)\le h(t)f(x)+h(1-t)f(y). \tag{1.5}$$

Definition 1.9. [39] Let $f:K\subseteq\mathbb{R}\to\mathbb{R}$ be a non-negative function, we say that $f:K\to\mathbb{R}$ is a *tgs*-convex function on K, if the inequality

$$f\left((1-t)x+ty\right)\le t(1-t)\left[f(x)+f(y)\right] \tag{1.6}$$

holds for all $x,y\in K$ and $t\in(0,1)$. We say that f is *tgs*-concave if $(-f)$ is *tgs*-convex.

Definition 1.10. [28] A function: $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is said to be m-MT-convex, if f is positive and for $\forall x,y\in I$, and $t\in(0,1)$, with $m\in[0,1]$, satisfies the following inequality

$$f\left(tx+m(1-t)y\right)\le\frac{\sqrt{t}}{2\sqrt{1-t}}f(x)+\frac{m\sqrt{1-t}}{2\sqrt{t}}f(y). \tag{1.7}$$

Definition 1.11. [33] Let $K\subseteq\mathbb{R}$ be an open m-invex set with respect to $\eta:K\times K\times(0,1]\to\mathbb{R}$ and $h_1,h_2:[0,1]\to[0,+\infty)$ are continuous functions. A function $f:K\to\mathbb{R}$ is said to be generalized (m,h_1,h_2)-preinvex with respect to η, if

$$f\left(mx+t\eta(y,x,m)\right)\le mh_1(t)f(x)+h_2(t)f(y) \tag{1.8}$$

is valid for all $x,y\in K$ and $t\in[0,1]$, with some fixed $m\in(0,1]$. If the inequality (1.8) reverses, then f is said to be generalized (m,h_1,h_2)-preincave on K.

Definition 1.12. [11] A set $K\subseteq\mathbb{R}^n$ is named as m-invex with respect to the mapping $\eta:K\times K\to\mathbb{R}^n$ for some fixed $m\in(0,1]$, if $mx+t\eta(y,mx)\in K$ holds for each $x,y\in K$ and any $t\in[0,1]$.

Remark 1.13. In Definition 1.12, under certain conditions, the mapping $\eta(y,mx)$ could reduce to $\eta(y,x)$. For example when $m=1$, then the m-invex set degenerates an invex set on K.

We are in position to introduce the notion of generalized relative semi-(r; m, p, q, h_1, h_2)-preinvex mappings.

Definition 1.14. Let $K\subseteq\mathbb{R}$ be an open m-invex set with respect to the mapping $\eta:K\times K\times(0,1]\to\mathbb{R}$. Suppose $h_1,h_2:[0,1]\to[0,+\infty)$ and $\varphi:I\to K$ are continuous. A mapping $f:K\to(0,+\infty)$ is said to be generalized relative semi-(r; m, p, q, h_1, h_2)-preinvex, if

$$\begin{aligned}&f\left(m\varphi(x)\right)+t\eta\left(\varphi(y),\varphi(x),m\right)\\&\le M_r\left(h_1(t),h_2(t);mf(x),f(y),p,q\right)\end{aligned} \tag{1.9}$$

holds for all $x,y\in I$ and $t\in[0,1]$, for any fixed $p,q>-1$ and some fixed $m\in(0,1]$, where

$$M_r\left(h_1(t),h_2(t);mf(x),f(y),p,q\right)$$
$$:=\begin{cases}\left[mh_1^p(t)f^r(x)+h_2^q(t)f^r(y)\right]^{\frac{1}{r}}, & \text{if } r\neq 0;\\ \left[mf(x)\right]^{h_1^p(t)}\left[f(y)\right]^{h_2^q(t)}, & \text{if } r=0,\end{cases}$$

is the weighted power mean of order r for positive numbers $f(x)$ and $f(y)$.

Remark 1.15. In Definition 1.14, if we choose $r=p=q=1$ and $\varphi(x)=x$, $\forall x\in I$, then we get Definition 1.11. If we choose $r=p=q=1$, $\varphi(x)=x$, $\forall x\in I$ and $h_1(t)=\frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t)=\frac{m\sqrt{t}}{2\sqrt{1-t}}$, then we get MT_m-preinvex function [15,17].

Remark 1.16. For $r=p=q=1$, let us discuss some special cases in Definition 1.14 as follows.

(I) If taking $h_1(t)=(1-t)^s$, $h_2(t)=t^s$ for $s\in(0,1]$, then we get generalized relative semi-(m, s)-Breckner-preinvex mappings.

(II) If taking $h_1(t)=h_2(t)=1$, then we get generalized relative semi-(m, P)-preinvex mappings.

(III) If taking $h_1(t)=(1-t)^{-s}$, $h_2(t)=t^{-s}$ for $s\in(0,1]$, then we get generalized relative semi-(m, s)-Godunova-Levin-Dragomir-preinvex mappings.

(IV) If taking $h_1(t)=h(1-t)$, $h_2(t)=h(t)$, then we get generalized relative semi-(m, h)-preinvex mappings.

(V) If taking $h_1(t)=h_2(t)=t(1-t)$, then we get generalized relative semi-(m, tgs)-preinvex mappings.

(VI) If taking $h_1(t)=\frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t)=\frac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized relative semi-m-MT-preinvex mappings.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

Let see the following example of a generalized relative semi-(r; m, p, q, h_1, h_2)-preinvex mappings which is not convex.

Example 1.17. Let taking $m=r=\frac{1}{2},\ h_1(t)=t^l,\ h_2(t)=t^s,$ for all $l,s\in[0,1]$, for any fixed $p,q\geq 1$ and $\varphi(x)=x.$ Consider the mapping $f:[0,+\infty)\to[0,+\infty)$ as follows

$$f(x)=\begin{cases} x, & 0\leq x\leq 1;\\ 1, & x>1.\end{cases}$$

Define a bifunction $\eta:[0,+\infty)\times[0,+\infty)\times\left\{\frac{1}{2}\right\}\to\mathbb{R}$ by

$$\eta(y,x,m)=\begin{cases} -y, & 0\leq y\leq 1;\\ x+y, & y>1.\end{cases}$$

Then f is generalized relative semi-$\left(\frac{1}{2};\frac{1}{2},p,q,t^l,t^s\right)$-preinvex mapping for any fixed $p,q\geq 1$ and for all $l,s\in[0,1]$. But f is not preinvex with respect to η and also it is not convex (consider $x=0,\ \ y=2$ and $t\in(0,1]$).

Motivated by the above literatures, the main objective of this article is to establish some new estimates on generalizations of Ostrowski type inequalities associated with differentiable generalized relative semi-(r; m, p, q, h_1, h_2)-preinvex mappings on m-invex sets. It is pointed out that some new special cases will be deduced from main results of the article.

2. Main Results

In this section, in order to prove our main results regarding some generalizations of Ostrowski type inequalities for differentiable generalized relative semi-(r; m, p, q, h_1, h_2)-preinvex mappings, we need the following new integral identity.

Lemma 2.1. *Let $\varphi:I\to K$ be a continuous function. Suppose $K\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta:K\times K\times(0,1]\to\mathbb{R}$ for some fixed $m\in(0,1]$ and $\eta(\varphi(b),\varphi(a),m)>0$. Assume that*

$$f:K=\left[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)\right]\to\mathbb{R}$$

be a differentiable mapping on K° and $f'\in L_1(K)$. Then for any two complex numbers $\lambda_1(x),\lambda_2(x)$ and α, $k>0$, the following integral identity holds:

$$\begin{aligned}
&f(x)+\frac{1}{\eta(\varphi(b),\varphi(a),m)}\\
&\times\left\{\lambda_1(x)\left[\left(\frac{k}{k+\alpha}\right)\left((m\varphi(a))^{\frac{\alpha}{k}+1}-x^{\frac{\alpha}{k}+1}\right)\right.\right.\\
&\qquad\left.+(m\varphi(a))(x-m\varphi(a))\right]\\
&+\lambda_2(x)\left[\left(\frac{k}{k+\alpha}\right)\left(x^{\frac{\alpha}{k}+1}-\begin{pmatrix} m\varphi(a)\\ +\eta(\varphi(b),\varphi(a),m)\end{pmatrix}^{\frac{\alpha}{k}+1}\right)\right.\\
&-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))\\
&\quad\left.\times\left(x-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))\right)\right]\\
&+f(m\varphi(a))\left[m\varphi(a)-(m\varphi(a))^{\frac{\alpha}{k}}\right]\\
&+f(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))\\
&\quad\times\begin{bmatrix}(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))^{\frac{\alpha}{k}}\\ -(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))\end{bmatrix}\\
&\left.-\frac{\alpha}{k}\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}t^{\frac{\alpha}{k}-1}f(t)\,dt\right\}\\
&=\frac{1}{\eta(\varphi(b),\varphi(a),m)}\left\{\int_{m\varphi(a)}^{x}\left(t^{\frac{\alpha}{k}}-m\varphi(a)\right)\begin{bmatrix}f'(t)\\ -\lambda_1(x)\end{bmatrix}dt\right.\\
&+\int_{x}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}\left(t^{\frac{\alpha}{k}}-\begin{pmatrix} m\varphi(a)\\ +\eta(\varphi(b),\varphi(a),m)\end{pmatrix}\right)\\
&\qquad\left.\times\left[f'(t)-\lambda_2(x)\right]dt\right\}.
\end{aligned}\tag{2.1}$$

Proof. Integrating by parts and changing the variable of definite integrals yield

Proof. Integrating by parts and changing the variable of definite integrals yield

$$\begin{aligned}
&\frac{1}{\eta(\varphi(b),\varphi(a),m)}\\
&\left\{\int_{m\varphi(a)}^{x}\left(t^{\frac{\alpha}{k}}-m\varphi(a)\right)\left[f'(t)-\lambda_1(x)\right]dt\right.\\
&+\int_{x}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}\left(t^{\frac{\alpha}{k}}-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))\right)\\
&\qquad\left.\times\left[f'(t)-\lambda_2(x)\right]dt\right\}\\
&=\frac{1}{\eta(\varphi(b),\varphi(a),m)}\left\{\left(t^{\frac{\alpha}{k}}-m\varphi(a)\right)\right.\\
&\qquad\times\left[f(t)-\lambda_1(x)t\right]\Big|_{m\varphi(a)}^{x}\\
&\quad-\frac{\alpha}{k}\int_{m\varphi(a)}^{x}t^{\frac{\alpha}{k}-1}\left[f(t)-\lambda_1(x)t\right]dt\\
&-\left(t^{\frac{\alpha}{k}}-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))\right)\\
&\quad\times\left[f(t)-\lambda_2(x)t\right]\Big|_{x}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}\\
&\left.-\frac{\alpha}{k}\int_{x}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}t^{\frac{\alpha}{k}-1}\left[f(t)-\lambda_2(x)t\right]dt\right\}
\end{aligned}$$

$$
\begin{aligned}
&= f(x)+\frac{1}{\eta(\varphi(b),\varphi(a),m)} \\
&\quad\times\left\{\lambda_1(x)\left[\left(\frac{k}{k+\alpha}\right)\left((m\varphi(a))^{\frac{\alpha}{k}+1}-x^{\frac{\alpha}{k}+1}\right)\right.\right. \\
&\qquad\left.+(m\varphi(a))(x-m\varphi(a))\right] \\
&+\lambda_2(x)\left[\left(\frac{k}{k+\alpha}\right)\left(x^{\frac{\alpha}{k}+1}-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))^{\frac{\alpha}{k}+1}\right)\right. \\
&-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)) \\
&\quad\left.\times(x-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)))\right] \\
&+f(m\varphi(a))\left[m\varphi(a)-(m\varphi(a))^{\frac{\alpha}{k}}\right] \\
&+f(m\varphi(a))+\eta(\varphi(b),\varphi(a),m) \\
&\quad\times\left[(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))^{\frac{\alpha}{k}}\right. \\
&\qquad\left.-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))\right] \\
&\left.-\frac{\alpha}{k}\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} t^{\frac{\alpha}{k}-1}f(t)dt\right\}.
\end{aligned}
$$

This completes the proof of our lemma.

Remark 2.2. In Lemma 2.1, if we choose $\alpha=k=m=1$, $\eta(\varphi(y),\varphi(x),m)=\varphi(y)-m\varphi(x)$ and $\varphi(x)=x,\quad \forall x\in I$, we get ([7], Lemma 8).

Throughout this paper we denote

$$
\begin{aligned}
&I_{f,\eta,\varphi,\lambda_1(x),\lambda_2(x)}(x;\alpha,k,m,a,b) \\
&:=\frac{1}{\eta(\varphi(b),\varphi(a),m)} \\
&\quad\times\left\{\int_{m\varphi(a)}^{x}\left(t^{\frac{\alpha}{k}}-m\varphi(a)\right)\left[f'(t)-\lambda_1(x)\right]dt\right. \\
&+\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}\left(t^{\frac{\alpha}{k}-1}-\begin{pmatrix} m\varphi(a) \\ +\eta(\varphi(b),\varphi(a),m)\end{pmatrix}\right) \\
&\qquad\left.\times\left[f'(t)-\lambda_2(x)\right]dt\right\}.
\end{aligned}
$$

Corollary 2.3. *With the assumption in Lemma* 2.1, *we have for any* $\lambda_1(x)=\lambda_2(x)=\lambda(x)\in\mathbb{R}$ *that*

$$
\begin{aligned}
&I_{f,\eta,\varphi,\lambda(x)}(x;\alpha,k,m,a,b) \\
&= f(x)+\frac{1}{\eta(\varphi(b),\varphi(a),m)} \\
&\quad\times\left\{\lambda(x)\left[\left(\frac{k}{k+\alpha}\right)\left((m\varphi(a))^{\frac{\alpha}{k}+1}-x^{\frac{\alpha}{k}+1}\right)\right.\right. \\
&\qquad\left.+(m\varphi(a))(x-m\varphi(a))\right. \\
&+\left(\frac{k}{k+\alpha}\right)\left(x^{\frac{\alpha}{k}+1}-\begin{pmatrix} m\varphi(a) \\ +\eta(\varphi(b),\varphi(a),m)\end{pmatrix}^{\frac{\alpha}{k}+1}\right) \\
&-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)) \\
&\quad\left.\times(x-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)))\right] \\
&+f(m\varphi(a))\left[m\varphi(a)-(m\varphi(a))^{\frac{\alpha}{k}}\right] \\
&+f(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)) \\
&\quad\times\left[(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))^{\frac{\alpha}{k}}\right. \\
&\qquad\left.-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))\right] \\
&\left.-\frac{\alpha}{k}\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} t^{\frac{\alpha}{k}-1}f(t)dt\right\}.
\end{aligned}
\tag{2.2}
$$

Remark 2.4. If we take $\lambda(x)=f'(x)$ in (2.2), then we get

$$
\begin{aligned}
&I_{f,\eta,\varphi,f'(x)}(x;\alpha,k,m,a,b) \\
&= f(x)+\frac{1}{\eta(\varphi(b),\varphi(a),m)} \\
&\quad\times\left\{f'(x)\left[\left(\frac{k}{k+\alpha}\right)\left((m\varphi(a))^{\frac{\alpha}{k}+1}-x^{\frac{\alpha}{k}+1}\right)\right.\right. \\
&\qquad+(m\varphi(a))(x-m\varphi(a)) \\
&+\left(\frac{k}{k+\alpha}\right)\left(x^{\frac{\alpha}{k}+1}-\begin{pmatrix} m\varphi(a) \\ +\eta(\varphi(b),\varphi(a),m)\end{pmatrix}^{\frac{\alpha}{k}+1}\right) \\
&-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)) \\
&\quad\left.\times(x-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)))\right] \\
&+f(m\varphi(a))\left[m\varphi(a)-(m\varphi(a))^{\frac{\alpha}{k}}\right] \\
&+f(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)) \\
&\quad\times\left[(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))^{\frac{\alpha}{k}}\right. \\
&\qquad\left.-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))\right] \\
&\left.-\frac{\alpha}{k}\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} t^{\frac{\alpha}{k}-1}f(t)dt\right\}.
\end{aligned}
\tag{2.3}
$$

Remark 2.5. If we take $\lambda_1(x)=\dfrac{f(x)-f(a)}{x-a}$ and $\lambda_2(x)=\dfrac{f(b)-f(x)}{b-x}$ in (2.1), then we get

$$f(x)+\frac{1}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\left\{\left(\frac{f(x)-f(a)}{x-a}\right)\left[\left(\frac{k}{k+\alpha}\right)\left((m\varphi(a))^{\frac{\alpha}{k}+1}-x^{\frac{\alpha}{k}+1}\right)\right.\right.$$
$$\left.+(m\varphi(a))(x-(m\varphi(a)))\right]$$
$$+\left(\frac{f(b)-f(x)}{b-x}\right)$$
$$\times\left[\left(\frac{k}{k+\alpha}\right)\left(x^{\frac{\alpha}{k}+1}-\begin{pmatrix}m\varphi(a)\\+\eta(\varphi(b),\varphi(a),m)\end{pmatrix}^{\frac{\alpha}{k}+1}\right)\right.$$
$$-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))$$
$$\left.\times(x-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)))\right]$$
$$+f(m\varphi(a))\left[m\varphi(a)-(m\varphi(a))^{\frac{\alpha}{k}}\right]$$
$$+f(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))$$
$$\times\left[(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))^{\frac{\alpha}{k}}\right.$$
$$\left.-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))\right]$$
$$\left.-\frac{\alpha}{k}\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}t^{\frac{\alpha}{k}-1}f(t)dt\right\}$$
$$=\frac{1}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\left\{\int_{m\varphi(a)}^{x}\left(t^{\frac{\alpha}{k}}-m\varphi(a)\right)\left[f'(t)-\frac{f(x)-f(a)}{x-a}\right]dt\right.$$
$$+\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}\left(t^{\frac{\alpha}{k}}-\begin{pmatrix}m\varphi(a)\\+\eta(\varphi(b),\varphi(a),m)\end{pmatrix}\right)$$
$$\left.\times\left[f'(t)-\frac{f(b)-f(x)}{b-x}\right]dt\right\}. \tag{2.4}$$

Remark 2.6. If we take $\lambda(x)=0$ in (2.2), then we get generalized Montgomery's identity, i.e.

$$I_{f,\eta,\varphi}(x;\alpha,k,m,a,b)$$
$$=f(x)+\frac{1}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\left\{f(m\varphi(a))\left[m\varphi(a)-(m\varphi(a))^{\frac{\alpha}{k}}\right]\right.$$
$$+f(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))$$
$$\times\left[(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))^{\frac{\alpha}{k}}\right.$$
$$\left.-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))\right]$$
$$\left.-\frac{\alpha}{k}\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}t^{\frac{\alpha}{k}-1}f(t)dt\right\}. \tag{2.5}$$

Using relation (2.1), the following results can be obtained for the corresponding version for power of the first derivative.

Theorem 2.7. *Let $\alpha,k>0$, $0<r\le 1$ and $p_1,p_2>-1$. Let $K\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta:K\times K\times(0,1]\to\mathbb{R}$ for some fixed $m\in(0,1]$. Suppose $h_1,h_2:[0,1]\to[0,+\infty)$ and $\varphi:I\to K$ are continuous functions and $\lambda_1(x),\lambda_2(x)$ are two complex numbers. Assume that $f:K\to(0,+\infty)$ be a differentiable function on K°, where $\eta(\varphi(b),\varphi(a),m)>0$. If $f'(x)^q$ is generalized relative semi-(r; m, p_1, p_2, h_1, h_2)-preinvex mappings, $q>1$, $p^{-1}+q^{-1}=1$, then the following inequality holds:*

$$I_{f,\eta,\varphi,\lambda_1(x),\lambda_2(x)}(x;\alpha,k,m,a,b)\ge\frac{2^{\frac{q-1}{q}}}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\left\{B_{\varphi}^{\frac{1}{p}}(x;\alpha,k,m,p,a)\left[\eta(\varphi(b),\varphi(a),m)\right.\right.$$
$$\times\left[mf'(a)^{rq}I_{\eta,\varphi}^{r}(h_1(t);x,r,m,a,b,p_1)\right.$$
$$\left.+f'(b)^{rq}I_{\eta,\varphi}^{r}(h_2(t);x,r,m,a,b,p_2)\right]^{\frac{1}{r}}$$
$$\left.+|\lambda_1(x)|^q(x-m\varphi(a))\right]^{\frac{1}{q}}$$
$$+C_{\eta,\varphi}^{\frac{1}{p}}(x;\alpha,k,m,p,a,b)\left[\eta(\varphi(b),\varphi(a),m)\right.$$
$$\times\left[mf'(a)^{rp}\bar{I}_{\eta,\varphi}^{r}(h_1(t);x,r,m,a,b,p_1)\right.$$
$$\left.+f'(b)^{rq}\bar{I}_{\eta,\varphi}^{r}(h_2(t);x,r,m,a,b,p_2)\right]^{\frac{1}{r}}$$
$$\left.\left.+|\lambda_2(x)|^q(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)\right]^{\frac{1}{q}}\right\}, \tag{2.6}$$

where

$$B_{\varphi}(x;\alpha,k,m,p,a):=\int_{m\varphi(a)}^{x}\left|t^{\frac{\alpha}{k}}-m\varphi(a)\right|^{p}dt,$$

$$C_{\eta,\varphi}(x;\alpha,k,m,p,a,b)$$
$$:=\int_{x}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}\left|t^{\frac{\alpha}{k}}-\begin{pmatrix}m\varphi(a)\\+\eta(\varphi(b),\varphi(a),m)\end{pmatrix}\right|^{p}dt,$$

$$I_{\eta,\varphi}\left(h_i(t);x,r,m,a,b,p_i\right) := \int_x^{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}} h_i^{\frac{p_i}{r}}(t)dt,\ \forall i=1,2,$$

and

$$\overline{I}_{\eta,\varphi}\left(h_i(t);x,r,m,a,b,p_i\right) := \int_{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}}^1 h_i^{\frac{p_i}{r}}(t)dt,\ \forall i=1,2.$$

Proof. Suppose that $q>1$ and $0<r\le 1$. From Lemma 2.1, generalized relative semi-(r; m, p_1, p_2, h_1, h_2)-preinvexity of $f'(x)^q$, Hölder inequality, Minkowski inequality, properties of the modulus and using the elementary inequality $(c+d)^\theta \le 2^{\theta-1}\left(c^\theta+d^\theta\right)$ where $\theta>1$, $c,d\ge 0$, we have

$$\left|I_{f,\eta,\varphi,\lambda_1(x),\lambda_2(x)}(x;\alpha,k,m,a,b)\right|$$
$$\le \frac{1}{\eta(\varphi(b),\varphi(a),m)} \times\left\{\int_{m\varphi(a)}^x \left|t^{\frac{\alpha}{k}}-m\varphi(a)\right|\left|f'(t)-\lambda_1(t)\right|dt\right.$$
$$+\int_x^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}\left|t^{\frac{\alpha}{k}}-\begin{pmatrix}m\varphi(a)\\ +\eta(\varphi(b),\varphi(a),m)\end{pmatrix}\right|$$
$$\left.\times\left|f'(t)-\lambda_2(t)\right|dt\right\}$$
$$\le \frac{1}{\eta(\varphi(b),\varphi(a),m)}\left\{\left(\int_{m\varphi(a)}^x\left|t^{\frac{\alpha}{k}}-m\varphi(a)\right|^p dt\right)^{\frac{1}{p}}\right.$$
$$\times\left(\int_{m\varphi(a)}^x\left|f'(t)-\lambda_1(t)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_x^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}\left|t^{\frac{\alpha}{k}}-\begin{pmatrix}m\varphi(a)\\ +\eta(\varphi(b),\varphi(a),m)\end{pmatrix}\right|^p dt\right)^{\frac{1}{p}}$$
$$\left.\times\left(\int_x^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}\left|f'(t)-\lambda_2(t)\right|^q dt\right)^{\frac{1}{q}}\right\}$$
$$\le \frac{1}{\eta(\varphi(b),\varphi(a),m)}\left\{B_\varphi^{\frac{1}{p}}(x;\alpha,k,m,p,a)\right.$$
$$\times\left(\int_{m\varphi(a)}^x 2^{q-1}\left(f'(t)^q+\left|\lambda_1(x)\right|^q\right)dt\right)^{\frac{1}{q}}$$
$$+C_{\eta,\varphi}^{\frac{1}{p}}(x;\alpha,k,m,p,a,b)$$
$$\left.\times\left(\int_x^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}2^{q-1}\begin{pmatrix}f'(t)^q\\ +\left|\lambda_2(x)\right|^q\end{pmatrix}dt\right)^{\frac{1}{q}}\right\}$$
$$=\frac{2^{\frac{q-1}{q}}}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\left\{B_\varphi^{\frac{1}{p}}(x;\alpha,k,m,p,a)\Big[\eta(\varphi(b),\varphi(a),m)\right.$$
$$\times\int_0^{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}}\left(f'\left(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)\right)\right)^q dt$$
$$+\left|\lambda_1(x)\right|^q\left(x-m\varphi(a)\right)\Big]^{\frac{1}{q}}$$
$$+C_{\eta,\varphi}^{\frac{1}{p}}(x;\alpha,k,m,p,a,b)$$
$$\times\Big[\eta(\varphi(b),\varphi(a),m)$$
$$\times\int_{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}}^1\left(f'\begin{pmatrix}m\varphi(a)\\ +t\eta(\varphi(b),\varphi(a),m)\end{pmatrix}\right)^q dt$$
$$\left.+\left|\lambda_2(x)\right|^q\left(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x\right)\Big]^{\frac{1}{q}}\right\}$$
$$\le\frac{2^{\frac{q-1}{q}}}{\eta(\varphi(b),\varphi(a),m)}\left\{B_\varphi^{\frac{1}{p}}(x;\alpha,k,m,p,a)\right.$$
$$\times\Big[\eta(\varphi(b),\varphi(a),m)$$
$$\times\int_0^{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}}\left[mh_1^{p_1}(t)f'(a)^{rq}+h_1^{p_2}(t)f'(b)^{rq}\right]^{\frac{1}{r}}dt$$
$$+\left|\lambda_1(x)\right|^q\left(x-m\varphi(a)\right)\Big]^{\frac{1}{q}}+C_{\eta,\varphi}^{\frac{1}{p}}(x;\alpha,k,m,p,a,b)$$
$$\times\Big[\eta(\varphi(b),\varphi(a),m)$$
$$\times\int_{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}}^1\begin{bmatrix}mh_1^{p_1}(t)f'(a)^{rq}\\ +h_2^{p_2}(t)f'(b)^{rq}\end{bmatrix}^{\frac{1}{r}}dt$$
$$\left.+\left|\lambda_2(x)\right|^q\left(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x\right)\Big]^{\frac{1}{q}}\right\}$$
$$\le\frac{2^{\frac{q-1}{q}}}{\eta(\varphi(b),\varphi(a),m)}\left\{B_\varphi^{\frac{1}{p}}(x;\alpha,k,m,p,a)\right.$$
$$\times\Big[\eta(\varphi(b),\varphi(a),m)$$

$$\times\left[\left(\int_0^{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}} m^{\frac{1}{r}} f'(a)^q h_1^{\frac{p_1}{r}}(t)dt\right)^r\right.$$

$$\left.+\left(\int_0^{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}} f'(b)^q h_2^{\frac{p_2}{r}}(t)dt\right)^r\right]^{\frac{1}{r}}$$

$$+\left|\lambda_1(x)\right|^q\left(x-m\varphi(a)\right)\Big]^{\frac{1}{q}}+C_{\eta,\varphi}^{\frac{1}{r}}(x;\alpha,k,m,p,a,b)$$

$$\times\left[\eta(\varphi(b),\varphi(a),m)\right.$$

$$\times\left[\left(\int_{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}}^1 m^{\frac{1}{r}} f'(a)^q h_1^{\frac{p_1}{r}}(t)dt\right)^r\right.$$

$$\left.+\left(\int_{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}}^1 f'(b)^q h_2^{\frac{p_2}{r}}(t)dt\right)^r\right]^{\frac{1}{r}}$$

$$\left.+\left|\lambda_2(x)\right|^q\left(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x\right)\Big]^{\frac{1}{q}}\right\}$$

$$=\frac{2^{\frac{q-1}{q}}}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left\{B_\varphi^{\frac{1}{p}}(x;\alpha,k,m,p,a)\left[\eta(\varphi(b),\varphi(a),m)\right.\right.$$

$$\times\left[mf'(a)^{rq} I_{\eta,\varphi}^r\left(h_1(t);x,r,m,a,b,p_1\right)\right.$$

$$\left.+f'(b)^{rq} I_{\eta,\varphi}^r\left(h_2(t);x,r,m,a,b,p_2\right)\right]^{\frac{1}{r}}$$

$$+\left|\lambda_1(x)\right|^q\left(x-m\varphi(a)\right)\Big]^{\frac{1}{q}}$$

$$+C_{\eta,\varphi}^{\frac{1}{p}}(x;\alpha,k,m,p,a,b)\left[\eta(\varphi(b),\varphi(a),m)\right.$$

$$\times\left[mf'(a)^{rp}\bar{I}_{\eta,\varphi}^r\left(h_1(t);x,r,m,a,b,p_1\right)\right.$$

$$\left.+f'(b)^{rp}\bar{I}_{\eta,\varphi}^r\left(h_2(t);x,r,m,a,b,p_2\right)\right]^{\frac{1}{r}}$$

$$\left.+\left|\lambda_2(x)\right|^q\left(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x\right)\Big]^{\frac{1}{q}}\right\}.$$

So, the proof of this theorem is complete.

We point out some special cases of Theorem 2.7.

Corollary 2.8. *In Theorem 2.7 for* $\lambda_1(x)=\lambda_2(x)=0$, *we have the following inequality for generalized relative semi-(r; m, p_1, p_2, h_1, h_2)-preinvex mappings*:

$$\left|I_{f,\eta,\varphi}(x;\alpha,k,m,a,b)\right|$$

$$\le\left(\frac{2}{\eta(\varphi(b),\varphi(a),m)}\right)^{\frac{q-1}{q}}$$

$$\times\left\{B_\varphi^{\frac{1}{p}}(x;\alpha,k,m,p,a)\right.$$

$$\times\left[mf'(a)^{rq} I_{\eta,\varphi}^r\left(h_1(t);x,r,m,a,b,p_1\right)\right.$$

$$\left.+f'(b)^{rq} I_{\eta,\varphi}^r\left(h_2(t);x,r,m,a,b,p_2\right)\right]^{\frac{1}{rq}} \quad (2.7)$$

$$+C_{\eta,\varphi}^{\frac{1}{p}}(x;\alpha,k,m,p,a,b)$$

$$\times\left[mf'(a)^{rp}\bar{I}_{\eta,\varphi}^r\left(h_1(t);x,r,m,a,b,p_1\right)\right.$$

$$\left.\left.+f'(b)^{rp}\bar{I}_{\eta,\varphi}^r\left(h_2(t);x,r,m,a,b,p_2\right)\right]^{\frac{1}{rq}}\right\}.$$

Corollary 2.9. *In Theorem 2.7 for* $h_1(t)=h(1-t)$ *and* $h_2(t)=h(t)$, *we have the following inequality for generalized relative semi-(r;m,p_1,p_2,h)-preinvex mappings*:

$$\left|I_{f,\eta,\varphi,\lambda_1(x),\lambda_2(x)}(x;\alpha,k,m,a,b)\right|$$

$$\le\frac{2^{\frac{q-1}{q}}}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left\{B_\varphi^{\frac{1}{p}}(x;\alpha,k,m,p,a)\left[\eta(\varphi(b),\varphi(a),m)\right.\right.$$

$$\times\left[mf'(a)^{rq} I_{\eta,\varphi}^r\left(h(1-t);x,r,m,a,b,p_1\right)\right.$$

$$\left.+f'(b)^{rq} I_{\eta,\varphi}^r\left(h(t);x,r,m,a,b,p_2\right)\right]^{\frac{1}{r}}$$

$$+\left|\lambda_1(x)\right|^q\left(x-m\varphi(a)\right)\Big]^{\frac{1}{q}}$$

$$+C_{\eta,\varphi}^{\frac{1}{p}}(x;\alpha,k,m,p,a,b)\left[\eta(\varphi(b),\varphi(a),m)\right.$$

$$\left[mf'(a)^{rq}\bar{I}_{\eta,\varphi}^r\left(h(1-t);x,r,m,a,b,p_1\right)\right.$$

$$\left.+f'(b)^{rq}\bar{I}_{\eta,\varphi}^r\left(h(t);x,r,m,a,b,p_2\right)\right]^{\frac{1}{r}}$$

$$\left.+\left|\lambda_2(x)\right|^q\left(\left(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)\right)-x\right)\Big]^{\frac{1}{q}}\right\}. \quad (2.8)$$

Corollary 2.10. *In Theorem 2.7 for* $h_1(t)=(1-t)^s$, $h_2(t)=t^s$, *we have the following inequality for generalized relative semi-(r;m,p_1,p_2,s)-Breckner-preinvex mappings*:

$$\left|I_{f,\eta,\varphi,\lambda_1(x),\lambda_2(x)}(x;\alpha,k,m,a,b)\right|$$
$$\leq \frac{2^{\frac{q-1}{q}}}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\left\{B_{\varphi}^{\frac{1}{r}}(x;\alpha,k,m,p,a)\left[\eta(\varphi(b),\varphi(a),m)\right.\right.$$
$$\times\left[mf'(a)^{rq}\left(\frac{r}{r+sp_1}\right)^r\right.$$
$$\times\left[1-\left(\frac{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}\right)^{\frac{sp_1}{r}+1}\right]^r$$
$$\left.+f'(b)^{rq}\left(\frac{r}{r+sp_2}\right)^r\left(\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}\right)^{r+sp_2}\right]^{\frac{1}{r}}$$
$$\left.+\left|\lambda_1(x)\right|^q(x-m\varphi(a))\right]^{\frac{1}{q}}$$
$$+C_{\eta,\varphi}^{\frac{1}{r}}(x;\alpha,k,m,p,a,b)\left[\eta(\varphi(b),\varphi(a),m)\right.$$
$$\times\left[mf'(a)^{rp}\left(\frac{r}{r+sp_1}\right)^r\right.$$
$$\times\left(\frac{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x}{\eta(\varphi(b),\varphi(a),m)}\right)^{r+sp_1}$$
$$+f'(b)^{rp}\left(\frac{r}{r+sp_2}\right)^r$$
$$\left.\times\left[1-\left(\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}\right)^{\frac{sp_2}{r}+1}\right]^r\right]^{\frac{1}{r}}$$
$$\left.\left.+\left|\lambda_2(x)\right|^q\left(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x\right)\right]^{\frac{1}{q}}\right\}. \quad (2.9)$$

Corollary 2.11. *In Theorem 2.7 for* $h_1(t)=(1-t)^{-s}$, $h_2(t)=t^{-s}$, *we have the following inequality for generalized relative semi-(r;m,p_1,p_2,-s)-Godunova-Levin-Dragomir-preinvex mappings*:

$$\left|I_{f,\eta,\varphi,\lambda_1(x),\lambda_2(x)}(x;\alpha,k,m,a,b)\right|$$
$$\leq \frac{2^{\frac{q-1}{q}}}{\eta(\varphi(b),\varphi(a),m)}\left\{B_{\varphi}^{\frac{1}{p}}(x;\alpha,k,m,p,a)\right.$$
$$\times\left[\eta(\varphi(b),\varphi(a),m)\left[mf'(a)^{rq}\left(\frac{r}{r-sp_1}\right)^r\right.\right.$$
$$\times\left[1-\left(\frac{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x}{\eta(\varphi(b),\varphi(a),m)}\right)^{1-\frac{sp_1}{r}}\right]^r$$
$$\left.+f'(b)^{rq}\left(\frac{r}{r-sp_2}\right)^r\left(\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}\right)^{r-sp_2}\right]^{\frac{1}{r}}$$
$$\left.+\left|\lambda_1(x)\right|^q(x-m\varphi(a))\right]^{\frac{1}{q}}$$
$$+C_{\eta,\varphi}^{\frac{1}{p}}(x;\alpha,k,m,p,a,b)\left[\eta(\varphi(b),\varphi(a),m)\right.$$
$$\times\left[mf'(a)^{rq}\left(\frac{r}{r-sp_1}\right)^r\right.$$
$$\times\left(\frac{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x}{\eta(\varphi(b),\varphi(a),m)}\right)^{r-sp_1}$$
$$+f'(b)^{rq}\left(\frac{r}{r-sp_2}\right)^r$$
$$\left.\times\left[1-\left(\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}\right)^{1-\frac{sp_2}{r}}\right]^r\right]^{\frac{1}{r}} \quad (2.10)$$
$$\left.\left.+\left|\lambda_2(x)\right|^q\left(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x\right)\right]^{\frac{1}{q}}\right\}.$$

Corollary 2.12. *In Theorem 2.7 for* $h_1(t)=h_2(t)=t(1-t)$, *we have the following inequality for generalized relative semi-(r;m,p_1,p_2,tgs)-preinvex mappings*:

$$\left|I_{f,\eta,\varphi,\lambda_1(x),\lambda_2(x)}(x;\alpha,k,m,a,b)\right|$$
$$\leq \frac{2^{\frac{q-1}{q}}}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\left\{B_{\varphi}^{\frac{1}{p}}(x;\alpha,k,m,p,a)\left[\eta(\varphi(b),\varphi(a),m)\right.\right.$$
$$\times\left[mf'(a)^{rq}\beta^r\left(\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)};\frac{p_1}{r}+1,\frac{p_1}{r}+1\right)\right.$$
$$\left.+f'(b)^{rq}\beta^r\left(\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)};\frac{p_2}{r}+1,\frac{p_2}{r}+1\right)\right]^{\frac{1}{r}}$$

$$+\left|\lambda_1(x)\right|^q\left(x-m\varphi(a)\right)\Big]^{\frac{1}{q}}$$

$$+C_{\eta,\varphi}^{\frac{1}{p}}\left(x;\alpha,k,m,p,a,b\right)\left[\eta\left(\varphi(b),\varphi(a),m\right)\right.$$
$$\times\Big[mf'(a)^{rq}\,\overline{I}_{\eta,\varphi}^{r}\left(t(1-t);x,r,m,a,b,p_1\right) \qquad (2.11)$$
$$+f'(b)^{rq}\,\overline{I}_{\eta,\varphi}^{r}\left(t(1-t);x,r,m,a,b,p_2\right)\Big]^{\frac{1}{r}}$$
$$+\left|\lambda_2(x)\right|^q\left(m\varphi(a)+\eta\left(\varphi(b),\varphi(a),m\right)-x\right)\Big]^{\frac{1}{q}}\Bigg\}.$$

where $\beta\left(\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)};\frac{p_1}{r}+1,\frac{p_1}{r}+1\right)$ *is incomplete beta function and*

$$m\varphi(a)<x\le m\varphi(a)+\eta\left(\varphi(b),\varphi(a),m\right).$$

Corollary 2.13. *In Theorem 2.7 for* $h_1(t)=\frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t)=\frac{\sqrt{t}}{2\sqrt{1-t}}$, *we have the following inequality for generalized relative semi-($r;m,p_1,p_2$)-MT-preinvex mappings:*

$$\left|I_{f,\eta,\varphi,\lambda_1(x),\lambda_2(x)}\left(x;\alpha,k,m,a,b\right)\right|$$
$$\le\frac{2^{\frac{q-1}{q}}}{\eta\left(\varphi(b),\varphi(a),m\right)}\Bigg\{B_{\varphi}^{\frac{1}{p}}\left(x;\alpha,k,m,p,a\right)$$
$$\times\Bigg[\eta\left(\varphi(b),\varphi(a),m\right)\Bigg[mf'(a)^{rq}\left(\frac{1}{2}\right)^{\frac{p_1}{r}}\beta^r$$
$$\times\left(\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)};1-\frac{p_1}{2r},1+\frac{p_1}{2r}\right)$$
$$+f'(b)^{rq}\left(\frac{1}{2}\right)^{\frac{p_2}{r}}\beta^r$$
$$\times\left(\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)};1+\frac{p_2}{2r},1-\frac{p_2}{2r}\right)\Bigg]^{\frac{1}{r}}$$
$$+\left|\lambda_1(x)\right|^q\left(x-m\varphi(a)\right)\Bigg]^{\frac{1}{q}}$$
$$+C_{\eta,\varphi}^{\frac{1}{p}}\left(x;\alpha,k,m,p,a,b\right)\left[\eta\left(\varphi(b),\varphi(a),m\right)\right.$$
$$\times\left[mf'(a)^{rq}\,\overline{I}_{\eta,\varphi}^{r}\left(\frac{\sqrt{1-t}}{2\sqrt{t}};x,r,m,a,b,p_1\right)\right.$$
$$+f'(b)^{rq}\,\overline{I}_{\eta,\varphi}^{r}\left(\frac{\sqrt{t}}{2\sqrt{1-t}};x,r,m,a,b,p_2\right)\Bigg]^{\frac{1}{r}}$$
$$+\left|\lambda_2(x)\right|^q\left(m\varphi(a)+\eta\left(\varphi(b),\varphi(a),m\right)-x\right)\Bigg]^{\frac{1}{q}}\Bigg\}. \qquad (2.12)$$

Theorem 2.14. *Let* $\alpha,k>0$, $0<r\le 1$ *and* $p_1,p_2>-1$. *Let* $K\subset\mathbb{R}$ *be an open m-invex subset with respect to* $\eta:K\times K\times(0,1]\to\mathbb{R}$ *for some fixed* $m\in(0,1]$. *Suppose* $h_1,h_2:[0,1]\to[0,+\infty)$ *and* $\varphi:I\to K$ *are continuous functions and* $\lambda_1(x),\lambda_2(x)$ *are two complex numbers. Assume that* $f:K\to(0,+\infty)$ *be a differentiable function on* K°, *where* $\eta\left(\varphi(b),\varphi(a),m\right)>0$. *If* $f'(x)^q$ *is generalized relative semi-(r; m, p_1, p_2, h_1, h_2)-preinvex mappings,* $q\ge 1$, *then the following inequality holds*:

$$\left|I_{f,\eta,\varphi,\lambda_1(x),\lambda_2(x)}\left(x;\alpha,k,m,a,b\right)\right|$$
$$\le\frac{2^{\frac{q-1}{q}}}{\eta\left(\varphi(b),\varphi(a),m\right)}$$
$$\times\Bigg\{B_{\varphi}^{1-\frac{1}{q}}\left(x;\alpha,k,m,p,a\right)\left[\eta\left(\varphi(b),\varphi(a),m\right)\right.$$
$$\times\Big[mf'(a)^{rq}\,I_{\eta,\varphi}^{r}\left(h_1(t);x,\alpha,k,r,m,a,b,p_1\right)$$
$$+f'(b)^{rq}\,I_{\eta,\varphi}^{r}\left(h_1(t);x,\alpha,k,r,m,a,b,p_2\right)\Big]^{\frac{1}{r}}$$
$$+\left|\lambda_1(x)\right|^q\left(B_{\varphi}\left(x;\alpha,k,m,1,a\right)\right)\Big]^{\frac{1}{q}}$$
$$+C_{\eta,\varphi}^{1-\frac{1}{q}}\left(x;\alpha,k,m,1,a,b\right)\left[\eta\left(\varphi(b),\varphi(a),m\right)\right.$$
$$\times\Big[mf'(a)^{rq}\,\overline{I}_{\eta,\varphi}^{r}\left(h_1(t);x,\alpha,k,r,m,a,b,p_1\right)$$
$$+f'(b)^{rq}\,\overline{I}_{\eta,\varphi}^{r}\left(h_2(t);x,\alpha,k,r,m,a,b,p_2\right)\Big]^{\frac{1}{r}} \qquad (2.13)$$
$$+\left|\lambda_2(x)\right|^q C_{\eta,\varphi}\left(x;\alpha,k,m,1,a,b\right)\Big]^{\frac{1}{q}}\Bigg\}.$$

where

$$I_{\eta,\varphi}\left(h_i(t);x,\alpha,k,r,m,a,b,p_i\right)$$
$$:=\int_0^{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}}\left|\begin{matrix}\left(\begin{matrix}m\varphi(a)\\+t\eta\left(\varphi(b),\varphi(a),m\right)\end{matrix}\right)^{\frac{\alpha}{k}}\\-m\varphi(a)\end{matrix}\right|h_i^{\frac{p_i}{r}}(t)\,dt,$$
$$\forall i=1,2,$$

$$\overline{I}_{\eta,\varphi}\left(h_i(t);x,\alpha,k,r,m,a,b,p_i\right)$$
$$:=\int_{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}}^{1}\left|\begin{matrix}\left(m\varphi(a)+t\eta\left(\varphi(b),\varphi(a),m\right)\right)^{\frac{\alpha}{k}}\\-\left(m\varphi(a)+\eta\left(\varphi(b),\varphi(a),m\right)\right)\end{matrix}\right|h_i^{\frac{p_i}{r}}(t)\,dt,$$

and $B_{\varphi}(x;\alpha,k,m,1,a)$, $C_{\eta,\varphi}(x;\alpha,k,m,1,a,b)$ *are defined as in Theorem 2.7.*

Proof. Suppose that $q\geq 1$ and $0<r\leq 1$. From Lemma 2.1, generalized relative semi-(r; m, p_1, p_2, h_1, h_2)-preinvexity of $f'(x)^q$, the well-known power mean inequality, Minkowski inequality, properties of the modulus and using the elementary inequality $(c+d)^{\theta}\leq 2^{\theta-1}\left(c^{\theta}+d^{\theta}\right)$ where $\theta>1,\ c,d\geq 0,$ we have

$$\left|I_{f,\eta,\varphi,\lambda_1(x),\lambda_2(x)}(x;\alpha,k,m,a,b)\right|$$

$$\leq\frac{1}{\eta(\varphi(b),\varphi(a),m)}\left\{\int_{m\varphi(\alpha)}^{x}\left|t^{\frac{\alpha}{k}}-m\varphi(\alpha)\right|\left|\begin{matrix}f'(t)\\-\lambda_1(x)\end{matrix}\right|dt\right.$$

$$+\int_{x}^{m\varphi(\alpha)+\eta(\varphi(b),\varphi(a),m)}\left|t^{\frac{\alpha}{k}}-\begin{pmatrix}m\varphi(\alpha)\\+\eta(\varphi(b),\varphi(a),m)\end{pmatrix}\right|$$

$$\left.\times\left|f'(t)-\lambda_2(x)\right|dt\right.$$

$$\leq\frac{1}{\eta(\varphi(b),\varphi(a),m)}\left\{\left(\int_{m\varphi(\alpha)}^{x}\left|t^{\frac{\alpha}{k}}-m\varphi(\alpha)\right|dt\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\int_{m\varphi(\alpha)}^{x}\left|t^{\frac{\alpha}{k}}-m\varphi(\alpha)\right|\left|f'(t)-\lambda_1(x)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{x}^{m\varphi(\alpha)+\eta(\varphi(b),\varphi(a),m)}\left|t^{\frac{\alpha}{k}}-\begin{pmatrix}m\varphi(\alpha)\\+\eta(\varphi(b),\varphi(a),m)\end{pmatrix}\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{x}^{m\varphi(\alpha)+\eta(\varphi(b),\varphi(a),m)}\left|t^{\frac{\alpha}{k}}-\left(m\varphi(\alpha)+\eta(\varphi(b),\varphi(a),m)\right)\right|\right.$$

$$\left.\left.\times\left|f'(t)-\lambda_2(x)\right|^q dt\right)^{\frac{1}{q}}\right\}$$

$$\leq\frac{1}{\eta(\varphi(b),\varphi(a),m)}\left\{B_{\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a)\right.$$

$$\times\left(\int_{m\varphi(\alpha)}^{x}\left|t^{\frac{\alpha}{k}}-m\varphi(\alpha)\right|2^{q-1}\left(f'(t)^q+\left|\lambda_1(x)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$+C_{\eta,\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a,b)$$

$$\times\left(\int_{x}^{m\varphi(\alpha)+\eta(\varphi(b),\varphi(a),m)}\left|t^{\frac{\alpha}{k}}-\begin{pmatrix}m\varphi(\alpha)\\+\eta(\varphi(b),\varphi(a),m)\end{pmatrix}\right|\right.$$

$$\left.\left.\times 2^{q-1}\left(f'(t)^q+\left|\lambda_2(t)\right|^q\right)dt\right)^{\frac{1}{q}}\right\}$$

$$=\frac{2^{\frac{q-1}{q}}}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left\{B_{\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a)\left[\eta(\varphi(b),\varphi(a),m)\right.\right.$$

$$\times\int_{x}^{\frac{x-m\varphi(\alpha)}{\eta(\varphi(b),\varphi(a),m)}}\left|\begin{matrix}\left(m\varphi(\alpha)+t\eta(\varphi(b),\varphi(a),m)\right)^{\frac{\alpha}{k}}\\-m\varphi(\alpha)\end{matrix}\right|$$

$$\times\left(f'\left(m\varphi(\alpha)+t\eta(\varphi(b),\varphi(a),m)\right)\right)^q dt$$

$$\left.+\left|\lambda_1(x)\right|^q B_{\varphi}(x;\alpha,k,m,1,a)\right]^{\frac{1}{q}}$$

$$+C_{\eta,\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a,b)\left[\eta(\varphi(b),\varphi(a),m)\right.$$

$$\times\int_{\frac{x-m\varphi(\alpha)}{\eta(\varphi(b),\varphi(a),m)}}^{1}\left|\begin{matrix}\left(m\varphi(\alpha)+t\eta(\varphi(b),\varphi(a),m)\right)^{\frac{\alpha}{k}}\\-\left(m\varphi(\alpha)+\eta(\varphi(b),\varphi(a),m)\right)\end{matrix}\right|$$

$$\times\left(f'\left(m\varphi(\alpha)+t\eta(\varphi(b),\varphi(a),m)\right)\right)^q dt$$

$$\left.\left.+\left|\lambda_2(x)\right|^q C_{\eta,\varphi}(x;\alpha,k,m,1,a,b)\right]^{\frac{1}{q}}\right\}$$

$$\leq\frac{2^{\frac{q-1}{q}}}{\eta(\varphi(b),\varphi(a),m)}\left\{B_{\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a)\right.$$

$$\times\left[\eta(\varphi(b),\varphi(a),m)\right.$$

$$\times\int_{0}^{\frac{x-m\varphi(\alpha)}{\eta(\varphi(b),\varphi(a),m)}}\left|\begin{matrix}\left(m\varphi(\alpha)+t\eta(\varphi(b),\varphi(a),m)\right)^{\frac{\alpha}{k}}\\-m\varphi(\alpha)\end{matrix}\right|$$

$$\times\left[mh_1^{p_1}(t)\,f'(a)^{rq}+h_2^{p_2}(t)\,f'(b)^{rq}\right]^{\frac{1}{r}}dt$$

$$\left.+\left|\lambda_1(x)\right|^q B_{\varphi}(x;\alpha,k,m,1,a)\right]^{\frac{1}{q}}$$

$$+C_{\eta,\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a,b)$$

$$\times\left[\eta(\varphi(b),\varphi(a),m)\right.$$

$$\times\int_{\frac{x-m\varphi(\alpha)}{\eta(\varphi(b),\varphi(a),m)}}^{1}\left|\begin{matrix}\left(m\varphi(\alpha)+t\eta(\varphi(b),\varphi(a),m)\right)^{\frac{\alpha}{k}}\\-\left(m\varphi(\alpha)+\eta(\varphi(b),\varphi(a),m)\right)\end{matrix}\right|$$

$$\times\left[mh_1^{p_1}(t)\,f'(a)^{rq}+h_2^{p_2}(t)\,f'(b)^{rq}\right]^{\frac{1}{r}}dt$$

$$\left.\left.+\left|\lambda_2(x)\right|^q C_{\eta,\varphi}(x;\alpha,k,m,1,a,b)\right]^{\frac{1}{q}}\right\}$$

$$\le \frac{2^{\frac{q-1}{q}}}{\eta(\varphi(b),\varphi(a),m)}\Bigg\{B_{\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a)$$

$$\times\Big[\eta(\varphi(b),\varphi(a),m)\Bigg[\Bigg(\int_0^{\frac{x-m\varphi(\alpha)}{\eta(\varphi(b),\varphi(a),m)}} m^{\frac{1}{r}} f'(a)^q$$

$$\times\left|\left(m\varphi(\alpha)+t\eta(\varphi(b),\varphi(a),m)\right)^{\frac{\alpha}{k}}-m\varphi(\alpha)\right| h_1^{\frac{p_1}{t}}(t)\,dt\Bigg)^r$$

$$+\Bigg(\int_0^{\frac{x-m\varphi(\alpha)}{\eta(\varphi(b),\varphi(a),m)}} f'(b)^q$$

$$\times\left|\left(m\varphi(\alpha)+t\eta(\varphi(b),\varphi(a),m)\right)^{\frac{\alpha}{k}}-m\varphi(\alpha)\right| h_2^{\frac{p_2}{t}}(t)\,dt\Bigg)\Bigg]^{\frac{1}{r}}$$

$$+\left|\lambda_1(x)\right|^q B_{\varphi}(x;\alpha,k,m,1,a)\Big]^{\frac{1}{q}}$$

$$+C_{\eta,\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a,b)\Big[\eta(\varphi(b),\varphi(a),m)$$

$$\times\Bigg[\Bigg(\int_{\frac{x-m\varphi(\alpha)}{\eta(\varphi(b),\varphi(a),m)}}^1 m^{\frac{1}{r}} f'(a)$$

$$\times\left|\begin{array}{l}\left(m\varphi(\alpha)+t\eta(\varphi(b),\varphi(a),m)\right)^{\frac{\alpha}{k}}\\ -\left(m\varphi(\alpha)+\eta(\varphi(b),\varphi(a),m)\right)\end{array}\right|$$

$$\times\Big[mh_1^{p_1}(t)f'(a)^{rq}+h_2^{p_2}(t)f'(b)^{rq}\Big]^{\frac{1}{r}}\,dt h_1^{\frac{p_1}{t}}(t)\,dt\Bigg)^r$$

$$+\left|\lambda_2(x)\right|^q C_{\eta,\varphi}(x;\alpha,k,m,1,a,b)\Big]^{\frac{1}{q}}\Bigg\}$$

$$=\frac{2^{\frac{q-1}{q}}}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\Bigg\{B_{\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a)\Big[\eta(\varphi(b),\varphi(a),m)$$

$$\times\Big[mf'(a)^{rq}I_{\eta,\varphi}^{r}(h_1(t);x,\alpha,k,r,m,a,b,p_1)$$

$$+f'(b)^{rq}I_{\eta,\varphi}^{r}(h_2(t);x,\alpha,k,r,m,a,b,p_2)\Big]^{\frac{1}{r}}$$

$$+\left|\lambda_1(x)\right|^q B_{\varphi}(x;\alpha,k,m,1,a)\Big]^{\frac{1}{q}}$$

$$+C_{\eta,\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a,b)\Big[\eta(\varphi(b),\varphi(a),m)$$

$$\times\Big[mf'(a)^{rq}\overline{I}_{\eta,\varphi}^{r}(h_1(t);x,\alpha,k,r,m,a,b,p_1)$$

$$+f'(b)^{rq}\overline{I}_{\eta,\varphi}^{r}(h_2(t);x,\alpha,k,r,m,a,b,p_2)\Big]^{\frac{1}{r}}$$

$$+\left|\lambda_2(x)\right|^q C_{\eta,\varphi}(x;\alpha,k,m,1,a,b)\Big]^{\frac{1}{q}}\Bigg\}.$$

So, the proof of this theorem is complete.

We point out some special cases of Theorem 2.14.

Corollary 2.15. *In Theorem 2.14 for* $\lambda_1(x)=\lambda_2(x)=0$, *we have the following inequality for generalized relative semi-(r; m, p_1, p_2, h_1, h_2)-preinvex mappings*:

$$\left|I_{f,\eta,\varphi}(x;\alpha,k,m,a,b)\right|$$

$$\le\left(\frac{2}{\eta(\varphi(b),\varphi(a),m)}\right)^{\frac{q-1}{q}}\Bigg\{B_{\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a)$$

$$\times\Big[mf'(a)^{rq}I_{\eta,\varphi}^{r}(h_1(t);x,\alpha,k,r,m,a,b,p_1)$$

$$+f'(b)^{rq}I_{\eta,\varphi}^{r}(h_2(t);x,\alpha,k,r,m,a,b,p_2)\Big]^{\frac{1}{rq}} \tag{2.14}$$

$$+C_{\eta,\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a,b)$$

$$\times\Big[mf'(a)^{rq}\overline{I}_{\eta,\varphi}^{r}(h_1(t);x,\alpha,k,r,m,a,b,p_1)$$

$$+f'(b)^{rq}\overline{I}_{\eta,\varphi}^{r}(h_2(t);x,\alpha,k,r,m,a,b,p_2)\Big]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 2.16. *In Theorem 2.14 for* $h_1(t)=h(1-t)$ *and* $h_2(t)=h(t)$, *we have the following inequality for generalized relative semi-(r; m, p_1, p_2, h)-preinvex mappings*:

$$\left|I_{f,\eta,\varphi,\lambda_1(x),\lambda_2(x)}(x;\alpha,k,m,a,b)\right|$$

$$\le\frac{2^{\frac{q-1}{q}}}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\Bigg\{B_{\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a)\Big[\eta(\varphi(b),\varphi(a),m)$$

$$\times\Big[mf'(a)^{rq}I_{\eta,\varphi}^{r}(h(1-t);x,\alpha,k,r,m,a,b,p_1)$$

$$+f'(b)^{rq}I_{\eta,\varphi}^{r}(h(t);x,\alpha,k,r,m,a,b,p_2)\Big]^{\frac{1}{r}}$$

$$+\left|\lambda_1(x)\right|^q B_{\varphi}(x;\alpha,k,m,1,a)\Big]^{\frac{1}{q}}$$

$$+C_{\eta,\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a,b)\Big[\eta(\varphi(b),\varphi(a),m)$$

$$\times\Big[mf'(a)^{rq}\overline{I}_{\eta,\varphi}^{r}(h(1-t);x,\alpha,k,r,m,a,b,p_1)$$

$$+ f'(b)^{rq}\,\overline{I}^{r}_{\eta,\varphi}\left(h(t);x,\alpha,k,r,m,a,b,p_2\right)\Big]^{\frac{1}{r}}$$
$$+\left|\lambda_2(x)\right|^q C_{\eta,\varphi}(x;\alpha,k,m,1,a,b)\Big]^{\frac{1}{q}}\Bigg\}. \quad (2.15)$$

Corollary 2.17. *In Theorem 2.14 for* $h_1(t)=(1-t)^s$, $h_2(t)=t^s$, *we have the following inequality for generalized relative semi-(r; m, p_1, p_2, s)-Breckner-preinvex mappings*:

$$\left|I_{f,\eta,\varphi,\lambda_1(x),\lambda_2(x)}(x;\alpha,k,m,a,b)\right|$$
$$\le \frac{2^{\frac{q-1}{q}}}{\eta(\varphi(b),\varphi(a),m)}\Bigg\{B_{\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a)$$
$$\times\Big[\eta(\varphi(b),\varphi(a),m)$$
$$\times\Big[mf'(a)^{rq}\,I^{r}_{\eta,\varphi}\left((1-t)^s;x,\alpha,k,r,m,a,b,p_1\right)$$
$$+ f'(b)^{rq}\,I^{r}_{\eta,\varphi}\left(t^s;x,\alpha,k,r,m,a,b,p_2\right)\Big]^{\frac{1}{r}}$$
$$+\left|\lambda_1(x)\right|^q B_{\varphi}(x;\alpha,k,m,1,a)\Big]^{\frac{1}{q}}$$
$$+C_{\eta,\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a,b)\Big[\eta(\varphi(b),\varphi(a),m)$$
$$\times\Big[mf'(a)^{rq}\,\overline{I}^{r}_{\eta,\varphi}\left((1-t)^s;x,\alpha,k,r,m,a,b,p_1\right)$$
$$+ f'(b)^{rq}\,\overline{I}^{r}_{\eta,\varphi}\left(t^s;x,\alpha,k,r,m,a,b,p_2\right)\Big]^{\frac{1}{r}}$$
$$+\left|\lambda_2(x)\right|^q C_{\eta,\varphi}(x;\alpha,k,m,1,a,b)\Big]^{\frac{1}{q}}\Bigg\}. \quad (2.16)$$

Corollary 2.18. *In Theorem 2.14 for* $h_1(t)=(1-t)^{-s}$, $h_2(t)=t^{-s}$, *we have the following inequality for generalized relative semi-(r; m, p_1, p_2, s)-Godunova-Levin-Dragomir-preinvex mappings*:

$$\left|I_{f,\eta,\varphi,\lambda_1(x),\lambda_2(x)}(x;\alpha,k,m,a,b)\right|$$
$$\le \frac{2^{\frac{q-1}{q}}}{\eta(\varphi(b),\varphi(a),m)}\Bigg\{B_{\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a)$$
$$\times\Big[\eta(\varphi(b),\varphi(a),m)$$
$$\times\Big[mf'(a)^{rq}\,I^{r}_{\eta,\varphi}\left((1-t)^{-s};x,\alpha,k,r,m,a,b,p_1\right)$$
$$+ f'(b)^{rq}\,I^{r}_{\eta,\varphi}\left(t^{-s};x,\alpha,k,r,m,a,b,p_2\right)\Big]^{\frac{1}{r}}$$
$$+\left|\lambda_1(x)\right|^q B_{\varphi}(x;\alpha,k,m,1,a)\Big]^{\frac{1}{q}}$$
$$+C_{\eta,\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a,b)\Big[\eta(\varphi(b),\varphi(a),m)$$
$$\times\Big[mf'(a)^{rq}\,\overline{I}^{r}_{\eta,\varphi}\left((1-t)^{-s};x,\alpha,k,r,m,a,b,p_1\right)$$
$$+ f'(b)^{rq}\,\overline{I}^{r}_{\eta,\varphi}\left(t^{-s};x,\alpha,k,r,m,a,b,p_2\right)\Big]^{\frac{1}{r}}$$
$$+\left|\lambda_2(x)\right|^q C_{\eta,\varphi}(x;\alpha,k,m,1,a,b)\Big]^{\frac{1}{q}}\Bigg\}. \quad (2.17)$$

Corollary 2.19. *In Theorem 2.14 for* $h_1(t)=h_2(t)=t(1-t)$, *we have the following inequality for generalized relative semi-(r; m, p_1, p_2, tgs)-preinvex mappings*:

$$\left|I_{f,\eta,\varphi,\lambda_1(x),\lambda_2(x)}(x;\alpha,k,m,a,b)\right|$$
$$\le \frac{2^{\frac{q-1}{q}}}{\eta(\varphi(b),\varphi(a),m)}\Bigg\{B_{\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a)$$
$$\times\Big[\eta(\varphi(b),\varphi(a),m)$$
$$\times\Big[mf'(a)^{rq}\,I^{r}_{\eta,\varphi}\left(t(1-t);x,\alpha,k,r,m,a,b,p_1\right)$$
$$+ f'(b)^{rq}\,I^{r}_{\eta,\varphi}\left(t(1-t);x,\alpha,k,r,m,a,b,p_2\right)\Big]^{\frac{1}{r}}$$
$$+\left|\lambda_1(x)\right|^q B_{\varphi}(x;\alpha,k,m,1,a)\Big]^{\frac{1}{q}}$$
$$+C_{\eta,\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a,b)\Big[\eta(\varphi(b),\varphi(a),m)$$
$$\times\Big[mf'(a)^{rq}\,\overline{I}^{r}_{\eta,\varphi}\left(t(1-t);x,\alpha,k,r,m,a,b,p_1\right)$$
$$+ f'(b)^{rq}\,\overline{I}^{r}_{\eta,\varphi}\left(t(1-t);x,\alpha,k,r,m,a,b,p_2\right)\Big]^{\frac{1}{r}}$$
$$+\left|\lambda_2(x)\right|^q C_{\eta,\varphi}(x;\alpha,k,m,1,a,b)\Big]^{\frac{1}{q}}\Bigg\}. \quad (2.18)$$

Corollary 2.20. *In Theorem 2.14 for* $h_1(t)=\frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t)=\frac{\sqrt{t}}{2\sqrt{1-t}}$, *we have the following inequality for generalized relative semi-(r; m, p_1, p_2)-MT-preinvex mappings*:

$$\left|I_{f,\eta,\varphi,\lambda_1(x),\lambda_2(x)}(x;\alpha,k,m,a,b)\right|$$
$$\le \frac{2^{\frac{q-1}{q}}}{\eta(\varphi(b),\varphi(a),m)}\Bigg\{B_{\varphi}^{1-\frac{1}{q}}(x;\alpha,k,m,1,a)$$
$$\times\Big[\eta(\varphi(b),\varphi(a),m)$$
$$\times\Big[mf'(a)^{rq}\,I^{r}_{\eta,\varphi}\left(\frac{\sqrt{1-t}}{2\sqrt{t}};x,\alpha,k,r,m,a,b,p_1\right)$$

$$+f'(b)^{rq} I^r_{\eta,\varphi}\left(\frac{\sqrt{t}}{2\sqrt{1-t}};x,\alpha,k,r,m,a,b,p_2\right)\Bigg]^{\frac{1}{r}}$$

$$+\left|\lambda_1(x)\right|^q B_\varphi(x;\alpha,k,m,1,a)\Bigg]^{\frac{1}{q}}$$

$$+C^{1-\frac{1}{q}}_{\eta,\varphi}(x;\alpha,k,m,1,a,b)$$

$$\times\left[\eta(\varphi(b),\varphi(a),m)\right)$$

$$\times\Bigg[mf'(a)^{rq}\,\overline{I}^r_{\eta,\varphi}\left(\frac{\sqrt{1-t}}{2\sqrt{t}};x,\alpha,k,r,m,a,b,p_1\right)$$

$$+f'(b)^{rq}\,\overline{I}^r_{\eta,\varphi}\left(\frac{\sqrt{t}}{2\sqrt{1-t}};x,\alpha,k,r,m,a,b,p_2\right)\Bigg]^{\frac{1}{r}}$$

$$+\left|\lambda_2(x)\right|^q C_{\eta,\varphi}(x;\alpha,k,m,1,a,b)\Bigg]^{\frac{1}{q}}\Bigg\}. \tag{2.19}$$

Remark 2.21. Using our Theorems 2.7 and 2.14 for different values of α, k, m, p_1, p_2, for some suitable continuous functions h_1, h_2, φ and complex numbers $\lambda_1(x)$, $\lambda_2(x)$, we can get some new Ostrowski type inequalities associated with generalized relative semi-($r; m, p_1, p_2, h_1, h_2$)-preinvex mappings.

Remark 2.22. Applying our Theorems 2.7 and 2.14, we can deduce some new inequalities using special means associated with generalized relative semi-($r; m, p_1, p_2, h_1, h_2$) -preinvex mappings.

3. Conclusions

In this article, we first presented a new integral identity concerning differentiable mappings defined on m-invex set. By using the notion of generalized relative semi-($r; m, p_1, p_2, h_1, h_2$)-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Ostrowski type inequalities are established. It is pointed out that some new special cases are deduced from main results of the article. Motivated by this new interesting class of generalized relative semi-($r; m, p_1, p_2, h_1, h_2$)-preinvex mappings we can indeed see to be vital for fellow researchers and scientists working in the same domain. We conclude that our methods considered here may be a stimulant for further investigations concerning Ostrowski, Hermite-Hadamard and Simpson type integral inequalities for various kinds of preinvex functions involving local fractional integrals, fractional integral operators, Caputo k-fractional derivatives, q-calculus, (p, q)-calculus, time scale calculus and conformable fractional integrals.

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