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# Obtaining of Some New Inequalities Using Functıonals for GA-Convex Functions

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**Abstract** In this paper, we get the fractional integral inequalities obtained for geometric arithmetically (GA) convex functions by using functionals. The left hand side of the Hermite-Hadamard and Hermite-Hadamard-Fejér inequalities obtained by using Hadamard fractional integrals for Geometric Arithmetically-convex functions was obtained via functionals. We conclude that some results obtained in this paper are the refinements of the earlier results.

***Keywords:*** *convex function, integral inequalities, GA-Convex function, functionals, Riemann-Liouville fractional integrals, Hadamard fractional integrals*

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## 1. Introduction

**Definition 1.1** A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f\left(tx+(1-t)y\right) \le tf\left(x\right)+(1-t)f\left(y\right)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature.

It is well known that theory of convex sets and convex functions play an important role in mathematics and the other pure and applied sciences.

If $f: I \to \mathbb{R}$ is a convex function on the interval $I$, then for any $a, b \in I$ with $a \neq b$ we have the following double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \quad (1.1)$$

This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping $f$. Both inequalities hold in the reversed direction if $f$ is concave. For some results which generalize, improve and extend the inequalities (1.1) we refer the reader to the recent papers (see [1-11]) and references therein.

**Definition 1.2** [10] Let $f: I \subseteq (0,\infty) \to \mathbb{R}$ is said to be GA-convex (geometric-arithmetically convex) if

$$f\left(x^t y^{1-t}\right) \le tf\left(y\right)+(1-t)f\left(x\right)$$

for all $x, y \in I$ and $t \in [0,1]$. If this inequality is reversed, then $f$ is said to be geometric arithmetically concave.

In [2], S. S. Dragomir proposed the following Hermite-Hadamard type inequalities which refine the first inequality of (1.1).

**Theorem 1.1 [2].** Let $f$ is convex on $[a,b]$. Then $H$ is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le H(0) \le H(t) \le H(1) \\ &\le \frac{1}{b-a}\int_a^b f(x)dx \end{aligned} \quad (1.2)$$

*where*

$$H(t)=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx.$$

An analogous result for convex functions which refines the second inequality of (1.1) is obtained by G. S. Yang and M. C. Hong in [9] as follows.

**Theorem 1.2** [9]. Let f is convex on $[a,b]$. Then $P$ is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)dx &= P(0) \le P(t) \le P(1) \\ &= \frac{f(a)+f(b)}{2} \end{aligned} \quad (1.3)$$

where

$$P(t)=\frac{1}{2(b-a)}\int_a^b\left[\begin{array}{l}f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)\\+f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)\end{array}\right]dx.$$

G. S. Yang and K. L. Tseng in [8] established some generalizations of (1.2) and (1.3) based on the following results.

**Theorem 1.3** [8] Let $f$: $[a,b]\to\mathbb{R}$ be a convex function, $0<\alpha<1$, $0<\beta<1$

$$A=\alpha a+(1-\alpha)b,$$

$$u_0=(b-a)\min\left\{\frac{\alpha}{1-\beta},\frac{1-\alpha}{\beta}\right\}$$

and let $h$ be defined by

$$h(t)=(1-\beta)f(A-\beta t)+\beta f(A+(1-\beta)t),$$

for $t\in[0,u_0]$. Then h is convex, increasing on $[0,u_0]$ and for all $t\in[0,u_0]$,

$$f(\alpha a+(1-\alpha)b)\le h(t)\le\alpha f(a)+(1-\alpha)f(b).$$

The weighted generalization of Hermite-Hadamard inequality for GA-convex functions is as follows [16]:

**Theorem 1.4** Let $f$:$I\subseteq(0,\infty)\to\mathbb{R}$ be a GA-convex function, $a$, $b\in I$ and $a<b$. If $g$: $[a,b]\to[0,\infty)$ is continuous and geometrically symmetric according to $\sqrt{ab}$, then

$$f\left(\sqrt{ab}\right)\int_a^b\frac{g(x)}{x}dx\le\int_a^b\frac{f(x)g(x)}{x}dx\le\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x}dx.$$

Specifically, if $g(x)=1$ is taken in this theorem, the following Hermite-Hadamard inequality is obtained for GA-convex functions:

$$f\left(\sqrt{ab}\right)\le\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\le\frac{f(a)+f(b)}{2}. \tag{1.4}$$

This inequality is also obtained in the case of $s=1$ specially in Theorem 3.1 and Theorem 3.3 in [14].

It is remarkable that M. Z. Sarıkaya et al. [7] proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

**Theorem 1.5** [7] Let $f$: $[a,b]\to\mathbb{R}$ be a positive function with $a<b$ and $f\in L_1[a,b]$. If $f$ is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[{}_RJ_{a+}^\alpha f(b)+{}_RJ_{b-}^\alpha f(a)\right]\le\frac{f(a)+f(b)}{2} \tag{1.5}$$

with $\alpha>0$.

We remark that the symbols $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha\ge0$ with $a\ge0$ which are defined by

$${}_RJ_{a+}^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\ x>a,$$

and

$${}_RJ_{b-}^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt,\ x<b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}dt.$

**Definition 1.3** [12] Let $f\in[a,b]$. The right-hand side and left-hand side Hadamard fractional integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha>0$ with $b>a\ge0$ are defined by

$${}_HJ_{b-}^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\left(\ln\frac{x}{t}\right)^{\alpha-1}f(t)\frac{dt}{t},\ x>a$$

and

$${}_HJ_{b-}^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\left(\ln\frac{t}{x}\right)^{\alpha-1}f(t)\frac{dt}{t},\ x<b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}dt.$

In this paper, we establish some new Hermite-Hadamard type inequalities for convex functions via Riemann-Liouville fractional integrals which refine the inequalities of (1.5).

Ruiyin Xiang [11] proved the following Lemma and Theorem for interesting inequalities of Hermite-Hadamard type inequalities for convex functions via Riemann-Liouville fractional integrals.

**Lemma 1.1** Let $f$: $[a,b]\to\mathbb{R}$ be a convex function and $h$ be defined by

$$h(t)=\frac{1}{2}\left[f\left(\left(\frac{a+b}{2}\right)-\frac{t}{2}\right)+f\left(\left(\frac{a+b}{2}\right)+\frac{t}{2}\right)\right].$$

Then $h(t)$ is convex, increasing on $[0,b-a]$ and for all $t\in[0,b-a]$,

$$f\left(\frac{a+b}{2}\right)\le h(t)\le\frac{f(a)+f(b)}{2}.$$

**Theorem 1.6** Let $f$:$[a,b]\to\mathbb{R}$ be a positive function with $a<b$ and $f\in L_1[a,b]$. If $f$ is a convex function on $[a,b]$, then $WH$ is convex and monotonically increasing on [0,1] and

$$f\left(\frac{a+b}{2}\right)=WH(0)\le WH(t)\le WH(1)=\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[{}_RJ_{a+}^\alpha f(b)+{}_RJ_{b-}^\alpha f(a)\right]$$

with $\alpha>0$, where

$$WH(t)=\frac{\alpha}{2(b-a)^\alpha}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)\times\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]dx.$$

We can rewrite as follows the theorem in [13] for s = 1:

**Theorem 1.7** Let $f:I\subseteq\mathbb{R}_+\longrightarrow\mathbb{R}$. Then the following statements are true:

(1) The function $f(x)$ is geometric-arithmetically convex on I if and only if $f(e^x)$ is convex on the interval $lnI=\{lnx|x\in I\}$, where it is assumed that $ln0=-\infty$.

(2) If $f(x)$ is decreasing and geometric-arithmetically convex on $I$, then it is convex on $I$.

(3) If $f(x)$ is increasing and convex on $I$, then it is also geometric-arithmetically convex on $I$.

For GA-convex functions Hermite-Hadamard inequalities obtained with the help of fractional integrals can be given as follows [15].

**Theorem 1.8** Let $f:I\subseteq(0,\infty)\rightarrow\mathbb{R}$ be a function, $a,b\in I$, $a<b$ and $f\in[a,b]$. If the function $f$ is GA-convex on $[a,b]$, then the following inequality for the fractional integrals hold:

$$f\left(\sqrt{ab}\right)\leq\frac{\Gamma(\alpha-1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[{}_{H}J_{a+}^{\alpha}f(b)+{}_{H}J_{b-}^{\alpha}f(a)\right]\leq\frac{f(a)+f(b)}{2} \tag{1.6}$$

with $\alpha>0$.

We have obtained the left sides of the (1.4) and (1.6) inequalities using a functional that we have defined. We have also obtained the left side of the Hermite-Hadamard-Fejér inequality through Hadamard fractional integrals for GA-convex functions.

## 2. The Left Hand Sides of the Hermite-Hadamard and Hermite-Hadamard-Fejér Inequalities via Functionals

**Theorem 2.1** Let $f:[a,b]\subset(0,\infty)\rightarrow\mathbb{R}$ a be geometric arithmetically (GA)-convex function, $0<\alpha<1$, $0<\beta<1$

$$A=\ln\left(a^{\alpha}b^{1-\alpha}\right),$$

$$u_0=(\ln b-\ln a)\min\left\{\frac{\alpha}{1-\beta},\frac{1-\alpha}{\beta}\right\}$$

and let $h$ be defined by

$$h(t)=(1-\beta)f\left(a^{\alpha}b^{1-\alpha}e^{-\beta t}\right)+\beta f\left(a^{\alpha}b^{1-\alpha}e^{(1-\beta)t}\right),$$

for $t\in[0,u_0]$. Then h is convex, increasing on $[0,u_0]$ and for all $t\in[0,u_0]$,

$$f\left(a^{\alpha}b^{1-\alpha}\right)\leq h(t)\leq\alpha f(a)+(1-\alpha)f(b).$$

**Proof.** We note that if $f:[a,b]\subset(0,\infty)\rightarrow\mathbb{R}$ is GA-convex according to Theorem 1.7 and $g$ is linear, then the composition $g=(f o exp)$ is convex on $[lna,lnb]$. According to the Theorem 1.3, also we note that a positive constant multiple of a convex function and a sum of two convex functions are convex, hence function

$$\begin{aligned}h(t)&=(1-\beta)g(A-\beta t)+\beta g(A+(1-\beta)t)\\&=(1-\beta)f\left(e^{A}e^{-\beta t}\right)+\beta f\left(e^{A}e^{(1-\beta)t}\right)\\&=(1-\beta)f\left(a^{\alpha}b^{1-\beta}e^{-\beta t}\right)\\&\quad+\beta f\left(a^{\alpha}b^{1-\alpha}e^{(1-\beta)t}\right)\end{aligned}$$

is convex and increasing on $[0,u_0]$ and for all $t\in[0,u_0]$,

$$\begin{aligned}&g(\alpha\ln a+(1-\alpha)\ln b)\\&\leq h(t)\leq\alpha g(\ln a)+(1-\alpha)f(\ln b)\end{aligned}$$

that is,

$$f\left(a^{\alpha}b^{1-\alpha}\right)\leq h(t)\leq\alpha f(a)+(1-\alpha)f(b).$$

**Lemma 2.1** Let $f:[a,b]\subset(0,\infty)\rightarrow\mathbb{R}$ be a GA-convex function and $h$ be defined by

$$h(t)=\frac{1}{2}\left[f\left(\sqrt{ab}e^{-\frac{t}{2}}\right)+f\left(\sqrt{ab}e^{\frac{t}{2}}\right)\right].$$

Then $h(t)$ is convex, increasing on $\left[0,\ln\frac{b}{a}\right]$ and for all $t\in\left[0,\ln\frac{b}{a}\right]$,

$$f\left(\sqrt{ab}\right)\leq h(t)\leq\frac{f(a)+f(b)}{2}.$$

**Proof.** If $f:[a,b]\subset(0,\infty)\rightarrow\mathbb{R}$ is geometric-arithmetically (GA) convex, then according to the Theorem 1.7 the function $g=f o exp$ is convex on $[lna,lnb]$. Then the function $h$ defined by

$$\begin{aligned}h(t)&=\frac{1}{2}\left[\begin{array}{l}g\left(\left[\frac{\ln a+\ln b}{2}\right]-\frac{t}{2}\right)\\+g\left(\left[\frac{\ln a+\ln b}{2}\right]+\frac{t}{2}\right)\end{array}\right]\\&=\frac{1}{2}\left[f\left(\sqrt{ab}e^{-\frac{t}{2}}\right)+f\left(\sqrt{ab}e^{\frac{t}{2}}\right)\right]\end{aligned}$$

is convex and increasing on $\left[0,\ln\frac{b}{a}\right]$ and for $\forall t\in\left[0,\ln\frac{b}{a}\right]$

$$g\left(\frac{\ln a+\ln b}{2}\right)\leq h(t)\leq\frac{g(\ln a)+g(\ln b)}{2}.$$

If substituting $g=f o exp$ in the above inequality, we have

$$f\left(\sqrt{ab}\right)\leq h(t)\leq\frac{f(a)+f(b)}{2}.$$

**Theorem 2.2** Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a function and $f\in L[a,b]$. If the function $f$ is GA-convex on $[a,b]$, then $H_\alpha$ defined by

$$H_\alpha(t)=\frac{\alpha}{2\left(\ln\frac{b}{a}\right)^\alpha}\int_a^b f\left(x^tG^{1-t}\right)\times\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{1}{x}dx$$

is convex and monotonically increasing on [0,1] and

$$f\left(\sqrt{ab}\right)=H_\alpha(0)\le H_\alpha(t)\le H_\alpha(1)=\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^\alpha}\left[{}_HJ_{\alpha+}^\alpha f(b)+{}_HJ_{b+}^\alpha f(a)\right].$$

**Proof. i)** Firstly, let $t_1$, $t_2$, $\beta\in[0,1]$. We need to show that

$$H_\alpha\left((1-\beta)t_1+\beta t_2\right)\le(1-\beta)H_\alpha(t_1)+\beta H_\alpha(t_2).$$

Using the definition of $H_\alpha$, we can write the following

$$H_\alpha\left((1-\beta)t_1+\beta t_2\right)=\frac{\alpha}{2\left(\ln\frac{b}{a}\right)^\alpha}\int_a^b f\left(x^{(1-\beta)t_1+\beta t_2}G^{1-(1-\beta)t_1-\beta t_2}\right)\times\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{1}{x}dx.$$

Since the function $f$ is geometric-arithmetically convex, we get

$$\begin{aligned}&f\left(x^{(1-\beta)t_1+\beta t_2}G^{(1-\beta)(1-t_2)+\beta(1-t_2)}\right)\\&=f\left[\left(x^{t_1}G^{1-t_1}\right)^{1-\beta}\left(x^{t_2}G^{1-t_2}\right)^{\beta}\right]\\&\le(1-\beta)f\left(x^{t_1}G^{1-t_1}\right)+\beta f\left(x^{t_2}G^{1-t_2}\right).\end{aligned}$$

So

$$\begin{aligned}&H_\alpha\left((1-\beta)t_1+\beta t_2\right)\\&\le(1-\beta)\left[\frac{\alpha}{2\left(\ln\frac{b}{a}\right)^\alpha}\int_a^b f\left(x^{t_1}G^{1-t_1}\right)\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{1}{x}dx\right]\\&+\beta\left[\frac{\alpha}{2\left(\ln\frac{b}{a}\right)^\alpha}\int_a^b f\left(x^{t_2}G^{1-t_2}\right)\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{1}{x}dx\right]\\&=(1-\beta)H_\alpha(t_1)+\beta H_\alpha(t_2),\end{aligned}$$

from which we get $H_\alpha$ is convex on [0,1].

**ii)** By elementary calculus, we have

$$\begin{aligned}H_\alpha(t)&=\frac{\alpha}{2\left(\ln\frac{b}{a}\right)^\alpha}\int_a^b f\left(x^{t_1}G^{1-t_1}\right)\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{1}{x}dx\\&=\frac{\alpha}{2\left(\ln\frac{b}{a}\right)^\alpha}\left[\int_a^{\sqrt{ab}} f\left(x^tG^{1-t}\right)\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{1}{x}dx\right.\\&\left.+\int_{\sqrt{ab}}^b f\left(x^tG^{1-t}\right)\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{1}{x}dx\right]\\&=\frac{\alpha}{2\left(\ln\frac{b}{a}\right)^\alpha}\left[\int_0^{\ln b-\ln a} f\left(Ge^{-\frac{tu}{2}}\right)\left[\left(\frac{\ln\frac{b}{a}}{2}+\frac{u}{2}\right)^{\alpha-1}+\left(\frac{\ln\frac{b}{a}}{2}-\frac{u}{2}\right)^{\alpha-1}\right]\frac{du}{2}\right.\\&\left.+\int_0^{\ln b-\ln a} f\left(Ge^{\frac{tu}{2}}\right)\left[\left(\frac{\ln\frac{b}{a}}{2}-\frac{u}{2}\right)^{\alpha-1}+\left(\frac{\ln\frac{b}{a}}{2}+\frac{u}{2}\right)^{\alpha-1}\right]\frac{du}{2}\right]\\&=\frac{\alpha}{4\left(\ln\frac{b}{a}\right)^\alpha}\int_0^{\ln b-\ln a} f\left(Ge^{\frac{tx}{2}}+Ge^{-\frac{tx}{2}}\right)\times\left[\left(\frac{\ln\frac{b}{a}}{2}-\frac{x}{2}\right)^{\alpha-1}+\left(\frac{\ln\frac{b}{a}}{2}+\frac{x}{2}\right)^{\alpha-1}\right]dx.\end{aligned}$$

It follows from Lemma 2.1 that function $h(x)=\frac{1}{2}\left[f\left(\sqrt{ab}e^{-\frac{tx}{2}}\right)+f\left(\sqrt{ab}e^{\frac{tx}{2}}\right)\right]$ is monotonically increasing on $[0,\ln b-\ln a]$. Since $\left[\left(\frac{\ln\frac{b}{a}}{2}-\frac{x}{2}\right)^{\alpha-1}+\left(\frac{\ln\frac{b}{a}}{2}+\frac{x}{2}\right)^{\alpha-1}\right]$ is nonnegative, hence $H_\alpha(t)$ is increasing on [0,1]. Finally, from

$$f\left(\sqrt{ab}\right)=H_\alpha(0)$$

and

$$H_{\alpha}(1)=\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[{}_{H}J_{a+}^{\alpha}f(b)+{}_{H}J_{b-}^{\alpha}f(a)\right],$$

we have completed the proof.

**Corollary 2.1** With assumptions in Theorem 2.2, if $\alpha=1$, we get

$$f\left(\sqrt{ab}\right)=H_1(0)\le H_1(t)\le H_1(1)$$
$$=\frac{1}{\ln\frac{b}{a}}\int_a^b f(x)\frac{1}{x}dx$$

where the function $H_1$ is defined as Theorem 2.2, which is just the result in Theorem 2.2.

**Remark 2.1** The inequality obtained in Theorem 3.2 gives us the left side of the inequality obtained in Theorem 1.8.

The next theorem is a generalization of Theorem 2.2:

**Theorem 2.3** Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a function and $f\in L[a,b]$. If the function $f$ is GA-convex on $[a,b]$ and $g$ is integrable, nonnegative and symmetrically according to $\sqrt{ab}$, that is $g\left(\frac{ab}{x}\right)=g(x)$ for all $x\in[a,b]$, then ${}_{g}H_{\alpha}(t)$ defined by

$${}_{g}H_{\alpha}(t)=\frac{\alpha}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\int_a^b f\left(x^tG^{1-t}\right)\times\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{g(x)}{x}dx$$

is convex and monotonically increasing on [0,1], for $\alpha>0$

$$f\left(\sqrt{ab}\right)={}_{g}H_{\alpha}(0)\le {}_{g}H_{\alpha}(t)\le {}_{g}H_{\alpha}(1)$$
$$=\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[{}_{H}J_{a+}^{\alpha}(fg)(b)+{}_{H}J_{b-}^{\alpha}(fg)(a)\right].$$

**Proof. i)** Firstly, let $t_1, t_2, \beta\in[0,1]$. We need to show that

$${}_{g}H_{\alpha}\left((1-\beta)t_1+\beta t_2\right)\le(1-\beta)\,{}_{g}H_{\alpha}(t_1)+\beta\,{}_{g}H_{\alpha}(t_2).$$

Using the definition of ${}_{g}H_{\alpha}$, we can write the following

$${}_{g}H_{\alpha}\left((1-\beta)t_1+\beta t_2\right)$$
$$=\frac{\alpha}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\int_a^b f\left(x^{(1-\beta)t_1+\beta t_2}G^{1-(1-\beta)t_1-\beta t_2}\right)\times\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{g(x)}{x}dx.$$

So,

$${}_{g}H_{\alpha}\left((1-\beta)t_1+\beta t_2\right)$$
$$\le(1-\beta)\left[\frac{\alpha}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\int_a^b f\left(x^{t_1}G^{1-t_1}\right)\times\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{g(x)}{x}dx\right]$$
$$\le\beta\left[\frac{\alpha}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\int_a^b f\left(x^{t_2}G^{1-t_2}\right)\times\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{g(x)}{x}dx\right]$$
$$=(1-\beta)\,{}_{g}H_{\alpha}(t_1)+\beta\,{}_{g}H_{\alpha}(t_2)$$

from which we get ${}_{g}H_{\alpha}$ is convex on [0,1].

**ii)** By elementary calculus, we have

$$g^{H_{\alpha}(t)}=\frac{\alpha}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\int_a^b f\left(x^tG^{1-t}\right)\times\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{g(x)}{x}dx$$

$$=\frac{\alpha}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[\begin{array}{l}\int_a^{\sqrt{ab}} f\left(x^tG^{1-t}\right)\times\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{g(x)}{x}dx\\ +\int_{\sqrt{ab}}^{b} f\left(x^tG^{1-t}\right)\times\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{g(x)}{x}dx\end{array}\right]$$

$$=\frac{\alpha}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[\begin{array}{l}\int_0^{\ln b-\ln a} f\left(\left(\frac{G}{e^{\frac{u}{2}}}\right)^{t}G^{1-t}\right)\times\left[\left(\ln\frac{be^{\frac{u}{2}}}{G}\right)^{\alpha-1}+\left(\ln\frac{Ge^{\frac{u}{2}}}{a}\right)^{\alpha-1}\right]\frac{g\left(Ge^{\frac{u}{2}}\right)}{2}du\\ +\int_0^{\ln b-\ln a} f\left(\left(Ge^{\frac{u}{2}}\right)^{t}G^{1-t}\right)\times\left[\left(\ln\frac{b}{Ge^{\frac{u}{2}}}\right)^{\alpha-1}+\left(\ln\frac{Ge^{\frac{u}{2}}}{a}\right)^{\alpha-1}\right]\frac{g\left(Ge^{\frac{u}{2}}\right)}{2}du\end{array}\right]$$

$$= \frac{\alpha}{2\left(\ln\frac{b}{a}\right)^{\alpha}} \left[ \begin{array}{l} \int_0^{\ln b-\ln a} f\left(Ge^{-\frac{tu}{2}}\right) \\ \times \left[\left(\frac{\ln\frac{b}{a}}{2}+\frac{u}{2}\right)^{\alpha-1} + \left(\frac{\ln\frac{b}{a}}{2}-\frac{u}{2}\right)^{\alpha-1}\right] \frac{g\left(Ge^{-\frac{u}{2}}\right)}{2} du \\ + \int_0^{\ln b-\ln a} f\left(Ge^{\frac{tu}{2}}\right) \\ \times \left[\left(\frac{\ln\frac{b}{a}}{2}-\frac{u}{2}\right)^{\alpha-1} + \left(\frac{\ln\frac{b}{a}}{2}+\frac{u}{2}\right)^{\alpha-1}\right] \frac{g\left(Ge^{\frac{u}{2}}\right)}{2} du \end{array}\right].$$

Since the function $g$ is simmetrically according to the $\sqrt{ab}$

$$g\left(Ge^{-\frac{u}{2}}\right) = g\left(\frac{ab}{Ge^{-\frac{u}{2}}}\right) = g\left(Ge^{\frac{u}{2}}\right).$$

So,

$$\begin{aligned} {}_gH_a(t) &= \frac{\alpha}{2\left(\ln\frac{b}{a}\right)^{\alpha}} \frac{1}{2} \int_0^{\ln b-\ln a} \left[ f\left(Ge^{\frac{tx}{2}}\right) + f\left(Ge^{-\frac{tx}{2}}\right)\right] \\ &\times g\left(Ge^{\frac{x}{2}}\right) \left[\left(\frac{\ln\frac{b}{a}}{2}-\frac{x}{2}\right)^{\alpha-1} + \left(\frac{\ln\frac{b}{a}}{2}+\frac{x}{2}\right)^{\alpha-1}\right] dx. \end{aligned}$$

It follows from Lemma 2.1 that

$$h(x) = \frac{1}{2}\left[ f\sqrt{ab}e^{-\frac{tx}{2}} + f\sqrt{ab}e^{\frac{tx}{2}}\right]$$

is monotonically increasing on $[0, lnb-lna]$. Since

$$g\left(Ge^{\frac{x}{2}}\right)\left[\left(\frac{\ln\frac{b}{a}}{2}+\frac{x}{2}\right)^{\alpha-1} + \left(\frac{\ln\frac{b}{a}}{2}-\frac{x}{2}\right)^{\alpha-1}\right]$$

is nonnegative, hence ${}_gH_{\alpha}(t)$ is increasing on [0,1]. Finally, from

$$\begin{aligned} f\left(\sqrt{ab}\right) &= {}_gH_{\alpha}(0) \le {}_gH_{\alpha}(t) \le {}_gH_{\alpha}(1) \\ &= \frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}} \left[ {}_HJ_{a+}^{\alpha}(fg)(b) + {}_HJ_{b-}^{\alpha}(fg)(a)\right]. \end{aligned}$$

Here, by an easy calculation we get

$$\begin{aligned} {}_gH_{\alpha}(0) &= f(G)\frac{\alpha}{2\left(\ln\frac{b}{a}\right)^{\alpha}} \int_a^b \left[\left(\ln\frac{b}{x}\right)^{\alpha-1} + \left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{g(x)}{x}dx \\ &= \frac{\alpha}{2\left(\ln\frac{b}{a}\right)^{\alpha}} \left[ {}_HJ_{a+}^{\alpha}g(b) + {}_HJ_{b-}^{\alpha}f(a)\right] \end{aligned}$$

and

$$\begin{aligned} &{}_gH_{\alpha}(1) \\ &= \frac{\alpha}{2\left(\ln\frac{b}{a}\right)^{\alpha}} \left[ \begin{array}{l} \int_a^b \frac{f(x)g(x)}{2}\left(\ln\frac{b}{x}\right)^{\alpha-1}dx \\ +\int_a^b \frac{f(x)g(x)}{2}\left(\ln\frac{x}{a}\right)^{\alpha-1}dx \end{array}\right] \\ &= \frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}} \left[ {}_HJ_{a+}^{\alpha}(fg)(b) + {}_HJ_{b-}^{\alpha}(fg)(a)\right]. \end{aligned}$$

This completes the proof of Theorem.

**Remark 2.2.** If $g(x)=1$ in Theorem 2.3, then the following equality holds:

$$\begin{aligned} H_{\alpha}(t) &= \frac{\alpha}{2\left(\ln\frac{b}{a}\right)^{\alpha}} \int_a^b f\left(x^tG^{1-t}\right) \\ &\times \left[\left(\ln\frac{b}{x}\right)^{\alpha-1} + \left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{1}{x}dx. \end{aligned}$$

## 3. Conclusion

In this paper, we obtain some new Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for geometric arithmetically convex functions via Hadamard fractional integrals. We conclude that the results obtained in this work are the refinements of the earlier results. An interesting topic is whether we can use the methods in this paper to establish the left side hand of Hermite-Hadamard and Hermite-Hadamard-Fejér inequalities for geometric arithmetically convex functions via Hadamard fractional integrals.

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