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Some Inequalities of the Hermite-Hadamard Type for Harmonically Quasi-Convex Functions

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Abstract In the paper, by Holder's integral inequality, the authors establish some Hermite-Hadamard type integral inequalities for harmonically quasi-convex functions.

Keywords: *Hermite-Hadamard inequality, harmonically quasi-convex function, Holder's integral inequality*

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1. Introduction

The following definitions for various convex functions are well known in the literature.

Definition 1.1 A function $f: I \subseteq R \to R$ is said to be convex if

$$f(\lambda x+(1-\lambda)y) \le \lambda f(x)+(1-\lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.2 ([1,2,3]) A function $f: I \subseteq R \to R$ is said to be quasi-convex if

$$f(\lambda x+(1-\lambda)y) \le \max\{f(x), f(y)\}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.3 ([4]) For $f:[0,b] \to R$ with $b>0$ and $m \in (0,1]$, if

$$f(\lambda x+m(1-\lambda)y) \le \lambda f(x)+m(1-\lambda)f(y)$$

is valid for all $x, y \in [0,b]$ and $\lambda \in [0,1]$, then we say that $f(x)$ is an m-convex function on $[0,b]$.

Definition 1.4 ([5]) Let $f:[0,b] \to R$ with $b>0$ and $(\alpha, m) \in (0,1]^2$. If

$$f(\lambda x+m(1-\lambda)y) \le \lambda^{\alpha} f(x)+m(1-\lambda^{\alpha})f(y)$$

is valid for all $x, y \in [0,b]$ and $\lambda \in [0,1]$, then we say that $f(x)$ is an (α, m)-convex function on $[0,b]$.

Definition 1.5 ([9]) A function $f: I \subseteq R_+ = (0,\infty) \to R$ is said to be a harmonically quasi-convex function on I if

$$f\left(\left(\frac{t}{x}+\frac{1-t}{y}\right)^{-1}\right) \le \max\{f(x), f(y)\}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

In [7,8], the following inequalities of Hermite-Hadamard type were established.

Theorem 1.1 ([7], Theorems 2.2 and 2.3]). *Let $f: I^{\circ} \subseteq R \to R$ be a differentiable mapping and $a, b \in I^{\circ}$* with $a<b$. Then

(i) if $|f'|$ is convex on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8};$$

(ii) if $|f'|^{p/(p-1)}$ is convex on $[a,b]$ for $p>1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{2(p+1)^{1/p}}\left(\frac{|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}}{2}\right)^{(p-1)/p}.$$

Theorem 1.2 ([8], Theorem 2.3]) Let $f: I \subseteq R \to R$ *be differentiable on* I° and $a, b \in I$ with $a<b$. If $|f'|^p$ *is s-convex on* $[a,b]$ for $p>1$, then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{4}\left(\frac{4}{p+1}\right)^{1/p}(|f'(a)|+|f'(b)|).$$

In this paper, we will create some integral inequalities of Hermite-Hadamard type for harmonically quasi-convex functions.

2. A Lemma

For creating some integral inequalities of Hermite-Hadamard type for harmonically quasi-convex functions, we need the following lemma.

Lemma 2.1 Let $f : I \subseteq R_{+} \to R$ be differentiable on I, $a,b \in I$ with $a<b$, and $\theta \in (0,1]$. If $f' \in L_1([a,b])$, then

$$\frac{H(\theta,a,b)-a}{2aH(\theta,a,b)}\int_0^1 \left[\begin{array}{l} (1-2t)\left(\frac{t}{a}+\frac{1-t}{H(\theta,a,b)}\right)^{-2} \\ f'\left(\left(\frac{t}{a}+\frac{1-t}{H(\theta,a,b)}\right)^{-1}\right) \end{array}\right] dt$$
$$=\frac{f(a)+f\left(H(\theta,a,b)\right)}{2} - \frac{aH(\theta,a,b)}{H(\theta,a,b)-a}\int_a^{H(\theta,a,b)}\frac{f(x)}{x^2}dx,$$

where $H(\theta,a,b)=\frac{ab}{\theta a+(1-\theta)b}$.

Proof Integrating by part and changing variables $x=\left(ta^{-1}+(1-t)[H(\theta,a,b)]^{-1}\right)^{-1}$ for $t \in [0,1]$ yield

$$\int_0^1 (1-2t)\left(\frac{t}{a}+\frac{1-t}{H(\theta,a,b)}\right)^{-2} f'\left(\left(\frac{t}{a}+\frac{1-t}{H(\theta,a,b)}\right)^{-1}\right)dt$$
$$=\frac{aH(\theta,a,b)}{H(\theta,a,b)-a}\left(f(a)+f\left(H(\theta,a,b)\right)\right) - \frac{2[aH(\theta,a,b)]^2}{[H(\theta,a,b)-a]^2} f'\left(\left(\frac{t}{a}+\frac{1-t}{H(\theta,a,b)}\right)^{-1}\right)dt$$
$$=\frac{aH(\theta,a,b)}{H(\theta,a,b)-a}\left(f(a)+f\left(H(\theta,a,b)\right)\right) - \frac{2[aH(\theta,a,b)]^2}{[H(\theta,a,b)-a]^2}\int_a^{H(\theta,a,b)}\frac{f(x)}{x^2}dx.$$

Lemma 2.1 is thus proved.

3. Some New Integral Inequalities of Hermite-Hadamard Type

Now we set off to create some integral inequalities of Hermite-Hadamard type for harmonically quasi-convex functions.

Theorem 3.1 Let $f : I \subseteq R_{+} \to R$ be a differentiable function, $a,b \in I$ with $a<b$, $\theta \in (0,1]$, and $|f'| \in L_1([a,b])$. If $|f'|$ is harmonically quasi-convex on $[a,b]$, then

$$\left| \begin{array}{l} \frac{f(a)+f\left(H(\theta,a,b)\right)}{2} \\ -\frac{aH(\theta,a,b)}{H(\theta,a,b)-a}\int_a^{H(\theta,a,b)}\frac{f(x)}{x^2}dx \end{array}\right|$$
$$\le \frac{\left\{ \begin{array}{l} [H(\theta,a,b)-a]^2 \\ +2aH(\theta,a,b)\left(\begin{array}{l} 2\ln H\left(\frac{1}{2},a,H(\theta,a,b)\right) \\ -\ln[aH(a,b)] \end{array}\right) \end{array}\right\}}{2[H(\theta,a,b)-a]} \times \max\left\{|f'(a)|,\left|f'\left(H(\theta,a,b)\right)\right|\right\}.$$

Proof Using Lemma 2.1 and the harmonic quasi-convexity of $|f'|$, we have

$$\left| \begin{array}{l} \frac{f(a)+f\left(H(\theta,a,b)\right)}{2} \\ -\frac{aH(\theta,a,b)}{H(\theta,a,b)-a}\int_a^{H(\theta,a,b)}\frac{f(x)}{x^2}dx \end{array}\right|$$
$$\le \frac{H(\theta,a,b)-a}{2aH(\theta,a,b)}\int_0^1 \left[\begin{array}{l} |1-2t|\left(\frac{t}{a}+\frac{1-t}{H(\theta,a,b)}\right)^{-2} \\ \left|f'\left(\left(\frac{t}{a}+\frac{1-t}{H(\theta,a,b)}\right)^{-1}\right)\right| \end{array}\right] dt$$
$$\le \frac{H(\theta,a,b)-a}{2aH(\theta,a,b)}\max\left\{|f'(a)|,\left|f'\left(H(\theta,a,b)\right)\right|\right\} \times \int_0^1 |1-2t|\left(\frac{t}{a}+\frac{1-t}{H(\theta,a,b)}\right)^{-2}dt$$
$$= \frac{\left\{ \begin{array}{l} [H(\theta,a,b)-a]^2 \\ +2aH(\theta,a,b)\left(\begin{array}{l} 2\ln H\left(\frac{1}{2},a,H(\theta,a,b)\right) \\ -\ln[aH(a,b)] \end{array}\right) \end{array}\right\}}{2[H(\theta,a,b)-a]} \times \max\left\{|f'(a)|,\left|f'\left(H(\theta,a,b)\right)\right|\right\}.$$

The proof of Theorem 3.1 is complete.

Theorem 3.2 Let $f : I \subseteq R_{+} \to R$ be a differentiable function, $a,b \in I$ with $a<b$, $\theta \in (0,1]$, and $|f'| \in L_1((0,b])$. If $|f'|^q$ is harmonically quasi-convex on $[a,b]$ and $q>1$, then

$$\left| \begin{array}{l} \frac{f(a)+f\left(H(\theta,a,b)\right)}{2} \\ -\frac{aH(\theta,a,b)}{H(\theta,a,b)-a}\int_a^{H(\theta,a,b)}\frac{f(x)}{x^2}dx \end{array}\right|$$
$$\le \frac{\left[\begin{array}{l} \left((q-1)[H(\theta,a,b)-a]\right)^{1-1/q} \\ \times\left([H(\theta,a,b)]^{2q-1}-a^{2q-1}\right)^{1/q} \end{array}\right]}{2[(q-1)aH(\theta,a,b)]^{1-1/q}} \max\left\{|f'(a)|,\left|f'\left(H(\theta,a,b)\right)\right|\right\}.$$

Proof Since $\left|f'\right|^q$ is harmonically quasi-convex on $[a,b]$, by Lemma 2.1 and Hölder's inequality, we have

$$\left|\frac{f(a)+f\left(H(\theta,a,b)\right)}{2}-\frac{aH(\theta,a,b)}{H(\theta,a,b)-a}\int_a^{H(\theta,a,b)}\frac{f(x)}{x^2}\mathrm{d}x\right|$$

$$\leq\frac{H(\theta,a,b)-a}{2aH(\theta,a,b)}\left(\int_0^1\left(\frac{t}{a}+\frac{1-t}{H(\theta,a,b)}\right)^{-2q}\left|f'\left(\left(\frac{t}{a}+\frac{1-t}{H(\theta,a,b)}\right)^{-1}\right)\right|^q\mathrm{d}t\right)^{1/q}$$

$$\times\left(\int_0^1|1-2t|^{q/(q-1)}\mathrm{d}t\right)^{1-1/q}$$

$$\leq\frac{H(\theta,a,b)-a}{2aH(\theta,a,b)}\left(\int_0^1\left(\frac{t}{a}+\frac{1-t}{H(\theta,a,b)}\right)^{-2q}\mathrm{d}t\right)^{1/q}$$

$$\times\left(\int_0^1|1-2t|^{q/(q-1)}\mathrm{d}t\right)^{1-1/q}$$

$$\times\max\left\{\left|f'(a)\right|,\left|f'\left(H(\theta,a,b)\right)\right|\right\}$$

$$=\frac{\left[\begin{array}{l}\left((q-1)[H(\theta,a,b)-a]\right)^{1-1/q}\\ \times\left([H(\theta,a,b)]^{2q-1}-a^{2q-1}\right)^{1/q}\end{array}\right]}{2[(2q-1)aH(\theta,a,b)]^{1-1/q}}$$

$$\times\max\left\{\left|f'(a)\right|,\left|f'\left(H(\theta,a,b)\right)\right|\right\}.$$

Theorem 3.2 is thus proved.

Theorem 3.3. Let $f:I\subseteq R_+\to R$ be a differentiable function, $a,b\in I$ with $a<b$, $\theta\in(0,1]$, and $|f'|\in L_1((0,b])$. If $\left|f'\right|^q$ is harmonically quasi-convex on $[a,b]$ for $q>1$, then

$$\left|\frac{f(a)+f\left(H(\theta,a,b)\right)}{2}-\frac{aH(\theta,a,b)}{H(\theta,a,b)-a}\int_a^{H(\theta,a,b)}\frac{f(x)}{x^2}\mathrm{d}x\right|$$

$$\leq\frac{\left\{\begin{array}{l}[H(\theta,a,b)-a]\\ \left(a^{2q/(q-1)}+[H(\theta,a,b)]^{2q/(q-1)}\right)^{1-1/q}\end{array}\right\}}{2^{(3q-1)/q}aH(\theta,a,b)}$$

$$\max\left\{\left|f'(a)\right|,\left|f'\left(H(\theta,a,b)\right)\right|\right\}.$$

Proof From the GA-inequality, we have

$$\left(\frac{t}{a}+\frac{1-t}{H(\theta,a,b)}\right)^{-2q/(q-1)}$$

$$\leq ta^{2q/(q-1)}+(1-t)[H(\theta,a,b)]^{2q/(q-1)}$$

for all $t\in[0,1]$. By Lemma 2.1 and the harmonic quasi-convexity of $\left|f'\right|^q$ and Hölder's inequality, we have

$$\left|\frac{f(a)+f\left(H(\theta,a,b)\right)}{2}-\frac{aH(\theta,a,b)}{H(\theta,a,b)-a}\int_a^{H(\theta,a,b)}\frac{f(x)}{x^2}\mathrm{d}x\right|$$

$$\leq\frac{H(\theta,a,b)-a}{2aH(\theta,a,b)}$$

$$\times\left(\int_0^1|1-2t|\left|f'\left(\left(\frac{t}{a}+\frac{1-t}{H(\theta,a,b)}\right)^{-1}\right)\right|^q\mathrm{d}t\right)^{1/q}$$

$$\times\left(\int_0^1|1-2t|\left(\frac{t}{a}+\frac{1-t}{H(\theta,a,b)}\right)^{-2q/(q-1)}\mathrm{d}t\right)^{1-1/q}$$

$$\leq\frac{H(\theta,a,b)-a}{2aH(\theta,a,b)}\left(\int_0^1|1-2t|\max\left\{\begin{array}{l}\left|f'(a)\right|^q,\\ \left|f'\left(H(\theta,a,b)\right)\right|^q\end{array}\right\}\mathrm{d}t\right)^{1/q}$$

$$\times\left(\int_0^1|1-2t|\left(\frac{t}{a}+\frac{1-t}{H(\theta,a,b)}\right)^{-2q/(q-1)}\mathrm{d}t\right)^{1-1/q}$$

$$=\frac{\left\{\begin{array}{l}[H(\theta,a,b)-a]\\ \times\left(a^{2q/(q-1)}+[H(\theta,a,b)]^{2q/(q-1)}\right)^{1-1/q}\end{array}\right\}}{2^{(3q-1)/q}aH(\theta,a,b)}$$

$$\max\left\{\left|f'(a)\right|,\left|f'\left(H(\theta,a,b)\right)\right|\right\}.$$

The proof of Theorem 3.3 is complete.

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