Turkish Journal of Analysis and Number Theory, 2017, Vol. 5, No. 6, 210-225
Available online at http://pubs.sciepub.com/tjant/5/6/3
©Science and Education Publishing
DOI:10.12691/tjant-5-6-3

Enumeration of 2-Wilf Classes of Four 4-letter Patterns

David Callan[1], Toufik Mansour[2,*]

[1]Department of Statistics, University of Wisconsin, Madison, WI
[2]Department of Mathematics, University of Haifa, Haifa, Israel
*Corresponding author: tmansour@univ.haifa.ac.il

Received September 13, 2017; Revised October 25, 2017; Accepted November 03, 2017

Abstract Let S_n be the symmetric group of all permutations of n letters. We show that there are precisely 64 Wilf classes consisting of exactly 2 symmetry classes of subsets of four 4-letter patterns.

Keywords: *pattern avoidance, Wilf-equivalence*

Cite This Article: David Callan, and Toufik Mansour, "Enumeration of 2-Wilf Classes of Four 4-letter Patterns." *Turkish Journal of Analysis and Number Theory*, vol. 5, no. 6 (2017): 210-225. doi: 10.12691/tjant-5-6-3.

1. Introduction

Let S_n be the set of all permutations of $[n]=\{1, 2, ..., n\}$. We say a permutation is *standard* if its support set is an initial segment of the positive integers, and for a permutation π whose support is any set of positive integers, $St(\pi)$ is the standard permutation obtained by replacing the smallest entry of π by 1, the next smallest entry by 2, and so on. For example, $St(2374)=1243$. A permutation π *avoids* a sequence τ if there is no subsequence ρ of π for which $St(\rho)=St(\tau)$. In this context, $St(\tau)$ is called a *pattern*. For a set T of patterns, we denote the set of permutations of S_n that avoid all the patterns in T by $S_n(T)$. For instance, Knuth [4] showed that

$$|S_n(\tau)|=C_n=\frac{1}{n+1}\binom{2n}{n},\ \tau\in\mathcal{S}_3,$$

where C_n denotes the nth *Catalan number*. The generating function for the Catalan numbers is given by $C(x)=\sum_{n\geq0}C_nx^n=\frac{1-\sqrt{1-4x}}{2x}$ and satisfies $C(x)=1+xC^2(x)$.

Two sets of patterns T and T' belong to the same *symmetry class* if and only if T' can be obtained by the action of the dihedral group of order eight – generated by the operations inverse, reverse and complement – on T. The sets of patterns T and R belong to the same *Wilf class* (or are *Wilf-equivalent*) if and only if $|S_n(T)|=|S_n(R)|$ for all $n\geq0$. Let wk be the number of distinct Wilf classes of subsets of exactly k patterns in S_4. Le [6] established that $w_2=38$. Recently, Callan, Mansour and Shattuck [2,3] studied the distinct Wilf classes of subsets of exactly three 4-letter patterns and showed that $w_3=242$. Mansour and Schork [7,8,9] showed that

$$w_6=8438,\ w_7=15392,\ w_8=19002,\ w_9=16393,$$
$$w_{10}=10624,\ w_{11}=5857,\ w_{12}=3044,\ w_{13}=1546,$$
$$w_{14}=786,\ w_{15}=393,\ w_{16}=198,\ w_{17}=105,\ w_{18}=55,$$
$$w_{19}=28,\ w_{20}=14,\ w_{21}=8,\ w_{22}=4,\ w_{23}=2,\ w_{24}=1.$$

Toward the goal of finding w_4, we establish the following result.

Theorem 1. *The number of Wilf classes consisting of exactly* 2 symmetry classes of subsets of 4 *patterns in* S_4 (2-*Wilf classes*) *is* 64.

2. Proof of Theorem 1

Using software of Kuszmaul [5] we determined all symmetry classes of 4 patterns in S_4 and the sequence $(|S_n(T)|)_{n=1,2,...,16}$ for a representative quadruple T in each symmetry class. Considering the pairs T, R with $(|S_n(T)|)_{n=1,2,...,16}=(|S_n(R)|)_{n=1,2,...,16}$, shows that there at most 64 2-Wilf classes of subsets of 4 patterns in S_4, see Table 1 in the appendix below. Then we used the insertion encoding algorithm (INSENC) [12] on the symmetry classes in Table 1 and successful outcomes, always a rational generating function, are referenced by "INSENC".

So, to prove Theorem 1, in the following subsections we find an explicit formula for the generating function $F_T(x)=\sum_{n\geq0}|S_n(T)|x^n$ whenever $F_T(x)$ is nonrational. The 15 cases where $F_T(x)$ is rational and INSENC fails are marked "EX" in Table 1; their proofs are omitted and left to the interested reader. The following notation and definitions will be useful. A permutation π expressed as $\pi = i_1\pi^{(1)}i_2\pi^{(2)}...i_m\pi^{(m)}$ where $i_1 < i_2 < ... < i_m$ and $j_m > \max\left(\pi^{(j)}\right)$ for $1 \le j \le m$ is said to have m *left-right maxima* (at $i_1, i_2, ..., i_m$). Given nonempty sets of numbers *S* and *T*, we will write $S < T$ to mean $\max(S) < \max(T)$ (with the inequality vacuously holding if S or T is empty). In this context, we will often denote singleton sets simply by the element in question. Also, for a number k, $S - k$ means the set $\{s - k : s \in S\}$.

2.1. Case 227

2.1.1. The Symmetry Class of {1342, 2314, 3412, 4231}

Note that all three patterns contain 231.

Lemma 2. *Let* T = {1342, 2314, 3412, 4231}. *The generating function for* T *-avoiders with* 2 *leftright maxima is given by*

$$G_2(x) = x^2C(x)^2 + \frac{x^3}{(1-x)^3}C(x)^2.$$

Proof. Let $H_d(x)$ be the generating function for *T*-avoiders $i\pi'n\pi''$ with 2 left-right maxima where π'' has *d* letters smaller than *i*. If d = 0, then π' and π'' independently avoid 231, and so $H_0(x) = x^2C(x)^2$. Now let $d \ge 1$ and $j_1, j_2, ..., j_d$ be the letters in π'' smaller than *i*. These letters occur in decreasing order (to avoid 3412). Since π avoids 1342, 4231, we can write π as

$$\pi = i\alpha_0\alpha_1 n\beta_0\beta_1...j_d\beta_1,$$

where $i > \alpha_0 > j_1 > ... > j_d > \alpha_1$. Since π avoids 2314, we also have $\beta_0 > \beta_1$. Note that α_0, β_0 are decreasing, and α_1, β_1 both avoid 231. Thus $H_d(x) = \frac{x^{d+2}}{(1-x)^2}C(x)^2$.

Summing over $d \ge 1$ and using the expression for $H_0(x)$, we obtain the required result.

Theorem 3. *Let* T = {1342, 2314, 3412, 4231}. *Then*

$$F_T(x) = C(x) + \frac{x^3}{(1-x)^3}C(x)^2 + \frac{x^4}{(1-x)^3(1-2x)}.$$

Proof. Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$, $G_1(x) = xC(x)$, and Lemma 2 gives $G_2(x)$. For $G_m(x)$ with $m \ge 3$, suppose $\pi = i_1\pi^{(1)}i_2\pi^{(2)}...i_m\pi^{(m)} \in S_n(T)$ has $m \ge 3$ left-right maxima. Since π avoids 1342 and 2314, we see that $\pi^{(s)} > i_{s-1}$ for all $s = 2, 3, ..., m-1$, and π^m can be written as $\alpha\beta$ with $\varepsilon > i_{m-1}$ and $\pi^{(1)} > \beta$ (to avoid 1342), and β is decreasing (to avoid 3412).

If $\beta = \emptyset$, then π avoids T if and only if each of $\pi^{(1)}, ..., \pi^{(m-1)}$, α avoids 231. If $\beta = \emptyset$, then $\pi^{(1)}, ..., \pi^{(m-1)}$, α are all decreasing. Thus, $G_m(x) = x^mC(x) + \frac{x^{m+1}}{(1-x)^{m+1}}$. Summing over $m \ge 3$, we obtain

$$F_T(x) = 1 - xC(x) - G_2(x) = x^3C(x)^4 + \frac{x^4}{(1-x)^3(1-2x)}.$$

Substituting for $G_2(x)$ and solving for $F_T(x)$, we complete the proof.

2.1.2. The Symmetry Class of {1342, 2314, 3412, 2431}

Again, all three patterns contain 231.

Lemma 4. *Let* T = {1342, 2314, 3412, 2431}. *The generating function for* T *-avoiders with* 2 *leftright maxima is given by*

$$G_2(x) = x^2C(x)^2 + \frac{x^3}{(1-x)^3}C(x)^2 + \frac{x^3}{1-x}\left(\frac{1-x}{1-2x} - \frac{1}{1-x}\right).$$

Proof. As in Lemma 2, let $G_2(x)$ be the generating function for *T*-avoiders $i\pi'n\pi''$ with 2 left-right maxima. Let *d* be the number letters smaller than *i* in π''. If d = 0, then π' and π'' independently avoid 231, and so the contribution is $x^2C(x)^2$. Now let $d \ge 1$ and $j_1, j_2, ..., j_d$ be the letters in π'' smaller than *i*. These letters occur in decreasing order (to avoid 3412). Since π avoids 1342, 2431, we can write π as

$$\pi = i\alpha_0\alpha_1 n\beta_0 j_1...j_d\beta_1,$$

where $i > \alpha_0 > j_1 > ... > j_d > \alpha_1$. Since π avoids 2314, we also have $\beta_0 > \beta_1$. Also, α_1, β_1 both avoid 231. If α_0 is decreasing, the contribution is $x^3C(x)^2/(1-x)^2$; otherwise, $\beta_1 = \alpha_1 = \emptyset$. which implies a contribution of $\frac{x^3}{1-x}\left(\frac{1-x}{1-2x} - \frac{1}{1-x}\right)$. Add all the contributions to complete the proof.

Theorem 5. *Let* T = {1342, 2314, 3412, 2431}. *Then*

$$F_T(x) = \left(1 + \frac{x^2}{(1-x)^2}\right)C(x) + \frac{x^2\left(x^2 + 2x - 1\right)}{(1-x)^3(1-2x)}.$$

Proof. Let $G_m(x)$ be the generating function for T -avoiders with m left-right maxima. Clearly, $G_0(x) = 1$, $G_1(x) = xF_T(x)$, and Lemma 4 gives $G_2(x)$. For $G_m(x)$ with $m \ge 3$, suppose $\pi = i_1\pi^{(1)}i_2\pi^{(2)}...i_m\pi^{(m)} \in S_n(T)$ has $m \ge 3$ left-right maxima. Since π avoids 1342, 2314, 3412, we see that $\pi^{(s)} > i_{s-1}$ for all $s = 2, 3, ..., m-1$, and $\pi^{(m)}$ can be written as $\alpha\beta$ with $\alpha > i_{m-1}$ and $\pi^{(1)} > \beta$ (to avoid 1342), and β is decreasing (to avoid 3412).

If $\beta = \emptyset$, then π avoids T if and only if each of $\pi^{(1)},...,\pi^{(m-1)}$, α avoids 231. If $\beta \neq \emptyset$, then $\alpha = \pi^{(2)} = ... = \pi^{(m-1)} \neq \emptyset$, and $\pi^{(1)}$ avoids both 132 and 231. Thus, by [4,11], we have

$$G_m(x) = x^m C(x)^m + \frac{x^{m+1}}{1-2x}.$$

Summing over $m \geq 3$, we obtain

$$F_T(x) - 1 - xF_T(x) - G_2(x) = x^3 C(x)^4 + \frac{x^4}{(1-x)(1-2x)}.$$

Substituting for $G_2(x)$ and solving for $F_T(x)$, we complete the proof.

2.2. Case 261

2.2.1. The Symmetry Class of {1342, 2314, 2413, 4231}

Note that all three patterns contain 231.

Theorem 6. *Let* T = {1342, 2314, 2413, 4231}. *Then*

$$F_T(x) = C(x) + \frac{x^3}{(1-x)^2} C^3(x) + \frac{x^4}{(1-x)^2(1-2x)} C(x)^2.$$

Proof. Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xC(x)$.

To find $G_2(x)$ suppose $\pi = i\pi' n\pi''$ has 2 left-right maxima. If $\pi'' > i$ then we have a contribution of $x^2 C(x)^2$. Otherwise, since π avoids T, we have that

$$\pi = i(i-1)...(i'+1)\alpha n(n-1)...(i+1)\beta i'\gamma,$$

where $\alpha < \beta < \gamma < i' < i$, i' is the maximal letter of π'' that smaller than i, and each of α, β, γ avoids 231. So we have a contribution of $x^3 C(x)^3 / (1-x)^2$. Hence,

$$G_2(x) = x^2 C(x)^2 + \frac{x^3}{(1-x)^2} C(x)^3.$$

For $G_m(x)$ with $m \geq 3$, suppose $\pi = i_1\pi^{(1)} i_2\pi^{(2)} ... i_m\pi^{(m)} \in S_n(T)$ has $m \geq 3$ left-right maxima. Since π avoids 1342, 2314, 2413, we see that $\pi^{(s)} > i_{s-1}$ for all $s = 2,3,...,m-1$, and $\pi^{(m)}$ can be written as $\alpha\beta$ with $\alpha > i_{m-1}$ and $\pi^{(1)} > \beta$ (to avoid 1342). If $\beta = \emptyset$, then π avoids T if and only if each of $\pi^{(1)},...,\pi^{(m-1)}$, α avoids 231. If $\beta \neq \emptyset$, then each of α, $\pi^{(1)},...,\pi^{(m-1)}$ is decreasing and β avoids 231. Thus,

$$G_m(x) = x^m C(x)^m + \frac{x^m}{(1-x)^m}(C(x) - 1).$$

Summing over $m \geq 3$, we obtain

$$F_T(x) - 1 - xC(x) - G_2(x) = x^3 C(x)^3 / (1 - xC(x)) + \frac{x^3}{(1-x)^2(1-2x)}(C(x) - 1).$$

Substitute for $G_2(x)$ and solve for $F_T(x)$ to complete the proof.

2.2.2. The Symmetry Class of {1324, 1342, 2143, 2431}

Note that all three patterns contain 132.

Theorem 7. *Let* T = {1324, 1342, 2143, 2431}. *Then*

$$F_T(x) = C(x) + \frac{x^3}{(1-x)^2} C(x)^3 + \frac{x^4}{(1-x)^2(1-2x)} C(x)^2.$$

Proof. Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$.

To find $G_2(x)$, suppose $\pi = i\pi' n\pi''$ has 2 left-right maxima. If $\pi' \neq \emptyset$ then we have a contribution of $x(F_T(x) - 1 - xF_T(X))$. Otherwise, by considering whether $i = 1$ or $i > 1$, we get contributions of $\frac{x^2(1-x)}{1-2x}$ or $\frac{x^3}{1-x}(C(x)-1) + x^2(F_T(x) - 1)$. Hence,

$$G_2(x) = x(F_T(x) - 1 - xF_T(x)) + \frac{x^3}{1-x}(C(x) - 1) + \frac{x^2(1-x)}{1-2x} + x^2(F_T(x) - 1).$$

For $G_m(x)$ with $m \geq 3$, suppose $\pi = i_1\pi^{(1)} i_2\pi^{(2)} ... i_m\pi^{(m)} \in S_n(T)$ has $m \geq 3$ left-right maxima. Since π avoids 1324, 1342, we see that $\pi^{(s)} < i_1$ for all $s = 2,3,...,m-1$, and $\pi^{(m)}$ can be written as $\alpha\beta$ with $\alpha >< i_1$ and $\beta > i_{m-1}$ (to avoid 2431). Note that $\pi^{(1)} > \pi^{(2)} ... \pi^{(m-1)}\alpha$ (to avoid 1324 and 1342). If $\pi^{(1)} \neq \emptyset$, then we have a contribution of $xG_{m-1}(x)$. Otherwise, $\beta \neq \emptyset$ (to avoid 2143) and π avoids T if and only if $\pi^{(1)},...,\pi^{(m-2)}$ avoids 132 and $\pi^{(m-1)} n\alpha$ avoids T, so we have a contribution of

$$x^{m-1} C^{m-3}(x)(C(x) - 1)(F_T(x) - 1).$$

Thus,

$$G_m(x) = xG_{m-1}(x) + x^{m-1} C(x)^{m-3}(C(x) - 1)(F_T(x) - 1).$$

Summing over $m \geq 3$, we obtain

$$F_T(x)-1-xF_T(x)-G_2(x)$$
$$=x\big(F_T(x)-1-xF_T(x)\big)+\frac{x^2\big(C(x)-1\big)\big(F_T(x)-1\big)}{1-xC(x)}.$$

Substitute for $G_2(x)$ and solve for $F_T(x)$ to complete the proof.

2.3. Case 398

2.3.1. The Symmetry Class of {2341, 4123, 1342, 1243}

Lemma 8. *Let T = {2341, 4123, 1342, 1243}. Let* A_d *(respectively,* B_d*,* C_d *and* D_d *) be the generating function for the number of T-avoiders of the form* $(n-1-d)\pi' n\pi''(n-1)\ldots\pi^{(d)}(n-d)\pi^{(d+1)}$ *(respectively, such that* $\pi'=\emptyset$*,* $n-2-d\in\pi^{(d)}$ *and* $n-2-d\in\pi^{(d+1)}$*). Then, for all* $d\geq 1$*,*

$$A_d = x^{2+d}+xA_d+\sum_{j=1}^{d-1}x^jB_{d+1-j}+C_d+D_d,$$

and

$$B_d = x^{d+2}+\sum_{j=0}^{d}x^jB_{d+1-j},$$
$$C_d = x^{d-1}\left(x^4C^2(x)+\frac{x^5}{(1-x)(1-2x)}\right),$$
$$D_d = x^{d+3}C^{d+3}(x).$$

Proof. Let us write an equation for the generating function A_d. Let $\pi=(n-1-d)\pi' n\pi^{(1)}(n-1)\ldots\pi^{(d)}(n-d)\pi^{(d+1)}$. If $n = d+2$ then we have a contribution of x^{d+2}. Thus, we can assume that $n > d + 2$. By considering the position of $n-2-d\geq 1$, we have

- If $n-2-d$ belongs to π' then $\pi'=(n-2-d)\pi''$, so π' avoids T if and only if

$$(n-2-d)\pi''(n-1)\pi^{(1)}(n-2)\ldots\pi^{(d)}(n-1-d)\pi^{(d+1)}$$

  avoids T. Thus we have a contribution of xA_d.

- If $n-2-d$ belongs to $\pi^{(j)}$, $1\leq j\leq d-1$, then since π avoids 1243 and 4123, we see that there is no letter smaller than $n-2-d$ on its left side. Thus, we have a contribution of x^jB_{d+1-j}.
- If $n-2-d$ belongs to $\pi^{(d)}$ (resp. $\pi^{(d+1)}$), then we have a contribution of C_d (resp. D_d).

Adding all contributions, we have, for $d\geq 1$,

$$A_d = x^{2+d}+xA_d+\sum_{j=1}^{d-1}x^jB_{d+1-j}+C_d+D_d.$$

By restricting the above proof to the case $\pi'=\emptyset$, we obtain that for all $d\geq 1$,

$$B_d = x^{d+2}+\sum_{j=0}^{d}x^jB_{d+1-j}.$$

It is not hard to see that $C_d = x^{d-1}C_1$ for all $d\geq 1$. To find a formula for C_1, we define $C_{1,e}$ to be the generating function for T-avoiders of the form

$$(n-1-d)\pi' n\pi^{(1)}(n-1)\ldots\pi^{(d)}(n-d)\pi^{(d+1)}$$

such that $n-2-d\in\pi^{(d)}$ and π' has e letters. Since π avoids 4123, these e letters are decreasing. By direct calculations, we see that $C_{1,0}=x^3\big(C(x)-1\big)$ and $C_{1,e}=x^{e+4}/(1-x)^{e+1}$ for all $e\geq 1$. Hence,

$$C_1=\sum\nolimits_{e\geq 0}C_{1,e}=x^4C^2(x)+\frac{x^5}{(1-x)(1-2x)}.$$

We leave $D_d = x^{d+3}C^{d+3}(x)$ as an exercise for the interested reader (Hint: consider the position of the letter $n-2-d$).

Lemma 9. *The generating function for T-avoiders of the form* $(n-1)\pi' n\pi''$ *is*

$$x(1+x)C(x)-\frac{x}{1-x}.$$

Proof. Let $a_n(i)$ be the number of permutations $(n-1)i\pi'\in S_n(T)$. Then, (details left to the reader)

$$a_n(i)=a_{n-1}(1)+\ldots+a_{n-1}(i)+\delta_{i=n-3}$$

with $a_n(n-1)=0$, $a_n(n)=C_{n-2}$ and $a_n(n-2)=C_{n-2}+C_{n-3}-1$, where $C_n=\frac{1}{n+1}\binom{2n}{n}$ is the Catalan number (number of 123-avoiders of length n).

Define $A_n(v)=\sum\nolimits_{i=1}^{n}a_n(i)v^{i-1}$. Then the above recurrence can be written as

$$A_n(v)=C_{n-2}v^{n-1}+\big(C_{n-2}+2C_{n-3}-1\big)v^{n-3}$$
$$+\frac{1}{1-v}\Big(A_{n-1}(v)-v^{n-3}A_{n-1}(1)\Big)+v^{n-4}$$

with $A_2(v)=1$ and $A_3(v)=1+v^2$.

Define $A(x,v)=\sum\nolimits_{n\geq 0}A_n(v)x^n$. Multiplying the last recurrence by x^n and summing over $n\geq 4$,

$$\left(1-\frac{x}{v(1-v)}\right)A(x/v,v)$$
$$=\frac{x^2}{v^2}+\frac{x^3}{v^3}\Big(v^2+1+2x\Big)C^2(x)$$
$$+\frac{x^4(v-1)}{v^4(1-x)}-\frac{x}{v^3(1-v)}A(x,1).$$

By taking $v = 1/C(x)$, we obtain

$$A(x,1) = x(1+x)C(x) - \frac{x}{1-x},$$

as required.

Lemma 10. *The generating function for T-avoiders with 2 left-right maxima is*

$$G_2(x) = \left(1-x+x^2\right)C(x) + \frac{\left(2x^2-2x+1\right)\left(x^3-2x^2+3x-1\right)}{(1-2x)(1-x)^3}.$$

Proof. Suppose $\pi = i\pi' n\pi'' \in S_n(T)$ with exactly 2 left-right maxima. Denote the contribution of the cases $i < n-1$ and $i = n-1$ by H and H' so that $G_2(x) = H + H'$.

To find a formula for H, we distinguish the cases π'' has a subsequence of two letters ab such that $i < a < b$, or not. In the former case, the contribution is $\frac{x^4}{(1-x)^2(1-2x)}$. In the latter case, by Lemma 8, the contribution is $\sum_{d\ge1} A_d$, where

$$A_d = \frac{1}{1-x}\left(\begin{array}{l} x^{d+2} + \sum_{j=1}^{d-1} x^{d+3}C^{d+2-j} \\ +x^{d-1}\left(x^4C^2(x) + \frac{x^5}{(1-x)(1-2x)}\right) \\ +x^{d+3}C^{d+3}(x) \end{array}\right).$$

Lemma 9 shows that $H' = x(1+x)C(x) - \frac{x}{1-x}$. Therefore,

$$G_2(x) = x(1+x)C(x) - \frac{x}{1-x} + \frac{x^4}{(1-x)^2(1-2x)} + \sum_{d\ge1} A_d,$$

and the result follows after algebraic simplification.

Theorem 11. *Let T = {2341, 4123, 1342, 1243}. Then*

$$F_T(x) = C(x) + \frac{x}{(1-x)^2}\left(C(x) - 1 - xC(x)\right) + \frac{x^4}{(1-2x)(1-x)^4}.$$

Proof. Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xC(x)$. Lemma 10 gives $G_2(x)$.

For $G_m(x)$ with $m \ge 3$, suppose

$$\pi = i_1\pi^{(1)}i_2\pi^{(2)}\ldots i_m\pi^{(m)} \in S_n(T)$$

has $m \ge 3$ left-right maxima. Since π avoids T, we see that $\pi^{(3)} = \ldots = \pi^{(m)} = \emptyset$. Thus, π avoids T if and only I $i_1\pi^{(1)}i_2\pi^{(2)}$ avoids T. Therefore, $G_m(x) = x^{m-2}G_2(x)$.

By summing over m, we have

$$F_T(x) = 1 + xC(x) + \frac{1}{1-x}G_2(x),$$

which completes the proof.

2.3.2. The Symmetry Class of {2341, 4123, 1342, 1234}

Theorem 12. *Let T =* {2341, 4123, 1342, 1234}. *Then*

$$F_T(x) = C(x) + \frac{x}{(1-x)^2}\left(C(x) - 1 - xC(x)\right) + \frac{x^4}{(1-2x)(1-x)^4}.$$

Proof. To write a formula for $F_T(x)$, let $\pi \in S_n(T)$. If π avoids 123, the contribution is $C(x)$. Otherwise, let $\pi_a\pi_b\pi_c$ be an occurrence of 123 such that a, $a+b$ and $a+b+c$ are minimal. If $\pi_c = n$, then π can be written as $(i-1)(i-2)\ldots(\pi_b+1)\pi' n(n-1)\ldots i$ such that π' has exactly 2 left-right maxima π_a and π_b and π' avoids 123. Thus, we have a contribution of

$$\frac{x}{(1-x)^2}\left(C(x) - 1 - xC(x)\right).$$

If $\pi_c < n$, then π can be written as

$$\pi = (i-1)(i-2)\ldots(\pi_b+1)\pi_a\alpha n(n-1)\ldots (\pi_c+1)\pi_b(\pi_b-1)\ldots(\pi_a+1)\beta\pi_c(\pi_c+1)\ldots i,$$

where $\pi_a\alpha$ and β are decreasing and $\beta < \pi_a$. Thus, we have a contribution of $\frac{x^4}{(1-2x)(1-x)^4}$. The result follows by adding contributions.

2.4. Case 457

2.4.1. The Symmetry Class of {2314, 3124, 1342, 1423}

Theorem 13. Let T = {2314, 3124, 1342, 1423}. *Then*

$$F_T(x) = \frac{C(x)}{1 - \frac{x^3C(x)}{(1-x)^2(1-2x)}}.$$

Proof. Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$.

To find $G_2(x)$, suppose $\pi = i\pi' n\pi'' \in S_n(T)$ with 2 left-right maxima. It is not hard to see that π avoids T if and only if π can be decomposed as $i\alpha' n\alpha''\beta$ such that $\alpha = i\alpha' n\alpha''$ is a permutation on $\{s+1, s+2, \ldots, n\}$ that avoids 123 and $\beta \in S_s(T)$, where α cannot be decomposed as $\gamma\gamma'$ with $\gamma > \gamma'$ and γ, γ' are not empty. Hence, $G_2(x) = x(C(x)-1)F_T(x)$.

For $G_m(x)$ with $m \geq 3$, suppose

$$\pi = i_1\pi^{(1)}i_2\pi^{(2)}\ldots i_m\pi^{(m)} \in S_n(T)$$

has m left-right maxima. Since π avoids T, we see that $\pi^{(j)} > i_{j-1}$ for all $j = 2,3,\ldots,m-1$ and $\pi^{(m)} = \beta\alpha$ with $i_m > \beta > i_m - 1$ and $\alpha < \pi^{(1)}$. Thus, π avoids T if and only if $\pi^{(1)},\ldots,\pi^{(m-1)}$, β are all decreasing and α avoids T. Hence, $G_m(x) = \frac{x^m}{(1-x)^m}F_T(x)$.

Summing over m, we have

$$F_T(x) = 1 + xF_T(x) + x(C(x)-1)F_T(x) + \frac{x^3}{(1-x)^2(1-2x)}F_T(x),$$

and solving for $F_T(x)$ completes the proof.

2.4.2. The Symmetry Class of {2413, 3142, 1324, 1243}

Theorem 14. *Let* T = {2413, 3142, 1324, 1243}. *Then*

$$F_T(x) = \frac{C(x)}{1 - \frac{x^3C(x)}{(1-x)^2(1-2x)}}.$$

Proof. Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$.

For $G_m(x)$ with $m \geq 2$, suppose

$$\pi = i_1\pi^{(1)}i_2\pi^{(2)}\ldots i_m\pi^{(m)} \in S_n(T)$$

has m left-right maxima. Since π avoids T, we see that $\pi^{(1)} > \pi^{(2)} > \ldots > \pi^{(m)} > \alpha$, where $\pi^{(m)} = \beta\alpha$, and $i_2 > \beta > i_1$. If $\beta = \emptyset$, then π avoids T if and only if $\pi^{(1)},\ldots,\pi^{(m-1)}$ avoids 132 and α avoids T. Otherwise, π avoids T is and only if $\pi^{(1)}$ is decreasing, $\pi^{(2)},\ldots,\pi^{(m-1)} = \emptyset$, β is not empty and avoids both 132 and 213, and α avoids T. Thus, by [4,11], we have

$$G_m(x) = x^mC^{m-1}(x)F_T(x) + \frac{x^m}{1-x}\left(\frac{1-x}{1-2x} - 1\right)F_T(x).$$

Summing over m, we have

$$F_T(x) = 1 + xF_T(x) + \frac{x^2C(x)}{1-xC(x)} + \frac{x^3}{(1-x)^2(1-2x)}F_T(x),$$

and solving for $F_T(x)$ completes the proof.

2.5. Case 471

2.5.1. The Symmetry Class of {2314, 1342, 1324, 4123}

Theorem 15. *Let* T = {2314, 1342, 1324, 4123}. *Then*

$$F_T(x) = C(x) + \frac{x^3}{(1-x)^5}C(x).$$

Proof. To write a formula for $F_T(x)$, let $\pi \in S_n(T)$. If π avoids 123, then we have a contribution of $C(x)$. Otherwise, let $\pi_a\pi_b\pi_c$ be an occurrence of 123 such that *a*, *a*+*b* and *a*+*b*+*c* are minimal. W consider the following four cases:

- $\pi_c = n$ and $a = 1$: $\pi = \pi_a\alpha\pi_b\pi_c\beta\gamma$ with $\pi_a > \alpha > \gamma$ and $\pi_c > \beta > \pi_b$, where α avoids $\{123, 132, 213\}$, β avoids $\{213, 231, 123\}$, and γ avoids 123. Thus, we have a contribution of $x^3K^2(x)C(x)$, where

$$K(x) = 1 + x/(1-x)^2,$$

see [4, 11].

- $\pi_c = n$ and $a \neq 1$. Here,

$$\pi = (i-1)(i-2)\ldots(\pi_b+1)\ldots i'\pi_b\pi_c(\pi_c-1)\ldots i_\gamma$$

where $\gamma < i'$ and γ avoids 123. Thus, we have a contribution of $x^4C(x)/(1-x)^3$.

- There is a letter x on the right side of π_c greater than π_c. Here, $\pi = \pi_a\alpha\pi_b\pi_c\beta\gamma$, where β avoids {4123, 213, 231}, $\pi_a > \alpha > \gamma$, $\beta > \pi_c$, α avoids {123, 132, 231}, and γ avoids 123. Thus, we have a contribution of $x^3K(x)(H(x)-1)C(x)$, where

$$H(x) = F_{\{213,231,4123\}}(x) = \frac{2x^2-2x+1}{(1-x)^3}.$$

- There is a letter x between π_a and π_b such that $x > \pi_c$, and there is no letter x on the right side of π_c greater than π_c. When $a = 1$, π has the form

$$\pi_a(\pi_a-1)\ldots in(n-1)\ldots(\pi_c+1)\pi_b\pi_c(\pi_c-1)\ldots(\pi_b+1)\gamma$$

where $\gamma > i$ and γ avoids 123, giving a contribution of $\frac{x^4}{(1-x)^2}K(x)C(x)$. When $a \neq 1$, π has the form

$$(i'-1)(i'-2)\ldots(\pi_b+1)\pi_a(\pi_a-1)\ldots in(n-1)\ldots$$
$$(\pi_c+1)\pi_b\pi_c(\pi_c-1)\ldots i'\gamma$$

where $\gamma < i$ and γ avoids 123, giving a contribution of $\frac{x^4}{(1-x)^4}C(x)$.

Adding all the contribution, we obtain $F_T(x)=C(x)+\frac{x^3}{(1-x)^5}C(x)$, as claimed.

2.5.2. The Symmetry Class of {2314, 1342, 4123, 1234}

Theorem 16. *Let T* = {2314, 1342, 4123, 1234}. *Then*

$$F_T(x)=C(x)+\frac{x^3}{(1-x)^5}C(x).$$

Proof. Let $\pi\in S_n(T)$. If π avoids 123, the contribution is $C(x)$. Otherwise, let $\pi_a\pi_b\pi_c$ be an occurrence of 123 such that a, *a* + b and *a* + *b* + *c* are minimal. In this case π can be written as

$$\pi=(i'-1)(i'-2)...(\pi_b+1)\pi_a(\pi_a-1)...i''n(n-1)...$$
$$(\pi_c+1)\pi_b(\pi_b-1)...(\pi_a+1)\pi_c(\pi_c-1)...i'\alpha,$$

where α avoids 123. Thus, the contribution is $\frac{x^3}{(1-x)^5}C(x)$, and the result follows.

2.6. Case 613

2.6.1. The Symmetry Class of {2413, 3142, 3412, 2341}

Theorem 17. *Let T* = {2413, 3142, 3412, 2341}. *Then*

$$F_T(x)=\frac{1-x^3/(1-x)^3-\sqrt{\left(1-x^3/(1-x)^3\right)^2-4x}}{2x}.$$

Proof. Let $G_m(x)$ be the generating function for *T*-avoiders with *m* left-right maxima. Clearly, $G_0(x)=1$ and $G_1(x)=xF_T(x)$.

Now suppose $m\ge 2$ and $\pi=i_1\pi^{(1)}...i_m\pi^{(m)}$ has *m* left-right maxima. Since π avoids 2341, we have $\pi^{(j)}>i_{j-2}$ for all *j*. If $\pi^{(2)}>i_1$, then the contribution is $xF_T(x)G_{m-1}(x)$. Otherwise, since π avoids 2413, we have $\pi^{(3)}...\pi^{(m)}>i_2$ and

$$\pi^{(1)}\pi^{(2)}=(i_1-1)(i_2-1)...i'(n-1)(n-2)...$$
$$(i_1+1)(i'-1)...21,$$

where $i_1\ge i'\ge 1$, which leads to contribution of $\frac{x^3}{(1-x)^3}G_{m-2}(x)$. Thus,

$$G_m(x)=xF_T(x)G_{m-1}(x)+\frac{x^3}{(1-x)^3}G_{m-2}(x),$$

for all $m\ge 2$. Summing over *m*, we find that

$$F_T(x)=1+xF_T(x)+xF_T(x)(F_T(x)-1)+\frac{x^3}{(1-x)^3}F_T(x),$$

and solving for $F_T(x)$ completes the proof.

2.6.2. The Symmetry Class of {2413, 3142, 1243, 1432}

Theorem 18. *Let T* = {2413, 3142, 1243, 1432}. *Then*

$$F_T(x)=\frac{1-x^3/(1-x)^3-\sqrt{\left(1-x^3/(1-x)^3\right)^2-4x}}{2x}.$$

Proof. Let $G_m(x)$ be the generating function for *T*-avoiders with m left-right maxima. Clearly, $G_0(x)=1$ and $G_1(x)=xF_T(x)$.

To find $G_2(x)$, suppose $\pi=i\pi'n\pi''$ has 2 left-right maxima. If $\pi''=\emptyset$, the contribution is $x^2F_T(x)$. If $\pi''>i$ and π'' is not empty, the contribution is $\frac{x^3}{(1-x)^2}$. If $i=n-1$ and π'' is not empty, the contribution is $x^2F_T(x)(F_T(x)-1)$. Otherwise, $i<n-1$ and π'' has a letter smaller than *i*, so we have a contribution of $\frac{x^3}{(1-x)^2}(F_T(x)-1)$. Hence,

$$G_2(x)=x^2F_T(x)+\frac{x^3}{(1-x)^2}+x^2F_T(x)(F_T(x)-1)$$
$$+\frac{x^3}{(1-x)^2}(F_T(x)-1).$$

Now suppose $m\ge 2$ and $\pi=i_1\pi^{(1)}...i_m\pi^{(m)}$ has *m* left-right maxima. Since π avoids 1243, we have $\pi^{(j)}<i_2$ for all *j*. If $\pi^{(m)}<i_1$, then the contribution is $xF_T(x)G_{m-1}(x)$. Otherwise, π has a letter between i_1 and i_2, so $\pi^{(2)}=...=\pi^{(m-1)}=\emptyset$, $\pi^{(1)}$ is decreasing and $\pi^{(m)}=\alpha\beta$ where $i_2>\alpha>i_1$ and α is nonempty increasing. Also, $\pi^{(1)}>\beta$ and β avoids T. Thus,

$$G_m(x)=xF_T(x)G_{m-1}(x)$$
$$+\frac{x^{m+1}}{(1-x)^2}F_T(x)$$

for all $m\ge 3$. Summing over *m*, we find that the generating function $F_T(x)$ satisfies

$$F_T(x)=1+xF_T(x)+G_2(x)$$
$$+xF_T(x)(F_T(x)-1-xF_T(x))+\frac{x^4}{(1-x)^3}F_T(x).$$

Substituting for $G_2(x)$ and then solving for $F_T(x)$ completes the proof.

2.7. Case 633

2.7.1. The Symmetry Class of {2413, 3142, 1342, 2341}

Note that all four patterns contain 231.

Theorem 19. *Let T* = {2413, 3142, 1342, 2341}. *Then*

$$F_T(x)=\frac{(1-x)^2-x^2C(x)}{(1-x)^2-x\left(1-x+x^2\right)C(x)}.$$

Proof. Let $G_m(x)$ be the generating function for *T*-avoiders with m left-right maxima. Clearly, $G_0(x)=1$ and $G_1(x)=xF_T(x)$.

To find $G_2(x)$, suppose $\pi=i\pi'n\pi''$ has 2 left-right maxima. If $\pi''>i$, then we have a contribution of $x^2F_T(x)C(x)$, where $C(x)$ counts the number of 231-avoiders of length *n*. Otherwise, π can be written as $\pi=i(i-1)...(i'+1)n(n-1)...(i+1)\pi'''$, where π''' is nonempty and avoids *T*, which gives contribution of $\frac{x^2}{(1-x)^2}(F_T(x)-1)$. Hence,

$$G_2(x)=x^2F_T(x)C(x)+\frac{x^2}{(1-x)^2}(F_T(x)-1).$$

Now let $m\geq 3$ and suppose $\pi=i_1\pi^{(1)}...i_m\pi^{(m)}$ has m left-right maxima. Then π avoids *T* if and only if $\pi^{(j)}>i_{j-1}$ for $j=3,4,...,m,$ and $i_1\pi^{(1)}i_2\pi^{(2)}$ avoids *T* and $\pi^{(j)}$ avoids 231 for all $j=3,4,...,m.$ Thus,

$$G_m(x)=x^{m-2}G_2(x)C(x)^{m-2}.$$

Summing over *m*, we find that the generating function $F_T(x)$ satisfies

$$F_T(x)=1+xF_T(x)+\frac{1}{1-xC(x)}\left(x^2F_T(x)C(x)+\frac{x^2}{(1-x)^2}(F_T(x)-1)\right),$$

and solving for $F_T(x)$ completes the proof.

2.7.2. The Symmetry Class of {2143, 1324, 1342, 1423}

Note that all four patterns contain 132.

Theorem 20. *Let T* = {2143, 1324, 1342, 1423}. *Then*

$$F_T(x)=\frac{(1-x)^2-x^2C(x)}{(1-x)^2-x\left(1-x+x^2\right)C(x)}.$$

Proof. Let $G_m(x)$ be the generating function for *T*-avoiders with *m* left-right maxima. Clearly, $G_0(x)=1$ and $G_1(x)=xF_T(x)$.

To find $G_2(x)$, suppose $\pi=i\pi'n\pi''$ has 2 left-right maxima. If π'' has a letter greater than *i*, then π can be written as $\pi=in(n-1)...(i+2)\alpha(i+1)\beta$ where $\alpha(i+1)\beta$ avoids *T*, and so we have a contribution of $\frac{x^2}{(1-x)^2}(F_T(x)-1)$. Otherwise, that is, if *i* = *n* − 1, we have a contribution of $x(F_T(x)-1)$. Thus,

$$G_2(x)=x(F_T(x)-1)+\frac{x^2}{(1-x)^2}(F_T(x)-1).$$

Now let $m\geq 3$ and suppose $\pi=i_1\pi^{(1)}...i_m\pi^{(m)}$ has m left-right maxima. If $\pi^{(m)}<i_1$ then

$$\pi^{(1)}>\pi^{(2)}>...>\pi^{(m-1)}\pi^{(m)}$$

and $\pi^{(j)}$ avoids 132 for $j=1,2,...,m-2$ and $\pi^{(m-1)}i_m\pi^{(m)}$ avoids *T*, which leads to contribution of $x^{m-1}(F_T(x)-1)C(x)^{m-1}$, where $C(x)$ counts 132-avoiders of length *n*. Otherwise, $\pi^{(m)}$ has a letter greater than i_{m-1}, so π can be written as

$$\pi=i_1(i_1+1)...(i_1+m-2)n(n-1)...(i_1+m)\alpha(i_1+m-1)\beta$$

where $\alpha(i_1+m-1)\beta$ avoids *T*, which gives a contribution of $\frac{x^m}{1-x}(F_T(x)-1)$. Thus,

$$G_m(x)=x^{m-1}(F_T(x)-1)C(x)^{m-1}+\frac{x^m}{1-x}(F_T(x)-1).$$

Summing over *m*, we find that the generating function $F_T(x)$ satisfies

$$F_T(x)=1+xF_T(x)+\frac{x(F_T(x)-1)}{1-xC(x)}+\frac{x^2}{(1-x)^2}(F_T(x)-1),$$

and solving for $F_T(x)$ completes the proof.

2.8. Case 639

2.8.1. The Symmetry Class of {3142, 2314, 1342, 2431}

Theorem 21. *Let* T = {3142, 2314, 1342, 2431}. *Then*

$$F_T(x)=\frac{\left(1-4x+3x^2+x^3+x^2(1-x)^2C(x)\right)C(x)}{(1-x)\left(1-3x+x^2\right)}.$$

Proof. Let $G_m(x)$ be the generating function for *T*-avoiders with m left-right maxima. Clearly, $G_0(x)=1$ and $G_1(x)=xF_T(x)$.

To find $G_2(x)$, suppose $\pi=i\pi'n\pi''$ has 2 left-right maxima. If $\pi''>i$, then we have a contribution of $x^2C(x)^2$, where $C(x)$ counts the number of 231-avoiders of length *n*. Otherwise, π'' has a letter smaller than *i*. If π' is decreasing and $\pi''<i$ then $\pi'>\pi''$ and the contribution is $x^2(C(x)-1)/(1-x)$, and if π' is sequence and π'' has a letter greater than *i*, we have a contribution of $\frac{x^2}{1-x}(F_T(x)-1)$. If π' is not decreasing, then we have a contribution $x^2(L-1/(1-x))(F_T(x)-1)$, where $L=\frac{1-x}{1-2x}$ is the generating function for {231, 132}-avoiders [11]. Thus,

$$G_2(x)=x^2C^2(x)+\frac{x^2}{1-x}(C(x)-1)^2$$
$$+\frac{x^2}{1-x}(F_T(x)-1)+x^2(L-1/(1-x))(F_T(x)-1).$$

To find $G_3(x)$, suppose $\pi=i\pi'i'\pi''n\pi'''(i<i'<n)$ has 3 left-right maxima. If $\pi''=\emptyset$, then by considering whether π''' has a letter smaller than *i* or not, we get a contribution of $x^3C(x)^2+x^3(C(x)-1)$. Otherwise, we have a contribution of $x^3C(x)^2(C(x)-1)$. Thus,

$$G_3(x)=x^3C(x)^2+x^3(C(x)-1)+x^3C(x)^2(C(x)-1).$$

Now let $m\geq 4$ and suppose $\pi=i_1\pi^{(1)}...i_m\pi^{(m)}$ has *m* left-right maxima. Since π avoids *T*, we have $\pi^{(j)}>i_{j-1}$ for $j=2,3,...,m-1$ and $\pi^{(m)}$ has no letters between i_1 and i_{m-1}. By considering whether $\pi^{(m-1)}$ is empty or not, we get a contribution of

$$G_m(x)=xG_{m-1}(x)+x^mC(x)^{m-1}(C(x)-1).$$

Summing over *m*, we find that

$$F_T(x)=1+xF_T(x)+G_2(x)$$
$$+x(F_T(x)-1-xF_T(x)-G_2(x))$$
$$+\frac{x^4C(x)^3(C(x)-1)}{1-xC(x)}.$$

Substituting for $G_2(x)$ and solving for $F_T(x)$ completes the proof.

2.8.2. The Symmetry Class of {2413, 2431, 2314, 1342}

Theorem 22. *Let* T = {2413, 2431, 2314, 1342}. *Then*

$$F_T(x)=\frac{\left(1-4x+3x^2+x^3+x^2(1-x)^2C(x)\right)C(x)}{(1-x)\left(1-3x+x^2\right)}.$$

Proof. Let $G_m(x)$ be the generating function for *T*-avoiders with *m* left-right maxima. Clearly, $G_0(x)=1$ and $G_1(x)=xF_T(x)$.

To find $G_2(x)$, suppose $\pi=i\pi'n\pi''$ has 2 left-right maxima. The contribution of the case $i=n-1$ is $x(F_T(x)-1)$. If $i<n-1$, then $\pi'<i<\pi''<n$ and π'' is not empty and π', π'' both avoid 231, and so the contribution is $x^2C(x)(C(x)-1)$. Hence,

$$G_2(x)=x(F_T(x)-1)+x^2C(x)(C(x)-1).$$

Now let $m\geq 3$ and suppose $\pi=i_1\pi^{(1)}...i_m\pi^{(m)}$ has *m* left-right maxima. Since π avoids *T*, we have either $\pi^{(1)}<i_1<\pi^{(2)}<i_2<...<\pi^{(m-1)}<i_m$ with $\pi^{(m-1)}\neq\emptyset$, or $\pi^{(m)}=\alpha\beta$ with

$$\beta<\pi^{(1)}<i_1<...<\pi^{(m-2)}<i_{m-2}<\alpha<i_m$$

and $\pi^{(m-1)}=\emptyset$. The contribution of first case is $x^mC(x)^{m-1}(C(x)-1)$ and the contribution of the second case is $x^mC(x)^{m-1}+x^mL(F_T(x)-1)$, for the cases β is empty or not, where $L=F_{\{132,231\}}(x)=\frac{1-x}{1-2x}$ [11]. Hence,

$$G_m(x)=x^mC(x)^{m-1}(C(x)-1)$$
$$+x^mC(x)^{m-1}+x^mL(F_T(x)-1).$$

Summing over m, we obtain

$$F_T(x)=1+xF_T(x)+x(F_T(x)-1)$$
$$+x^2C(x)(C(x)-1)+\frac{x^3C(x)^3}{1-xC(x)}+\frac{x^3}{1-x}L(F_T(x)-1),$$

and solving for $F_T(x)$ completes the proof.

2.9. Case 732

2.9.1. The Symmetry Class of {3412, 3421, 2413, 3241}

Theorem 23. *Let* T = {3412, 3421, 2413, 3241}. *Then*

$$F_T(x)=C\left(\frac{x(1-x)^2}{1-2x}\right).$$

Proof. Let $G_m(x)$ be the generating function for *T*-avoiders with *m* left-right maxima. Clearly, $G_0(x)=1$ and $G_1(x)=xF_T(x)$.

To find a formula for $G_m(x)$ with $m\geq 2$, suppose $\pi=i_1\pi^{(1)}...i_m\pi^{(m)}\in S_n(T)$ has *m* leftright maxima. If $\pi^{(m)}<i_{m-1}$, the contribution is $xF_T(x)G_{m-1}(x)$. Otherwise, since π avoids {3412, 3421}, there exists $s\in[1,m-1]$ such that $\pi^{(m)}$ contains a letter between i_{s-1} and i_s. Since π avoids {3412, 3421, 2413}, we have that $\pi^{(s+1)}=...=\pi^{(m-1)}=\emptyset$, $i_s-1>\pi^{(s)}>i_{s-1}$, and $\pi^{(m)} = \alpha(i_s - 1)$ where π avoids {231, 213}. Thus, for fixed s, we have a contribution of $x^{m+2-s}LF_T(x)G_{s-1}(x)$. Therefore,

$$G_m(x) = xF_T(x)G_{m-1}(x) + \sum_{s=1}^{m-1} x^{m+2-s}LF_T(x)G_{s-1}(x).$$

Summing over $m \geq 2$, we obtain

$$F_T(x) - 1 - xF_T(x)$$
$$= xF_T(x)\left(F_T(x) - 1\right) + \frac{x^3}{1-x}LF_T(x)^2,$$

and solving for $F_T(x)$ completes the proof.

2.9.2. The Symmetry Class of {3142, 1342, 1423, 1243}

Theorem 24. *Let* T = {3142, 1342, 1423, 1243}. *Then*

$$F_T(x) = C\left(\frac{x(1-x)^2}{1-2x}\right).$$

Proof. Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$.

To find a formula for $G_2(x)$, suppose $\pi = i\pi' n\pi''$ has 2 left-right maxima. If $\pi'' < i$ then $\pi' > \pi''$ and we have a contribution of $x^2F_T(x)^2$. Otherwise, π can be written as $\pi = i(i-1)...i' n\alpha^{(n)}(n-1)\alpha^{(n-1)}...(i+1)\alpha^{(i+1)}$ where $i' > \alpha^{(n)}...\alpha^{(i+3)} > \alpha^{(i+1)}$, and $\alpha^{(n)}...\alpha^{(i+3)}$ is decreasing, and $\alpha^{(i+2)}, \alpha^{(i+1)}$ each avoids T. So we have a contribution of $\frac{x^3}{(1-x)\left(1-\frac{x}{1-x}\right)}F_T(x)^2$. Hence,

$$G_2(x) = x^2F_T(x)^2 + \frac{x^3}{1-2x}F_T(x)^2.$$

To find a formula for $G_m(x)$ with $m \geq 3$, suppose $\pi = i_1\pi^{(1)}...i_m\pi^{(m)} \in S_n(T)$ has m left-right maxima. If $\pi^{(m)} = \emptyset$, the contribution is $xG_{m-1}(x)$. Otherwise, $\emptyset \neq \pi^{(m)} < i_1\pi^{(1)}...i_{m-1}\pi^{(m-1)}$, leading to a contribution of $xG_{m-1}(x)\left(F_T(x)-1\right)$. Thus,

$$G_m(x) = xG_{m-1}(x) + xG_{m-1}(x)\left(F_T(x)-1\right)$$
$$= xG_{m-1}(x)F_T(x).$$

By summing over $m \geq 3$, we obtain

$$F_T(x) - 1 - xF_T(x)$$
$$-\left(x^2F_T(x)^2 + \frac{x^3}{1-2x}F_T(x)^2\right)$$
$$= x\left(F_T(x) - 1 - xF_T(x)\right)F_T(x).$$

and solving for $F_T(x)$ completes the proof.

2.10. Case 842

2.10.1. The Symmetry Class of {2134, 2314, 2341, 1423}

Define $a_n = |S_n(T)|$ and $a_n(i_1, i_2, ..., i_s)$ to be the number of permutations $\pi = i_1, i_2, ..., i_s\pi'$ in $S_n(T)$. Define $a_n^- = \sum_{i=2}^n a_n^-(i) = \sum_{i=2}^n \sum_{j=1}^{i-1} a_n(i,j)$ and $a_n^+ = \sum_{i=1}^{n-1}\sum_{j=i+1}^n a_n(i,j)$.

Since π avoids 2314 and 2331, $a_n(i,j) = 0$ for $2 \leq i < j \leq n-1$. Also, it is not hard to see $a_n(1,j) = j-1$ for $j = 3,4,...,n-1$, $a_n(1,n) = a_n(2,1) = 1$, and

$$a_n(1,2) = a_n(n,1)$$
$$= \left|S_{n-2}(\{312, 2341, 2314, 2134\})\right|$$
$$= \binom{n-1}{3} + 1.$$

Thus,

$$a_n(1) = \binom{n-1}{3} + \binom{n-1}{2} + 1.$$

If $\pi = in\pi' \in S_n(T)$ with $2 \leq i \leq n-1$, then the leftmost letter of π' is either $< i$ or $= n-1$. Thus, $a_n(i,n) = a_{n-1}(i)$ for $i = 2,3,...,n-1$. Similarly, $a_n(i,i-1) = a_{n-1}(i-1)$ for all $i = 2,3,...,n$. Moreover, $a_n(n) = a_n(n-1) = a_{n-1}$. Thus,

$$a_n^+ = \sum_{i=2}^{n-1} a_{n-1}(i) + \binom{n-1}{3} + \binom{n-1}{2} + 1$$
$$= a_{n-1} + \binom{n-1}{2}.$$

Hence,

$$A^+(x) := \sum_{n\geq 2} a_n^+ x^n = x\left(A(x) - 1\right) + \frac{x^3}{(1-x)^3}.$$

If $\pi = i1\pi' \in S_n(T)$ with $3 \leq i \leq n-2$, then π contains the subword $n(n-1)...(i+1)$. Hence, $a_n(i,1) = a_{n-1}(i,1)$ for $i = 3,4,...,n-2$. If $\pi = ij\pi' \in S_n(T)$ with $2 \leq j < i \leq n-2$, then

$$a_n(i,j) = a_n(i,j,n) + \sum_{k=1}^{j-1} a_n(i,j,k)$$
$$= a_{n-1}(i,j) + \sum_{k=1}^{j-1} a_{n-1}(j,k).$$

Thus, $a_n^-(2) = 1$, $a_n^-(n) = a_{n-1}$,

$$a_n^-(n-1) = a_{n-1} - a_n(n-1,n) = a_{n-1} - a_{n-2},$$

and

$$a_n^-(i)=\sum_{j=1}^{i-1}a_{n-1}(i,j)=a_{n-1}^-(i)+\ldots+a_{n-1}^-(1).$$

Define $A_n^-(v)=\sum_{i=1}^{n}a_n^-(i)v^{i-1}$. Then

$$A_n^-(v)=v^{n-1}a_{n-1}+v^{n-2}(a_{n-1}-a_{n-2})$$
$$=\sum_{j=1}^{n-1}\frac{v^{j-1}-v^{n-2}}{1-v}a_{n-1}^-(j),$$

which is equivalent to

$$A_n^-(v)=v^{n-1}a_{n-1}+v^{n-2}(a_{n-1}-a_{n-2})+\frac{1}{1-v}\left(A_n^-(v)-v^{n-2}A_{n-1}^-(1)\right)$$

for $n\geq 2$, with $A_1^-(v)=0$. Define

$$A^-(x,v)=\sum_{n\geq 1}A_n^-(v)x^n.$$

Multiplying the last recurrence by $\frac{x^n}{v^n}$ and summing over $n\geq 2$, we obtain

$$A^-(x/v,v)=\frac{x(1+v)}{v^2}(A(x)-1)-\frac{x^2}{v^2}A(x)+\frac{x}{v^2(1-v)}(vA^-(x/v,v)-A^-(x,1)).$$

Taking $v=1/C(x)$, we obtain

$$A^-(x,1)=x(1+C(x)-xC(x))A(x)-x(1+C(x)).$$

Now, since $A(x,1)=1+x+A^-(x,1)+A^+(x)$, we obtain the following result.

Theorem 25. *Let T* = {2134, 2314, 2341, 1423}. *Then*

$$F_T(x)=C(x)+\frac{x^3}{(1-x)^3}C(x)^3.$$

2.10.2. The Symmetry Class of {1234, 2314, 2341, 1423}

Theorem 26. *Let T* = {1234, 2314, 2341, 1423}. *Then*

$$F_T(x)=C(x)+\frac{x^3}{(1-x)^3}C(x)^3.$$

Proof. To write a formula for $F_T(x)$, let $\pi\in S_n(T)$. If π avoids 123, then we have a contribution of $C(x)$. Otherwise, let $\pi_a\pi_b\pi_c$ be an occurrence of 123 such that *a*, *a*+*b* and *a*+*b*+*c* are minimal. Since π avoids *T*, we see that π can be written as

$$\pi^{(0)}j_1\pi^{(1)}\ldots j_m\pi^{(m)}\pi_a(\pi_a-1)\ldots 1\pi_a\pi_b(\pi_b-1)\ldots i'\,\pi_c i_1 i_2\ldots i_p(i'-1)(i'-2)\ldots(\pi_a+1),$$

where $\{j_1,j_2,\ldots,j_m,i_1,i_2,\ldots,i_p\}=\{\pi_c-1,\pi_c-2,\ldots,\pi_b+1\}$. Since $\pi^{(0)}j_1\pi^{(1)}..j_m\pi^{(m)}$ avoids 123 and has the form $\pi^{(0)}\pi'\pi''$ with $\pi^{(0)}>\pi'>i_1>\pi''$, the contribution is

$$C(x)\frac{1}{1-xC(x)}\frac{1}{1-\frac{x}{1-xC(x)}}\frac{x^3}{(1-x)^3}=\frac{x^3}{(1-x)^3}C(x)^3.$$

Therefore, by adding all the contribution, we obtain $F_T(x)=C(x)+\frac{x^3}{(1-x)^3}C(x)^3$, as claimed.

2.11. Case 874

2.11.1. The Symmetry Class of {2134, 1342, 2341, 1423}

Define $r_n=|S_n(213,\ 1342,\ 2341,\ 1423)|$ and set $R(x)=\sum_{n\geq 0}r_nx^n$. Note that $\pi=\pi'1\pi''$ avoids {213, 1342, 2341, 1423} if and only if $\pi'>\pi''$, π' avoids {123, 213} and π'' avoids {312, 213, 231}. Thus, by [11], we have

$$R(x)=1+\frac{x(1-x)}{1-2x}\left(\frac{1}{1-x}+\frac{x^2}{(1-x)^2}\right)$$
$$=1+\frac{x}{1-2x}+\frac{x^3}{(1-x)(1-2x)}$$

and so $r_n=3\cdot 2^{n-2}-1$ for all $n\geq 2$.

Define $a_n^+=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}a_n(i,j)$, where $a_n(i_1,i_2,\ldots,i_s)$ is the number of permutations of the form $\pi=i_1i_2\ldots i_s\pi'$ in $S_n(T)$.

Lemma 27. *For all* $n\geq 3$,

$$a_n^+=a_{n-1}+n-2+r_1+r_2+\ldots+r_{n-3}.$$

Proof. Let $\pi=ij\pi'$ in $S_n(T)$. If $i=1$ and $j\geq 4$ then π' is increasing, so $a_n(1,j)=1$ for all $j=4,5,\ldots,n$. Since $a_n(1,2)=|S_{n-2}(2134,\ 231,\ 312)|$, we have $a_n(1,2)=\binom{n-2}{2}+1$, and for $2\leq i<j\leq n-1$, $a_n(i,j)=r_{i-1}$. Thus,

$$a_n^+=\binom{n-2}{2}+1+\sum_{j=3}^{n-1}(r_0+r_1+\ldots+r_{j-2})+\sum_{i=1}^{n-1}a_n(i,n).$$

Let us write a formula for $b_n(i)=a_n(i,n)$. Let $\pi=in\pi'\in S_n(T)$ with $2\leq i\leq n-1$. Then either $\pi=in(n-1)\pi''$ or $\pi=in(n-1)\beta$ with $\alpha\neq\emptyset$. The first case contributes $a_{n-1}(i,n-1)$. For the second case, since π avoids 1423 we see that $\alpha=j\alpha'<i$, which leads to a contribution of $a_{n-1}(i,1)+\ldots+a_{n-1}(i,i-1)$. Hence, for all $2\leq i\leq n-1$,

$$b_n(i) = a_{n-1}(i,n-1) + a_{n-1}(i,1) + \ldots + a_{n-1}(i,i-1)$$
$$= a_{n-1}(i) - (a_{n-1}(i,i+1) + \ldots + a_{n-1}(i,n-2))$$
$$= a_{n-1}(i) - (n-2-i) r_{i-1} \delta_{i \le n-2}.$$

Since T contains 1432, $b_n(1) = 1$. Set

$$b_n = \sum_{i=1}^{n-1} a_n(i,n).$$

Then

$$b_n - 1 = a_{n-1} - \left(\binom{n-3}{2} + n - 3\right) - \sum_{i=2}^{n-2} (n-2-i) r_{i-1},$$

which, since $a_{n-1}(n-1) = a_{n-2}$, implies

$$b_n = a_{n-1} - \binom{n-2}{2} - \sum_{i=2}^{n-2} (n-2-i) r_{i-1}.$$

Hence,

$$a_n^+ = a_{n-1} + \sum_{j=2}^{n-1} (r_0 + r_1 + \ldots + r_{j-2}) - \sum_{i=2}^{n-2} (n-2-i) r_{i-1}.$$

which implies that $a_n^+ = a_{n-1} + n - 2 + r_1 + r_2 + \ldots + r_{n-3}$, a required.

Define $a_n^- = \sum_{i=2}^{n} \sum_{j=1}^{i-1} a_n(i,j)$.

Lemma 28. *We have*

$$\sum_{n \ge 2} a_n^- x^n = x(1 + C(x) - xC(x)) F_T(x) - x - xC(x).$$

Proof. Let us write an equation for a_n^-. Since $a_n(n) = a_n(n-1) = a_{n-1}$, we have

$$a_n^- = 2a_{n-1} + \sum_{i=2}^{n-2} \sum_{j=1}^{i-1} a_n(i,j).$$

Let $\pi = ij\pi' \in S_n(T)$ with $1 \le j < i \le n-2$. We claim

$$a_n(i,j) = a_{n-1}(i,j) + a_{n-1}(j,1) + \ldots + a_{n-1}(j,j-1), \quad (1)$$

for $1 \le j < i \le n-2$.

To prove (1), let $\pi = ij\pi' \in S_n(T)$ with $1 \le j < i \le n-2$. If $j = 1$, then π contains the subword $n(n-1)\ldots(i+1)$, so by removing the letter $n-1$, we obtain $a_n(i,1) = a_{n-1}(i,1)$, which implies that (1) holds for $j = 1$. Now, assume that $\pi = ij\pi' \in S_n(T)$ with $2 \le j < i \le n-2$. Note that $\pi = ijk\pi''$ avoids T if and only if $jk\pi''$ avoids T, for all $k < j$, So $a_n(i,j,k) = a_{n-1}(j,k)$, for all $1 \le k < j < i \le n-2$. Thus,

$$a_n(i,j) = \sum_{k=1}^{j-1} a_{n-1}(j,k) + \sum_{k=j+1}^{i-1} a_n(i,j,k) + a_n(i,j,n).$$

Since $\pi = ijk\pi'' \in S_n(T)$, $j+1 \le k \le i-1$, has the form

$$ijk(k-1)\ldots(j+1)\alpha n(n-1)\ldots(i+1)(i-1)\ldots(k+1)$$

the number such permutations is given by

$$d_{j-1} = |S_{j-1}(\{213, 1342, 2341, 1423\})|.$$

Thus,

$$a_n(i,j) = \sum_{k=1}^{j-1} a_{n-1}(j,k) + (i-1-j) d_{j-1} + a_n(i,j,n),$$

which leads to

$$a_n(i,j) - a_{n-1}(i,j)$$
$$= \sum_{k=1}^{j-1} a_{n-1}(j,k) a_n(i,j,n)$$
$$- \sum_{k=1}^{j-1} a_{n-2}(j,k) - a_{n-1}(i,j,n-1).$$

An avoider $\pi = ijn\pi' \in S_n(T)$ has the form $\pi = ijnk\pi''$ with $k < j$ or $\pi = ijn(n-1)\pi''$, and so

$$a_n(i,j,n) = \sum_{k=1}^{j-1} a_{n-2}(j,k) + a_{n-1}(i,j,n-1).$$

Hence,

$$a_n(i,j) = a_{n-1}(i,j) + \sum_{k=1}^{j-1} a_{n-1}(j,k),$$

as required.

Define $a_n^-(i) = \sum_{j=1}^{i-1} a_n(i,j)$. So (1) is equivalent to

$$a_n(i,j) = a_{n-1}(i,j) + a_{n-1}^-(i,j),$$

and by summing over $j = 1, 2, \ldots, i-1$, we have

$$a_n^-(i) = a_{n-1}^-(i) + a_{n-1}^-(1) + \ldots + a_{n-1}^-(i-1).$$

Also, $a_n^-(n) = a_{n-1}$ and $a_n^-(n-1) = a_{n-1} - a_{n-2}$. To solve this recurrence, we define $A_n^-(v) = \sum_{i=1}^{n} a_n^-(i) v^{i-1}$. Then, we have

$$A_n^-(v) = a_{n-1} v^{n-1} + (a_{n-1} - a_{n-2}) v^{n-2} + \frac{A_{n-1}^-(v) - v^{n-2} A_{n-1}^-(1)}{1-v} \quad (2)$$

with $A_2^-(v) = v$.

Define $A^-(x,v) = \sum_{n \ge 2} A_n^-(v) x^n$. Multiplying (2) by x^n / v^n and summing over n ≥ 3, we obtain

$$\left(1 - \frac{x}{v(1-v)}\right) A^-(x/v, v)$$
$$= x/v(F_T(x) - 1) + x/v^2(F_T(x) - 1)$$
$$- x^2/v^2 F_T(x) - \frac{x}{v^2(1-v)} A^-(x,1).$$

By taking $v = 1/C(x)$, we have

$$A^-(x,1) = x(1 + C(x) - xC(x)) F_T(x) - x - xC(x).$$

By Lemma 27,

$$\sum_{n\geq 2} a_n^+ x^n = x^2 + x\left(F_T(x) - 1 - x\right) + \frac{x^4}{(1-x)^2} + \frac{x^3}{1-x}R(x).$$

with $R(x) = 1 + \frac{x}{1-2x} + \frac{x^3}{(1-x)(1-2x)}$. Thus, by Lemma 28, we obtain

$$F_T(x) = 1 + x(1 + x + C(x) - xC(x))F_T(x) - xC(x) - x + \frac{x^4}{(1-x)^2} + \frac{x^3((1-x)^2 + x^3)}{(1-x)^2(1-2x)}.$$

Solving for $F_T(x)$, we get the following result.

Theorem 29. *Let T* = {2134, 1342, 2341, 1423}. *Then*

$$F_T(x) = \frac{x^3(1+x) + (1-2x)(1-x) - x(1-2x)C(x)}{(1-2x)\left(1-2x-x(1-x)\right)C(x)}.$$

2.11.2. The Symmetry Class of {2134, 1342, 2341, 1243}

Let T = {2134, 1342, 2341, 1243}. Define

$$b_n^+ = \sum_{i=1}^{n-1}\sum_{j=n+1}^{n} b_n(i,j)$$

and

$$b_n = \sum_{i=1}^{n-1}\sum_{j=1}^{n} b_n(i,j),$$

where $b_n(i_1, i_2, \ldots, i_s)$ is the number of permutations $\pi = i_1 i_2 \ldots i_s \pi'$ in $S_n(T)$.

Lemma 30. *For all* $n \geq 3$,

$$b_n^+ = n - 3 + b_{n-1} + r_0 + r_1 + \ldots + r_{n-3}.$$

Proof. The permutation $\pi = in\pi'$ in $S_n(T)$ if and only if $i\pi'$ in $S_{n-1}(T)$. Thus, $b_n(i,n) = b_{n-1}(i)$ for all $i = 1, 2, \ldots, n-1$. If $\pi = ij\pi' \in S_n$ with $2 \leq i < j \leq n-2$ and $(i,j) \neq (1,2)$, then π contains either 1243 or 2341, so $b_n(i,j) = 0$. If $\pi = 12\pi' \in S_n(T)$, then π' is increasing, and so $b_n(1,2) = 1$. If $\pi = 1j\pi' \in S_n(T)$ with $3 \leq j \leq n-2$, then π contains 2134, thus $b_n(1,j) = 0$. If $\pi = 1(n-1)\pi' \in S_n(T)$, then $\pi = 1(n-1)\pi''n$ where π'' avoids {213, 132, 2341}, so $b_n(1, n-1) = n-3$.

Now, assume that $\pi = i(n-1)\pi' \in S_n(T)$ with $2 \leq i \leq n-2$. Then $\pi = i(n-1)\ldots(i+1)\pi''n$, where π'' avoids {213, 1342, 2341, 1243}. Thus, $b_n(i, n-1) = r_{i-1}$ as defined in Sec. 2.11.1.

Hence,

$$b_n^+ = 1 + n - 3 + b_{n-1}(1) + \ldots + b_{n-1}(n-1) + r_1 + r_2 + \ldots + r_{n-3}.$$

Lemma 31. *We have*

$$\sum_{n\geq 2} b_n^- x^n = x\left(1 + C(x) - xC(x)\right)F_T(x) - x - xC(x).$$

Proof. Define $b_n^-(i) = \sum_{j=1}^{i=1} b_n(i,j)$ and

$$B_n^-(v) = \sum_{i=1}^{n} b_n^-(i)v^{i-1}.$$

By similar arguments as in the proof of Lemma 28, we obtain

$$b_n^-(i) = b_{n-1}^-(1) + \ldots + b_{n-1}^-(i),$$

with $b_{n-1}^-(n) = b_{n-1}$ and $b_n^-(n-1) = b_{n-1} - b_{n-2}$. So $B_2^-(v) = v$ and, for $n \geq 3$,

$$B_n^-(v) = b_{n-1}v^{n-1} + \left(b_{n-1} - b_{n-2}\right)v^{n-2} + \frac{B_{n-1}^-(v) + v^{n-2}B_{n-1}^-(1)}{1-v}. \quad (3)$$

Define $B^-(x,v) = \sum_{n\geq 2} B_n^-(v)x^n$. Multiplying (3) by x^n / v^n and summing over $n \geq 3$, we obtain

$$\left(1 - \frac{x}{v(1-v)}\right)B^-(x/v, v) = x/v\left(F_T(x) - 1\right) + x/v^2\left(F_T(x) - 1\right) - x^2/v^2 F_T(x) - \frac{x}{v^2(1-v)}B^-(x,1).$$

By taking $v = 1/C(x)$, we have

$$B^-(x,1) = x\left(1 + C(x) - xC(x)\right)F_T(x) - x - xC(x).$$

By Lemma 30, we have

$$\sum_{n\geq 2} b_n^+ x^n = x^2 + \frac{x^4}{(1-x)^2} + \frac{x^3}{1-x}R(x) + xF_T(x).$$

Thus, by Lemma 31, we obtain

$$F_T(x) = 1 + x^2 + \frac{x^4}{(1-x)^2} + \frac{x^3}{1-x}R(x) + x\left(2 + C(x) - xC(x)\right)F_T(x) - xC(x).$$

Solving for $F_T(x)$, we get the following result.

Theorem 32. *Let T* = {2134, 1342, 2341, 1243}. *Then*

$$F_T(x) = \frac{x^3(1+x) + (1-2x)(1-x) - x(1-2x)C(x)}{(1-2x)\left(1-2x-x(1-x)C(x)\right)}.$$

References

[1] T. Arıkan, E. Kılı¸c and T. Mansour, A Wilf class composed of 19 symmetry classes of quadruples of 4-letter patterns, *Notes on Number Theory and Discrete Mathematics. 23:3* (2017), 79-99.

[2] D. Callan, T. Mansour and M. Shattuck, Wilf classification of triples of 4-letter patterns I, *Discr. Math. Theor. Comput. Sci. 19* (2017) #5, 35pp.

[3] D. Callan, T. Mansour and M. Shattuck, Wilf classification of triples of 4-letter patterns II, *Discr. Math. Theor. Comput. Sci. 19* (2017) #6, 44pp.

[4] D. E. Knuth, *The Art of Computer Programming*, 3rd edition, Addison Wesley, Reading, MA, 1997.

[5] W. Kuszmaul, Fast algorithms for finding pattern avoiders and counting pattern occurrences in permutations, Preprint, arXiv:1509.08216, 2015.

[6] I. Le, Wilf classes of pairs of permutations of length 4, *Electron. J. Combin. 12* (2005), #R25.

[7] T. Mansour and M. Schork, Wilf classification of subsets of four letter patterns, *J. Combin. Number Theory 8* (2016), 1-129.

[8] T. Mansour and M. Schork, Wilf classification of subsets of eight and nine four-letter patterns, *J. Combin. Number Theory 8* (2016), 257-283.

[9] T. Mansour and M. Schork, Wilf classification of subsets of six and seven four-letter patterns, preprint.

[10] T. Mansour and A. Vainshtein, Restricted 132-avoiding permutations, *Adv. in Appl. Math. 26* (2001), 258-269.

[11] R. Simion and F. W. Schmidt, Restricted permutations, *European J. Combin. 6* (1985), 383-406.

[12] V. Vatter, Finding regular insertion encodings for permutation classes, *J. Symbolic Comput. 47:3* (2012), 259-265.

Appendix

Table 1. 2-Wilf classes of four 4-letter patterns

No.	Pattern set T	Generating function $F_T(x)$	Thm./[Ref]
227	$\{1342, 2314, 3412, 4231\}$ $\{1342, 2314, 3412, 2431\}$	$C(x)+\frac{x^3}{(1-x)^3}C^2(x)+\frac{x^4}{(1-x)^3(1-2x)}$	Thm. 3 Thm. 5
261	$\{1342, 2314, 2413, 4231\}$ $\{1324, 1342, 2143, 2431\}$	$C(x)+\frac{x^3}{(1-x)^2}C^3(x)+\frac{x^4}{(1-x)^2(1-2x)}C^2(x)$	Thm. 6 Thm. 7
398	$\{2341, 4123, 1342, 1243\}$ $\{2341, 4123, 1342, 1234\}$	$C(x)+\frac{x}{(1-x)^2}(C(x)-1-xC(x))+\frac{x^4}{(1-2x)(1-x)^4}$	Thm. 11 Thm. 12
430	$\{2314, 4312, 3412, 1342\}$ $\{2314, 4312, 3142, 1342\}$	$\frac{x^8-14x^6+37x^5-63x^4+62x^3-33x^2+9x-1}{(x^2-3x+1)(2x-1)^2(x-1)^3}$	INSENC INSENC
436	$\{2314, 4312, 1432, 1324\}$ $\{4312, 3124, 1342, 1234\}$	$\frac{-(x^6-9x^4+11x^3-11x^2+5x-1)}{(x-1)^6}$	INSENC INSENC
439	$\{2314, 4312, 1432, 1243\}$ $\{2314, 4312, 1342, 1234\}$	$\frac{2x^7-4x^6-2x^5+9x^4-11x^3+11x^2-5x+1}{(x-1)^6}$	INSENC INSENC
443	$\{2314, 4312, 1342, 1243\}$ $\{2134, 4312, 3124, 1432\}$	$\frac{x^7-4x^6-x^5+9x^4-11x^3+11x^2-5x+1}{(x-1)^6}$	INSENC INSENC
457	$\{2314, 3124, 1342, 1423\}$ $\{2413, 3142, 1324, 1243\}$	$\frac{C(x)}{1-\frac{x^3C(x)}{(1-x)^2(1-2x)}}$	Thm. 13 Thm. 14
464	$\{2314, 4132, 1342, 1423\}$ $\{2134, 3142, 4132, 1432\}$	$\frac{x^6-6x^5+15x^4-24x^3+19x^2-7x+1}{(2x-1)(x^2-3x+1)(x-1)^3}$	INSENC INSENC
471	$\{2314, 1342, 1324, 4123\}$ $\{2314, 1342, 4123, 1234\}$	$C(x)+\frac{x^3}{(1-x)^5}C(x)$	Thm. 15 Thm. 16
475	$\{4213, 2413, 2134, 1342\}$ $\{3142, 2314, 2341, 1423\}$	$\frac{-(3x^5-10x^4+20x^3-18x^2+7x-1)}{(x^2-3x+1)^2(x-1)^2}$	INSENC EX
512	$\{4213, 4312, 3412, 1342\}$ $\{4213, 3412, 1342, 4123\}$	$\frac{x^5+2x^4-3x^3+5x^2-4x+1}{(3x-1)(x-1)(x^2-x+1)}$	INSENC INSENC
513	$\{4213, 4312, 3142, 1342\}$ $\{4213, 3412, 4132, 1342\}$	$\frac{x^5-5x^4+8x^3-9x^2+5x-1}{(x-1)^2(3x-1)(x^2-x+1)}$	INSENC INSENC
530	$\{4213, 3412, 1342, 1423\}$ $\{2413, 4312, 1342, 4123\}$	$\frac{x^5-4x^4+7x^3-9x^2+5x-1}{(2x^3-4x^2+4x-1)(x-1)^2}$	INSENC INSENC
582	$\{2413, 2143, 3142, 1234\}$ $\{2134, 3124, 1432, 1243\}$	$\frac{-(x^3+2x-1)(x-1)}{(3x-1)(x^2-x+1)}$	INSENC INSENC
587	$\{2413, 2134, 4312, 1432\}$ $\{2134, 4312, 3142, 1243\}$	$\frac{-(3x^5+5x^4-4x^3+7x^2-4x+1)}{(x-1)^5}$	INSENC INSENC
588	$\{2413, 2134, 4312, 1342\}$ $\{3412, 3142, 4132, 1234\}$	$\frac{-(x^6-9x^5+20x^4-22x^3+16x^2-6x+1)}{(x-1)^7}$	INSENC INSENC
595	$\{3412, 3142, 4132, 1234\}$ $\{3142, 4123, 1243, 1234\}$	$\frac{x^8-7x^7+31x^6-74x^5+106x^4-88x^3+41x^2-10x+1}{(2x-1)(x^2-3x+1)^2(x-1)^3}$	INSENC INSENC
604	$\{2413, 4312, 3142, 1234\}$ $\{2143, 2134, 4312, 1243\}$	$\frac{-(2x^7+x^6+4x^5+5x^4-4x^3+7x^2-4x+1)}{(x-1)^5}$	INSENC INSENC
605	$\{2413, 4312, 3124, 1432\}$ $\{3412, 3124, 1432, 1423\}$	$\frac{-(3x^6-13x^5+24x^4-29x^3+20x^2-7x+1)}{(x^2-3x+1)(x-1)^5}$	INSENC INSENC
608	$\{2413, 4312, 1432, 1342\}$ $\{3124, 4132, 1432, 1342\}$	$\frac{-(x^5+x^4-7x^2+5x-1)}{(x-1)(x^2-3x+1)(x^2+2x-1)}$	INSENC INSENC
613	$\{2413, 3412, 3142, 2341\}$ $\{2413, 3142, 1432, 1243\}$	$\frac{1-t-\sqrt{(1-t)^2-4x}}{2x}, t=\frac{x^3}{(1-x)^3}$	Thm. 17 Thm. 18
621	$\{2413, 3142, 3124, 1342\}$ $\{2143, 2134, 3142, 1243\}$	$\frac{-(x^2-3x+1)^2}{(x-1)(x^4-4x^3+10x^2-6x+1)}$	INSENC INSENC
627	$\{2413, 3142, 4132, 1234\}$ $\{2134, 3124, 1432, 4123\}$	$\frac{x^5-3x^4+6x^3-9x^2+5x-1}{(x^2-3x+1)(x-1)^3}$	INSENC INSENC
629	$\{2413, 3142, 1432, 4123\}$ $\{2134, 1432, 1243, 1234\}$	$\frac{2x^3+x^2+2x-1}{2x^3+3x-1}$	INSENC INSENC

No.	Pattern set T	Generating function $F_T(x)$	Thm./[Ref]
633	{2413, 3142, 4123, 1423} {2143, 1342, 1324, 1423}	$\frac{(1-x)^2-x^2C(x)}{(1-x)^2-x(1-x+x^2)C(x)}$	Thm. 19 Thm. 20
638	{2413, 3124, 4132, 1432} {2413, 3124, 4132, 1342}	$\frac{-(x^6+3x^5-10x^4+20x^3-18x^2+7x-1)}{(x-1)^2(x^2-3x+1)^2}$	INSENC INSENC
639	{3142, 2314, 2431, 1342} {2413, 2431, 2314, 1342}	$\frac{(1-4x+3x^2+x^3+x^2(1-x)^2C(x))C(x)}{(1-x)(1-3x+x^2)}$	Thm. 21 Thm. 22
644	{2413, 3124, 1432, 4123} {3142, 1432, 1324, 4123}	$\frac{2x^5-3x^4+4x^3-8x^2+5x-1}{(x^3-2x^2+3x-1)(x^2-3x+1)}$	INSENC INSENC
677	{2143, 2134, 1342, 4123} {2143, 3124, 1342, 4123}	$\frac{3x^5-4x^4+11x^3-13x^2+6x-1}{(x^2-3x+1)(2x-1)(x-1)^2}$	INSENC INSENC
708	{2143, 3412, 3124, 1243} {3412, 3124, 1432, 1324}	$\frac{-(11x^6-31x^5+49x^4-48x^3+27x^2-8x+1)}{(x-1)^5(2x-1)^2}$	INSENC INSENC
709	{2143, 3412, 2314, 2431} {4312, 3412, 3124, 1432}	$\frac{-(x^6-3x^5+4x^4-11x^3+13x^2-6x+1)}{(x-1)^2(x^2-3x+1)(2x-1)}$	EX INSENC
710	{2143, 3412, 4132, 1324} {2143, 3412, 1432, 1324}	$\frac{2x^6-9x^5+15x^4-24x^3+19x^2-7x+1}{(1-2x)(1-3x+x^2)(1-x)^3}$	EX EX
715	{2143, 3412, 1342, 4123} {3412, 1432, 4123, 1243}	$\frac{x^5-5x^4+8x^3-10x^2+5x-1}{(2x-1)(x-1)^4}$	INSENC INSENC
719	{2143, 3412, 2341, 1342} {3412, 4132, 1432, 1243}	$\frac{x^8-x^7-13x^6+37x^5-63x^4+62x^3-33x^2+9x-1}{(1-2x)^2(x^2-3x+1)(x-1)^3}$	EX EX
725	{2143, 3142, 1432, 4123} {2143, 4132, 1342, 4123}	$\frac{x^6-3x^5+5x^4-7x^3+9x^2-5x+1}{(x^3-x^2+3x-1)(x-1)^3}$	INSENC INSENC
728	{2143, 3142, 1342, 1234} {4312, 3142, 1342, 4123}	$\frac{(x^3-2x^2+3x-1)(x-1)}{(2x^4-7x^3+8x^2-5x+1)}$	INSENC INSENC
732	{3412, 3421, 2413, 3241} {3142, 1342, 1423, 1243}	$C\left(\frac{x(1-x)^2}{1-2x}\right)$	Thm. 23 Thm. 24
742	{2143, 3124, 1432, 1243} {3124, 1432, 1342, 1243}	$\frac{-(x^2-3x+1)}{x^4+x^3-3x^2+4x-1}$	INSENC INSENC
748	{2143, 3124, 4123, 1243} {3124, 1432, 1324, 4123}	$\frac{-(7x^3+6x-1-12x^2+2x^5)}{(x^2-3x+1)(2x-1)^2}$	INSENC INSENC
772	{2143, 1324, 4123, 1243} {2134, 2413, 1324, 2341}	$\frac{(2x-1)(2x^4-7x^2+5x-1)}{(x-1)(x^2+x-1)(x^2-3x+1)^2}$	INSENC EX
773	{2143, 4123, 1423, 1234} {2134, 3142, 4132, 1243}	$\frac{x^6-3x^5+2x^4-8x^3+12x^2-6x+1}{(1-x)(x^2-3x+1)^2}$	INSENC INSENC
781	{2134, 4312, 3142, 1234} {4312, 3142, 1243, 1234}	$\frac{x^8+x^7+2x^6-4x^5-5x^4+4x^3-7x^2+4x-1}{(x-1)^5}$	INSENC INSENC
787	{2134, 4312, 1432, 1342} {4312, 3142, 1324, 1234}	$\frac{x^8+x^7-4x^5-5x^4+4x^3-7x^2+4x-1}{(x-1)^5}$	INSENC INSENC
801	{2134, 3412, 3142, 4132} {4312, 3142, 1432, 1243}	$\frac{3x^5-12x^4+23x^3-19x^2+7x-1}{(2x-1)^3(x-1)^2}$	INSENC INSENC
802	{2134, 3412, 3142, 1432} {2134, 3412, 1432, 1423}	$\frac{(x^2-x+1)(3x^4-7x^3+10x^2-5x+1)}{(1-x)^7}$	INSENC INSENC
806	{2134, 3412, 4132, 1432} {3412, 4132, 1432, 1234}	$\frac{x^7+x^6-2x^5-5x^4+4x^3-7x^2+4x-1}{(1-x)^5}$	INSENC INSENC
822	{2134, 3142, 4132, 1423} {2134, 3142, 4123, 1423}	$\frac{-(x^5+7x^3-12x^2+6x-1)}{(2x-1)^2(x^2-3x+1)}$	INSENC EX
830	{2134, 3142, 1324, 1243} {2134, 3142, 1243, 1234}	$\frac{(2x-1)^3}{(x^4-6x^2+5x-1)(x-1)^2}$	INSNEC INSENC
842	{2134, 2314, 2341, 1423} {1234, 2314, 2341, 1423}	$\frac{(1-x)(1-x+x^2)C(x)-x}{(1-x)^3}$	Thm. 25 Thm. 26
871	{2134, 1432, 1423, 1243} {3142, 1432, 1342, 4123}	$\frac{x^4-x^3+x^2-3x+1}{x^4-2x^3+3x^2-4x+1}$	INSENC INSENC
874	{2134, 1342, 4123, 1423} {2134, 4123, 1423, 1243}	$\frac{x^3(1+x)+(1-2x)(1-x)-x(1-2x)C(x)}{(1-2x)(1-2x-x(1-x)C(x))}$	Thm. 29 Thm. 32
880	{4312, 3412, 3142, 1432} {3412, 3142, 4132, 1432}	$\frac{6x-11x^2-2x^4+6x^3+x^5-1}{(x^2-3x+1)(x^3-3x^2+4x-1)}$	INSENC EX
888	{4312, 3412, 4132, 1342} {4312, 3142, 4132, 1342}	$\frac{x^4-2x^3+7x^2-5x+1}{(2x-1)(x^3-3x^2+4x-1)}$	INSENC INSENC
900	{4312, 3412, 1342, 4123} {3412, 4132, 1342, 4123}	$\frac{x^6-4x^5+12x^4-17x^3+14x^2-6x+1}{(x-1)^2(2x^4-7x^3+8x^2-5x+1)}$	INSENC INSENC
915	{4312, 3142, 1432, 1324} {4312, 3142, 1342, 1324}	$\frac{4x^6-15x^5+35x^4-42x^3+26x^2-8x+1}{(x-1)^3(2x-1)^3}$	INSENC INSENC
917	{4312, 3142, 1432, 1423} {4312, 3142, 1342, 1423}	$\frac{x^6-4x^5+13x^4-21x^3+18x^2-7x+1}{(x-1)(x^2-3x+1)(2x^3-4x^2+4x-1)}$	INSENC INSENC
926	{4312, 3142, 1423, 1243} {4312, 4132, 1324, 4123}	$\frac{x^8-2x^7-14x^6+49x^5-77x^4+68x^3-34x^2+9x-1}{(2x-1)^3(x-1)^4}$	INSENC EX

No.	Pattern set T	Generating function $F_T(x)$	Thm./[Ref]
934	$\{4312, 3124, 1432, 4123\}$ $\{3412, 3124, 4132, 1342\}$	$\frac{2x^7-10x^6+22x^5-39x^4+43x^3-26x^2+8x-1}{(2x-1)(x^2-3x+1)(x-1)^4}$	INSENC INSENC
954	$\{4312, 4132, 1324, 1243\}$ $\{4312, 1432, 1324, 1243\}$	$\frac{2x^9-14x^8-x^7+60x^6-126x^5+145x^4-102x^3+43x^2-10x+1}{(2x-1)^3(x-1)^5}$	INSENC INSENC
976	$\{4312, 1342, 4123, 1423\}$ $\{4132, 1342, 4123, 1243\}$	$\frac{(x^2-x+1)(3x^3-8x^2+5x-1)}{(1-x)^3(2x^3-4x^2+4x-1)}$	INSENC INSENC
998	$\{3412, 3142, 1324, 1243\}$ $\{3412, 4132, 1423, 1243\}$	$\frac{-9x^7+46x^6-99x^5+125x^4-95x^3+42x^2-10x+1}{(x^2-3x+1)(1-2x)^2(1-x)^4}$	EX EX
1006	$\{3412, 3124, 1324, 1243\}$ $\{3412, 1324, 1423, 1234\}$	$\frac{-7x^7+34x^6-77x^5+97x^4-75x^3+35x^2-9x+1)}{(1-2x)^2(1-x)^6}$	EX EX
1045	$\{1324, 1342, 2341, 2413\}$ $\{3124, 1432, 4123, 1423\}$	$\frac{x^5-3x^4+15x^3-17x^2+7x-1}{(x^2-3x+1)^2(2x-1)}$	EX INSENC