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On Generalization of Dragomir's Inequalities

Hüseyin Budak*, Mehmet Zeki Sarıkaya

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey
*Corresponding author: hsyn.budak@gmail.com

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Abstract In this paper, we establish some generalization of weighted Ostrowski type integral inequalities for functions of bounded variation.

Keywords: *Function of bounded variation, Ostrowski type inequalities, Riemann-Stieltjes integrals*

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1. Introduction

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)|<\infty$. Then we have the inequality

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\\&\le\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty,\end{aligned}\tag{1}$$

for all $x\in[a,b]$ [16]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.*

Definition 1. Let $P\ :\ a=x_0<x_1<\ldots<x_n=b$ be any partition of $[a,b]$ and let $\Delta f(x_i)=f(x_{i+1})-f(x_i)$ Then $f(x)$ is said to be of bounded variation if the sum

$$\sum_{i=1}^n|\Delta f(x_i)|$$

is bounded for all such partitions.

Let f be of bounded variation on $[a,b]$, and $\sum(P)$ denotes the sum $\sum_{i=1}^n|\Delta f(x_i)|$ corresponding to the partition P of $[a,b]$. The number

$$\bigvee_a^b(f) := \sup\left\{\sum(P)\ :\ P\in P([a,b])\right\},$$

is called the total variation of f on $[a,b]$. Here $P([a,b])$ denotes the family of partitions of $[a,b]$.

In [7], Dragomir proved following Ostrowski type inequalities related functions of bounded variation:

Theorem 1. Let $f\ :\ [a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then

$$\begin{aligned}&\left|\int_a^b f(t)dt-(b-a)f(x)\right|\\&\le\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f)\end{aligned}$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

In [9], Dragomir gave a simple proof of following Lemma:

Lemma 1. Let $f,u\ :\ [a,b]\to\mathbb{R}$. If f is continious on $[a,b]$ and u is bounded variation on $[a,b]$, then

$$\left|\int_a^b f(t)du(t)\right|\le\int_a^b|f(t)|\,d\left(\bigvee_a^t(u)\right)\le\max_{t\in[a,b]}|f(t)|\bigvee_a^b(u).$$

In [5], Dragomir obtained following Ostrowski type inequality for functions of bounded variation:

Theorem 2. Let $I_k\ :\ a=x_0<x_1<\ldots<x_k=b$ be a division of the interval $[a,b]$ and $\alpha_i\ (i=0,1,\ldots,k+1)$ be $k+2$ points so that $\alpha_0=a$, $\alpha_i\in[x_{i-1},x_i]$ $(i=1,\ldots,k)$, $\alpha_{k+1}=b$. If $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$, then we have the inequality:

$$\begin{aligned}&\left|\int_a^b f(x)dx-\sum_{i=0}^k(\alpha_{i+1}-\alpha_i)f(x_i)\right|\\&\le\left[\frac{1}{2}\upsilon(h)+\max_{i\in\{0,1,\ldots,k-1\}}\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right]\bigvee_a^b(f)\\&\le\upsilon(h)\bigvee_a^b(f)\end{aligned}\tag{2}$$

where $\upsilon(h):=\max\{h_i \mid i=0,...,n-1\}$, $h_i := x_{i+1}-x_i$ $(i=0,1,...,k-1)$ and $\bigvee_a^b(f)$ is the total variation of f on the interval $[a,b]$.

For some recent results connected with functions of bounded variation see [1,2,3,4,6,8,10-15,17-21].

The aim of this paper is to obtain some generalization of weighted Ostrowski type integral inequalities for functions of bounded variation.

2. Main Results

Firstly, we will give the following notations which are used in main Theorem:

Let $I_n : a=x_0<x_1<...<x_n=b$ be a partition of the interval $[a,b]$, $\alpha_i\,(i=0,1,...,n+1)$ be $n+2$ points so that $\alpha_0=a$, $\alpha_i\in[x_{i-1},x_i]$ $(i=1,...,n)$, $\alpha_{n+1}=b$. Let $w:[a,b]\to(0,\infty)$ be continious and positive mapping on (a,b), and

$$\upsilon(h):=\max\{h_i \mid i=0,...,n-1\},$$
$$h_i := x_{i+1}-x_i \ (i=0,1,...,n-1),$$
$$\upsilon(L):=\max\{L_i \mid i=0,...,n-1\},$$
$$L_i=\int_{x_i}^{x_{i+1}} w(u)du \ (i=0,1,...,n-1).$$

Theorem 3. If $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$, then we have the inequalities

$$\left|\sum_{i=0}^{n}\left(\int_{\alpha_i}^{\alpha_{i+1}} w(u)du\right)f(x_i)-\int_a^b f(t)w(t)dt\right|$$
$$\le \|w\|_{\infty,[a,b]}\left[\frac{1}{2}\upsilon(h)+\max_{i\in\{0,1,...,k-1\}}\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right]\bigvee_a^b(f) \quad (3)$$
$$\le \|w\|_{\infty,[a,b]}\,\upsilon(h)\bigvee_a^b(f)$$

and

$$\left|\sum_{i=0}^{n}\left(\int_{\alpha_i}^{\alpha_{i+1}} w(u)du\right)f(x_i)-\int_a^b f(t)w(t)dt\right|$$
$$\le\left[\frac{1}{2}v(L)+\max_{i\in\{0,1,...,n-1\}}\frac{1}{2}\left|\int_{x_i}^{\alpha_{i+1}} w(u)du-\int_{\alpha_{i+1}}^{x_{i+1}} w(u)du\right|\right]\bigvee_a^b(f) \quad (4)$$
$$\le v(L)\bigvee_a^b(f)$$

where $\bigvee_a^b(f)$ is the total variation of f on the interval $[a,b]$.

Proof. Let us consider the functions K defined by

$$K(t)=\begin{cases}\int_{\alpha_1}^{t} w(u)du, & t\in[a,x_1)\\ \int_{\alpha_2}^{t} w(u)du, & t\in[x_1,x_2)\\ \vdots & \\ \int_{\alpha_{n-1}}^{t} w(u)du, & t\in[x_{n-2},x_{n-1})\\ \int_{\alpha_n}^{t} w(u)du, & t\in[x_{n-1},b].\end{cases}$$

Integrating by parts , we obtain

$$\begin{aligned}&\int_a^b K(t)df(t)\\
&=\sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+1}} K(t)df(t)\right]\\
&=\sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+1}}\left(\int_{\alpha_{i+1}}^{t} w(u)du\right)df(t)\right]\\
&=\sum_{i=0}^{n-1}\left[\left(\int_{\alpha_{i+1}}^{x_{i+1}} w(u)du\right)f(x_{i+1})\right.\\
&\quad\left.+\left(\int_{x_i}^{\alpha_{i+1}} w(u)du\right)f(x_i)-\int_{x_i}^{x_{i+1}} f(t)w(t)dt\right]\\
&=\sum_{i=1}^{n}\left(\int_{\alpha_i}^{x_i} w(u)du\right)f(x_i)+\sum_{i=0}^{n-1}\left(\int_{x_i}^{\alpha_{i+1}} w(u)du\right)f(x_i)\\
&\quad-\int_a^b f(t)w(t)dt.\end{aligned} \quad (5)$$

In last equality in (5), we have

$$\begin{aligned}&\sum_{i=1}^{n}\left(\int_{\alpha_i}^{x_i} w(u)du\right)f(x_i)\\
&=\left(\int_{\alpha_n}^{b} w(u)du\right)f(b)+\sum_{i=1}^{n-1}\left(\int_{\alpha_i}^{x_i} w(u)du\right)f(x_i),\end{aligned} \quad (6)$$

and similarly

$$\begin{aligned}&\sum_{i=0}^{n-1}\left(\int_{x_i}^{\alpha_{i+1}} w(u)du\right)f(x_i)\\
&=\left(\int_{a}^{\alpha_1} w(u)du\right)f(a)+\sum_{i=1}^{n-1}\left(\int_{x_i}^{\alpha_{i+1}} w(u)du\right)f(x_i).\end{aligned} \quad (7)$$

Adding (6) and (7) in (5), we get the equality

$$\begin{aligned}
&\int_a^b K(t)df(t)\\
&=\left(\int_{\alpha_n}^{b} w(u)du\right)f(b)+\sum_{i=1}^{n-1}\left(\int_{\alpha_i}^{\alpha_{i+1}} w(u)du\right)f(x_i)\\
&+\left(\int_a^{\alpha_1} w(u)du\right)f(a)-\int_a^b f(t)w(t)dt\\
&=\sum_{i=0}^{n}\left(\int_{\alpha_i}^{\alpha_{i+1}} w(u)du\right)f(x_i)-\int_a^b f(t)w(t)dt.
\end{aligned} \tag{8}$$

On the other hand, taking modulus in (8) and using triangle inequality we have

$$\begin{aligned}
&\left|\sum_{i=0}^{n}\left(\int_{\alpha_i}^{\alpha_{i+1}} w(u)du\right)f(x_i)-\int_a^b f(t)w(t)dt\right|\\
&=\left|\int_a^b K(t)df(t)\right|\\
&=\left|\sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+1}}\left(\int_{\alpha_{i+1}}^{t} w(u)du\right)df(t)\right]\right|\\
&\le\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}\left(\int_{\alpha_{i+1}}^{t} w(u)du\right)df(t)\right|\\
&\le\|w\|_{\infty,[a,b]}\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}\left(t-\alpha_{i+1}\right)df(t)\right|
\end{aligned} \tag{9}$$

Using Lemma 1 in last inequality in (9), we have

$$\begin{aligned}
&\left|\int_{x_i}^{x_{i+1}}\left(t-\alpha_{i+1}\right)df(t)\right|\\
&\le\sup_{t\in[x_i,x_{i+1}]}\left|t-\alpha_{i+1}\right|\bigvee_{x_i}^{x_{i+1}}(f)\\
&=\max\left\{\alpha_{i+1}-x_i,x_{i+1}-\alpha_{i+1}\right\}\bigvee_{x_i}^{x_{i+1}}(f)\\
&=\left[\frac{1}{2}\left(x_{i+1}-x_i\right)+\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right]\bigvee_{x_i}^{x_{i+1}}(f).
\end{aligned} \tag{10}$$

Putting (10) in (9), we obtain

$$\begin{aligned}
&\left|\sum_{i=0}^{n}\left(\int_{\alpha_i}^{\alpha_{i+1}} w(u)du\right)f(x_i)-\int_a^b f(t)w(t)dt.\right|\\
&\le\|w\|_{\infty,[a,b]}\sum_{i=0}^{n-1}\left[\begin{matrix}\frac{1}{2}\left(x_{i+1}-x_i\right)\\+\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\end{matrix}\right]\bigvee_{x_i}^{x_{i+1}}(f)\\
&\le\|w\|_{\infty,[a,b]}\\
&\times\max_{i\in[0,\dots,n-1]}\left\{\begin{matrix}\frac{1}{2}\left(x_{i+1}-x_i\right)\\+\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\end{matrix}\right\}\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(f)\\
&\le\|w\|_{\infty,[a,b]}\left[\begin{matrix}\frac{1}{2}\upsilon(h)\\+\max_{i\in[0,\dots,n-1]}\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\end{matrix}\right]\bigvee_a^b(f).
\end{aligned} \tag{11}$$

This completes the proof of first inequality in (3).

On the other hand, in last inequality in (11), we have

$$\begin{aligned}
&\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\le\frac{1}{2}h_i\\
&\text{and}\max_{i\in[0,\dots,n-1]}\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\le\frac{1}{2}\upsilon(h).
\end{aligned} \tag{12}$$

Adding (12) in last inequality in (11), we obtain the inequality (3).

Finally, for proof of inequality (4), taking modulus in (8), we have

$$\begin{aligned}
&\left|\sum_{i=0}^{n}\left(\int_{\alpha_i}^{\alpha_{i+1}} w(u)du\right)f(x_i)-\int_a^b f(t)w(t)dt.\right|\\
&=\left|\int_a^b K(t)df(t)\right|\\
&=\left|\sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+1}}\left(\int_{\alpha_{i+1}}^{t} w(u)du\right)df(t)\right]\right|\\
&\le\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}\left(\int_{\alpha_{i+1}}^{t} w(u)du\right)df(t)\right|.
\end{aligned} \tag{13}$$

Using Lemma 1 for the last integral of (13), we have

$$\begin{aligned}
&\left|\int_{x_i}^{x_{i+1}}\left(\int_{\alpha_{i+1}}^{t} w(u)du\right)df(t)\right|\\
&\le\sup_{t\in[x_i,x_{i+1}]}\left|\int_{\alpha_{i+1}}^{t} w(u)du\right|\bigvee_{x_i}^{x_{i+1}}(f)\\
&=\max\left\{\int_{x_i}^{\alpha_{i+1}} w(u)du,\int_{\alpha_{i+1}}^{x_{i+1}} w(u)du\right\}\bigvee_{x_i}^{x_{i+1}}(f)\\
&=\left[\begin{matrix}\frac{1}{2}\int_{x_i}^{x_{i+1}} w(u)du\\+\frac{1}{2}\left|\int_{x_i}^{\alpha_{i+1}} w(u)du-\int_{\alpha_{i+1}}^{x_{i+1}} w(u)du\right|\end{matrix}\right]\bigvee_{x_i}^{x_{i+1}}(f).
\end{aligned} \tag{14}$$

Adding (14) in (13), we obtain

$$\left|\sum_{i=0}^{n}\left(\int_{\alpha_i}^{\alpha_{i+1}} w(u)du\right) f(x_i) - \int_a^b f(t)w(t)dt.\right|$$

$$\leq \sum_{i=0}^{n-1}\left[\begin{array}{l}\frac{1}{2}\int_{x_i}^{x_{i+1}} w(u)du \\ +\frac{1}{2}\left|\int_{x_i}^{\alpha_{i+1}} w(u)du - \int_{\alpha_{i+1}}^{x_{i+1}} w(u)du\right|\end{array}\right]\bigvee_{x_i}^{x_{i+1}}(f) \quad (15)$$

$$\leq \max_{i\in\{0,1,...,n-1\}}\left\{\begin{array}{l}\frac{1}{2}\int_{x_i}^{x_{i+1}} w(u)du \\ +\frac{1}{2}\left|\int_{x_i}^{\alpha_{i+1}} w(u)du - \int_{\alpha_{i+1}}^{x_{i+1}} w(u)du\right|\end{array}\right\}\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(f)$$

$$\leq \left[\begin{array}{l}\frac{1}{2}v(L) \\ +\max_{i\in\{0,1,...,n-1\}}\frac{1}{2}\left|\int_{x_i}^{\alpha_{i+1}} w(u)du - \int_{\alpha_{i+1}}^{x_{i+1}} w(u)du\right|\end{array}\right]\bigvee_a^b(f)$$

which completes the proof of first inequality in (4).

Using triangle inequality in last inequality in (15), we have

$$\left|\int_{x_i}^{\alpha_{i+1}} w(u)du - \int_{\alpha_{i+1}}^{x_{i+1}} w(u)du\right|$$
$$\leq \left|\int_{x_i}^{\alpha_{i+1}} w(u)du\right| + \left|\int_{\alpha_{i+1}}^{x_{i+1}} w(u)du\right|$$
$$= \int_{x_i}^{\alpha_{i+1}} w(u)du + \int_{\alpha_{i+1}}^{x_{i+1}} w(u)du = \int_{x_i}^{x_{i+1}} w(u)du$$

and

$$\max_{i\in\{0,1,...,n-1\}}\frac{1}{2}\left|\int_{x_i}^{\alpha_{i+1}} w(u)du - \int_{\alpha_{i+1}}^{x_{i+1}} w(u)du\right| \leq \frac{1}{2}v(L).$$

This completes the proof.

Remark 1. Under assumptions Theorem 3 with $w(u)=1$, the inequality (3) reduces inequality (2).

Remark 2. If $w(u)=h'(u)$ (differantiable with respect to u) in Theorem 3, then we have the inequality

$$\left|\sum_{i=0}^{n}\left(\int_{\alpha_i}^{\alpha_{i+1}} w(u)du\right) f(x_i) - \int_a^b f(t)w(t)dt\right|$$
$$\leq \left[\frac{1}{2}v(L) + \max_{i\in\{0,1,...,n-1\}}\left|h(\alpha_{i+1}) - \frac{x_i + x_{i+1}}{2}\right|\right]\bigvee_a^b(f) \quad (16)$$
$$\leq v(L)\bigvee_a^b(f)$$

which was proved by Kuei-Lin Tseng et al. in [20].

Remark 3. If we choose $w(u)=1$, $h(u)=u$ in (16), inequality reduces inequality (2).

Corollary 1. Under assumption Theorem 3, choosing $x_0=a, x_1=b$, $\alpha_0=a$, $\alpha_1=\alpha, \alpha_2=b$ in inequality (4) we obtain the inequality

$$\left|\begin{array}{l}\left(\int_a^\alpha w(u)du\right) f(a) \\ +\left(\int_\alpha^b w(u)du\right) f(b) - \int_a^b f(t)w(t)dt\end{array}\right|$$
$$\leq \left[\begin{array}{l}\frac{1}{2}\int_a^b w(u)du \\ +\frac{1}{2}\left|\int_a^\alpha w(u)du - \int_\alpha^b w(u)du\right|\end{array}\right]\bigvee_a^b(f) \quad (17)$$
$$\leq \left(\int_a^b w(u)du\right)\bigvee_a^b(f).$$

Remark 4.

1) In (17), if we take $\alpha=b$, then we have the "weighted left rectangle inequality"

$$\left|\left(\int_a^b w(u)du\right) f(a) - \int_a^b f(t)w(t)dt\right|$$
$$\leq \left(\int_a^b w(u)du\right)\bigvee_a^b(f).$$

2) If we take $\alpha=a$ in (17) then we have the "weighted right rectangle inequality"

$$\left|\left(\int_a^b w(u)du\right) f(b) - \int_a^b f(t)w(t)dt\right|$$
$$\leq \left(\int_a^b w(u)du\right)\bigvee_a^b(f).$$

3. Applications for Quadrature Rule

Let us consider the arbitrary division

$$I_n : a = x_0 < x_1 < ... < x_n = b$$

and let $w:[a,b]\to(0,\infty)$ be continious function with

$$\upsilon(L) := \max\{L_i |\ i=0,...,n-1\},$$
$$L_i = \int_{x_i}^{x_{i+1}} w(u)du \ \ (i=0,1,...,n-1).$$

Then the following Theorem holds.

Theorem 4. Let $f:Q\to\mathbb{R}$ is of bounded variatin on Q and $\xi_i\in[x_i,x_{i+1}]$ $(i=0,...,n-1)$. Then we have the quadrature formula:

$$\int_a^b f(t)w(t)dt = \sum_{i=0}^{n-1}\left(\int_{x_i}^{\xi_i} w(u)du\right)f(x_i)$$
$$+\sum_{i=0}^{n-1}\left(\int_{\xi_i}^{x_{i+1}} w(u)du\right)f(x_{i+1}) + R_w(I_n, f, w, \xi).$$

The remainder term $R_w(I_n, f, w, \xi)$ satisfies

$$\left|R_w(I_n, f, w, \xi)\right|$$
$$\leq \left[\frac{1}{2}\upsilon(L) + \max_{i\in\{0,1,\ldots,n-1\}}\frac{1}{2}\left|\int_{x_i}^{\xi_i} w(u)du - \int_{\xi_i}^{x_{i+1}} w(u)du\right|\right]\bigvee_a^b(f) \quad (18)$$
$$\leq \upsilon(L)\bigvee_a^b(f).$$

Proof. Applying Corollary 1 to interval $[x_i, x_{i+1}]$, we have the inequality

$$\left|\begin{array}{l}\left(\int_{x_i}^{\xi_i} w(u)du\right)f(x_i) \\ +\left(\int_{\xi_i}^{x_{i+1}} w(u)du\right)f(x_{i+1}) - \int_a^{x_{i+1}} f(t)w(t)dt\end{array}\right|$$
$$\leq \left[\begin{array}{l}\frac{1}{2}\int_{x_i}^{x_{i+1}} w(u)du \\ +\frac{1}{2}\left|\int_{x_i}^{\xi_i} w(u)du - \int_{\xi_i}^{x_{i+1}} w(u)du\right|\end{array}\right]\bigvee_{x_i}^{x_{i+1}}(f) \quad (19)$$

Summing the inequality (19) over i from 0 to $n-1$, then we have

$$\left|R_w(I_n, f, w, \xi)\right|$$
$$\leq \sum_{i=0}^{n-1}\left[\begin{array}{l}\frac{1}{2}\int_{x_i}^{x_{i+1}} w(u)du \\ +\frac{1}{2}\left|\int_{x_i}^{\xi_i} w(u)du - \int_{\xi_i}^{x_{i+1}} w(u)du\right|\end{array}\right]\bigvee_{x_i}^{x_{i+1}}(f)$$
$$\leq \max_{i\in\{0,1,\ldots,n-1\}}\left\{\begin{array}{l}\frac{1}{2}\int_{x_i}^{x_{i+1}} w(u)du \\ +\frac{1}{2}\left|\int_{x_i}^{\xi_i} w(u)du - \int_{\xi_i}^{x_{i+1}} w(u)du\right|\end{array}\right\}\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(f)$$
$$\leq \left[\begin{array}{l}\frac{1}{2}\upsilon(L) \\ + \max_{i\in\{0,1,\ldots,n-1\}}\frac{1}{2}\left|\int_{x_i}^{\xi_i} w(u)du - \int_{\xi_i}^{x_{i+1}} w(u)du\right|\end{array}\right]\bigvee_a^b(f)$$

This completes proof of the first inequality in (18).
Also, we have

$$\left|\int_{x_i}^{\xi_i} w(u)du - \int_{\xi_i}^{x_{i+1}} w(u)du\right|$$
$$\leq \left|\int_{x_i}^{\xi_i} w(u)du\right| + \left|\int_{\xi_i}^{x_{i+1}} w(u)du\right| \leq \int_{x_i}^{x_{i+1}} w(u)du$$

and

$$\max_{i\in\{0,1,\ldots,n-1\}}\frac{1}{2}\left|\int_{x_i}^{\xi_i} w(u)du - \int_{\xi_i}^{x_{i+1}} w(u)du\right| \leq \frac{1}{2}\upsilon(L)$$

which completes the proof.

Remark 5.

1) If we choose $\xi_i = x_{i+1}$, then we have the weighted left rectangle rule

$$\int_a^b f(t)w(t)dt$$
$$= \sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}} w(u)du\right)f(x_i) + R_{wL}(I_n, f, w).$$

The remainder $R_{wL}(I_n, f, w)$ satisfies

$$\left|R_{wL}(I_n, f, w)\right| \leq \upsilon(L)\bigvee_a^b(f).$$

2) Similarly, choosing $\xi_i = x_i$, we have the weighted right rectangle rule

$$\int_a^b f(t)w(t)dt$$
$$= \sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}} w(u)du\right)f(x_{i+1}) + R_{wR}(I_n, f, w).$$

And, the remainder term $R_{wR}(I_n, f, w)$ satisfies

$$\left|R_{wR}(I_n, f, w)\right| \leq \upsilon(L)\bigvee_a^b(f).$$

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