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Generalization of Fixed Point Theorems in Pseudocompact Tichonov Space

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Abstract In this paper we study some fixed point results in pseudocompact Tichnovo space using Edelstein type contractive conditions. The results presented in this paper include the generalization of some fixed point theorems established by Fisher and Pathak.

Keywords: *contraction mapping, fixed point, pseudocompact Tichnovo space.*

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1. Introduction

Fixed point theory is a fascinating topic for research in modern mathematics. In this direction the Banach contraction mapping theorem of 1922 popularly known as Banach contraction mapping principle is a rewarding result in analysis and fixed point theory. It has widespread applications in both pure and applied mathematics. The well known Banach [2] contraction mapping principle states that if "X is a complete metric space and $T : X \to X$ is a contraction mapping of X into itself then T has unique fixed point in X". This celebrated principle has been generalized by several authors. In 1961, Edelstein [6] introduced the concept of contractive mapping defined on compact metric spaces which is generalizetion of Banach contraction mapping principle. According to Edelstein "if T is a continuous mapping of a compact metric space X into itself satisfying $d(Tx,Ty) < d(x,y)$ for all $x, y \in X$, $x \neq y$, then T has unique fixed point in X". Edelstein's contractive mapping theorem has been extensively generalized and improved by several mathematicians for fixed points in several different ways viz, Bailey [1], Chatterjee [4], Ciric [5], Iseki [9], Kannan and Sharma [12], Pachpatte [15], Popa [17], Sahu [18], Sharma and Sahu [19] and Soni [20] on complete and compact metric spaces. The concept of fixed point results for contractive mappings in pseudocompact Tichonov spaces was introduced by Harinath [10]. Later on, Jain and Dixit [11] and Liu [13] also established fixed point results for several classes of contractive type mappings in pseudocompact Tichonov spaces. Inspired by the ideas of Fisher [7] and Pathak [16] the aim of the present paper is to prove the existence and uniqueness of fixed point results for self mapping in the setting of pseudocompact Tichonov spaces satisfying contractive type conditions.

The following fixed point theorems were proved in [7] and [16].

Theorem 1.1. [7] If T is a mapping of the complete metric space X into itself satisfying the inequality

$$\begin{aligned}&\left[d(Tx,Ty)\right]^2 \\ &\leq \alpha\left[d(x,Tx)d(y,Ty)\right]+\beta\left[d(x,Ty)d(y,Tx)\right]\end{aligned}$$

for all x, y in X where $0 \leq \alpha \leq 1$ and $0 \leq \beta$ then T has a fixed point.

Theorem 1.2. [16] Let P be a Pseudocompact Tichonov space and μ be a non-negative real valued continuous function over $P \times P$ ($P \times P$ is Tichnovo but need not be pseudocompact). Suppose μ also satisfies

$$(i)\ \begin{cases}\mu(x,x)=0 \text{ for all } x \in P \text{ and} \\ \mu(x,z) \leq \mu(x,y)+\mu(z,y) \text{ for all } x,y,z \in P\end{cases}$$

If S and T are two continuos self maps of P satisfying
(ii) $ST = TS$ and
(iii)

$$\begin{aligned}&\mu(STx,Sy) < \alpha_1\mu(Tx,y)+\alpha_2\mu(STx,Tx) \\ &+\alpha_3\mu(STx,y)+\alpha_4\mu(Tx,Sy)+\alpha_5\mu(Sy,y) \\ &+\alpha_6\left\{\frac{\mu(STx,Tx)\mu(Sy,y)}{\mu(Tx,y)}\right\} \\ &+\alpha_7\left\{\frac{\mu(Tx,Sy)\mu(STx,y)}{\mu(Tx,y)}\right\}\end{aligned}$$

for all distinct $x, y \in P$ with $Tx \neq y$, where $\alpha_3 \geq 0$, $\alpha_2+\alpha_3+\alpha_6 < 1$, $\alpha_1+\alpha_2+2\alpha_3+\alpha_5+\alpha_6 \leq 1$. Then S and T have a unique common fixed point in P which is unique whenever $\alpha_1+\alpha_3+\alpha_4+\alpha_7 \leq 1$.

2. Main Results

Theorem 2.1. Let P be a pseudocompact Tichonov space and d be a nonnegative real valued continuous function over $P \times P$ satisfying the conditions:

(*i*) $d(x,x) = 0 \forall x \in P.$

(*ii*) $d(x,y) \le d(x,z) + d(y,z) \forall x,y,z \in P.$

If S and T are continuous self maps of P satisfying

$$ST = TS \tag{2.1}$$

(*iii*)

$$\begin{aligned}
&\left[d(STx,Sy)\right]^2 \\
&< \alpha_1\left[d(Tx,y)d(Tx,STx)\right] \\
&+\alpha_2\left[\frac{\left(d(Tx,STx)\right)^2 d(y,Sy)}{d(Tx,y)}\right] \\
&+\alpha_3\left[d(STx,y)d(STx,Sy)\right] \\
&+\alpha_4\left[\frac{\left(d(Tx,y)\right)d(Tx,Sy)}{1+d(y,Sy)}\right] \\
&+\alpha_5\left[\frac{\left(1+d(Tx,y)\right)\left(d(Tx,STx)\right)d(Tx,y)}{1+d(y,Sy)}\right] \\
&+\alpha_6\left\{\left(\frac{d(y,Sy)d(STx,Tx)}{d(Tx,y)}\right)\right\}^2 \\
&+\alpha_7\left\{\left(\frac{d(y,STx)d(Tx,Sy)}{d(Tx,y)}\right)\right\}^2
\end{aligned} \tag{2.2}$$

for all distinct $x,y \in X$ with $Tx \ne y$ and $\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_6 < 1,$ then S and T have a common fixed point in X, which is unique whenever $\alpha_3 + \alpha_4 + \alpha_7 < 1.$

Proof. Define a function $\varphi : X \to R^+$ by $\varphi(x) = d(STx,Tx)$ for all $x \in X$. Clearly φ is continuous being the composite of three continuous functions S,T and d. Since X is compact, every real valued continuous function over X is bounded and attain its bounds. Thus there exists a point $v \in X$ such that $\varphi(v) = \inf\{\varphi(x) : x \in X\}.$ We now affirm that v is a fixed point for S. If not, let us suppose that $Sv \ne v,$ then using (2.2) we have

$$\begin{aligned}
&\left[\varphi(Sv)\right]^2 = \left[d(STSv,TSv)\right]^2 = \left[d(STSv,STv)\right]^2 \\
&< \alpha_1\left[d(TSv,Tv)d(TSv,STSv)\right] \\
&+\alpha_2\left[\frac{\left(d(TSv,STSv)\right)^2 d(Tv,STv)}{d(TSv,Tv)}\right] \\
&+\alpha_3\left[d(STSv,Tv)d(STSv,STv)\right] \\
&+\alpha_4\left[\frac{d(TSv,Tv)d(TSv,STv)}{\left(1+d(Tv,STv)\right)}\right] \\
&+\alpha_5\left[\frac{\left(1+d(TSv,Tv)\right)\left(d(TSv,STSv)\right)d(TSv,Tv)}{\left(1+d(Tv,STv)\right)}\right] \\
&+\alpha_6\left\{\frac{d(Tv,STv)d(STSv,TSv)}{d(TSv,Tv)}\right\}^2 \\
&+\alpha_7\left\{\frac{d(Tv,STSv)d(TSv,STv)}{d(TSv,Tv)}\right\}^2
\end{aligned}$$

$$\begin{aligned}
&\left[d(STSv,STv)\right] \\
&< \alpha_1\left[d(TSv,Tv) + \alpha_2 d(TSv,STSv)\right] \\
&+\alpha_3 d(STSv,Tv) + \alpha_5 d(TSv,Tv) + \alpha_6 d(STSv,TSv)
\end{aligned}$$

or

$$\begin{aligned}
&\left[1-(\alpha_2+\alpha_3+\alpha_6)\right]d(STSv,STv) \\
&< (\alpha_1+\alpha_3+\alpha_5)d(STv,Tv)
\end{aligned}$$

or

$$\left[1-(\alpha_2+\alpha_3+\alpha_6)\right]\varphi(Sv) < (\alpha_1+\alpha_3+\alpha_5)\varphi(v)$$

or

$$\varphi(Sv) < \frac{(\alpha_1+\alpha_3+\alpha_5)}{\left[1-(\alpha_2+\alpha_3+\alpha_6)\right]}\varphi(v)$$

or

$$\varphi(Sv) < \varphi(v)$$

which is contradiction because $d(STSv,STv) \ge 0.$ Hence $v \in P$ is a fixed point for $S,$ that is $S(v) = v.$ Using (2.1), we have

$$ST(v) = TS(v) = T(v). \tag{2.3}$$

Now we shall prove that $T(v) = v.$ If possible, let $T(v) \ne v,$ then by using (2.2) and (2.3), we have

$$\begin{aligned}
&\left[d(Tv,v)\right]^2 = \left[d(STv,Sv)\right]^2 \\
&< \alpha_1\left[d(Tv,v)d(Tv,STv)\right] + \alpha_2\left[\frac{\left(d(Tv,STv)\right)^2 d(v,Sv)}{d(Tv,v)}\right] \\
&+\alpha_3\left[d(STv,v)d(STv,Sv)\right] + \alpha_4\left[\frac{\left(d(Tv,v)\right)d(Tv,Sv)}{1+d(v,Sv)}\right] \\
&+\alpha_5\left[\frac{\left(1+d(Tv,v)\right)\left(d(Tv,STv)\right)^2}{1+d(v,Sv)}\right] \\
&+\alpha_6\left\{\frac{d(v,Sv)d(STv,Tv)}{d(Tv,v)}\right\}^2 + \alpha_7\left\{\frac{d(v,STv)d(Tv,Sv)}{d(Tv,v)}\right\}^2 \\
&< (\alpha_3+\alpha_4+\alpha_7)\left[d(Tv,v)\right]^2 \\
&< \left[d(Tv,v)\right]^2
\end{aligned}$$

which is a contradiction because $\alpha_3 + \alpha_4 + \alpha_7 < 1.$ Hence $v \in X$ is a fixed point of T i.e. $Tv = v.$

Uniqueness: To prove the uniqueness of v, if possible, let w be another fixed point for S and T i.e. $v = Sv = Tv$ and

$w = Sw = Tw \ (w \neq v)$. Then, using (2.2), we have

$$\begin{aligned}
&[d(v,w)]^2 = [d(STv,Sw)]^2 \\
&< \alpha_1[d(Tv,w)d(Tv,STv)] \\
&+\alpha_2\left[\frac{(d(Tv,STv))^2 d(w,Sw)}{d(Tv,w)}\right] \\
&+\alpha_3[d(STv,w)d(STv,Sw)] + \alpha_4\left[\frac{(d(Tv,w))d(Tv,Sw)}{1+d(w,Sw)}\right] \\
&+\alpha_5\left[\frac{(1+d(Tv,w))(d(Tv,STv))d(Tv,w)}{1+d(w,Sw)}\right] \\
&+\alpha_6\left\{\left(\frac{d(w,Sw)d(STv,Tv)}{d(Tv,w)}\right)\right\}^2 \\
&+\alpha_7\left\{\left(\frac{d(w,STv)d(Tv,Sw)}{d(Tv,w)}\right)\right\}^2 \\
&< (\alpha_3+\alpha_4+\alpha_7)[d(v,w)]^2
\end{aligned}$$

leading to a contradiction because $\alpha_3+\alpha_4+\alpha_7 < 1$, which proves that $v \in X$ is unique and this establishes the theorem.

Theorem (2.1) yields the following corollary.

Corollary 2.2 Let P be a pseudocompact Tichonov space and d be a non-negative real valued continuous function over $P \times P$ satisfying:

(*i*) $d(x,x) = 0 \forall x \in p$

(*ii*) $d(x,z) \leq d(x,z) + d(z,y) \forall x,y,z \in p$

and $S: P \times P$ is a continuous map satisfying the inequality

$$\begin{aligned}
&\left[d\left(S^2x,Sy\right)\right]^2 \\
&< \alpha_1\left[d(Sx,y)d\left(Sx,S^2x\right)\right] \\
&+\alpha_2\left[\frac{\left(d\left(Sx,S^2x\right)\right)^2 d(y,Sy)}{d(Sx,y)}\right] \\
&+\alpha_3\left[d\left(S^2x,y\right)d\left(S^2x,Sy\right)\right] + \alpha_4\left[\frac{(d(Sx,y))d(Sx,Sy)}{1+d(y,Sy)}\right] \\
&+\alpha_5\left[\frac{(1+d(Sx,y))\left(d\left(Sx,S^2x\right)\right)d(Sx,y)}{1+d(y,Sy)}\right] \\
&+\alpha_6\left\{\left(\frac{(d(y,Sy))d\left(S^2x,Sx\right)}{d(Sx,y)}\right)\right\}^2 \\
&+\alpha_7\left\{\left(\frac{\left(d\left(y,S^2x\right)\right)d(Sx,Sy)}{d(Sx,y)}\right)\right\}^2
\end{aligned}$$

for all distinct $x,y \in X$ with $Sx \neq y$ and $\alpha_1+\alpha_2+2\alpha_3+\alpha_5+\alpha_6 < 1$, then S has a fixed point in P which is unique.

Proof If we take $T = S$, then theorem (2.1) shows that S has a unique fixed point in X.

Theorem 2.3. Let P be a pseudocompact Tichonov space and d be a non-negative real valued continuous function over $P \times P$ satisfy the conditions:

(*i*) $d(x,x) = 0 \forall x \in P$.

(*ii*) $d(x,y) \leq d(x,z) + d(y,z) \forall x,y,z \in P$.

If S and T are continuous self maps of P satisfying

$$ST = TS \tag{2.4}$$

$$\begin{aligned}
&[d(STx,Sy)]^2 \\
&< \alpha_1[d(Tx,y)d(Tx,STx)] \\
&+\alpha_2\left[\frac{(d(Tx,STx))^2 d(y,Sy)}{d(Tx,y)}\right] \\
&+\alpha_3[d(STx,y)d(STx,Sy)] \\
&+\alpha_4\left[\frac{(d(Tx,y))d(Tx,Sy)}{1+d(y,Sy)}\right] \\
&+\alpha_5\left[\frac{(1+d(Tx,y))(d(Tx,STx))d(Tx,y)}{1+d(y,Sy)}\right] \\
&+\alpha_6\left\{\left(\frac{d(y,Sy)d(STx,Tx)}{d(Tx,y)}\right)\right\}^2 \\
&+\alpha_7\left\{\left(\frac{d(y,STx)d(Tx,Sy)}{d(Tx,y)}\right)\right\}^2 \qquad (2.5) \\
&+\alpha_8 \max\left\{\begin{array}{l} d(y,Sy)d(Sy,STx), \\ d(y,STx)d(Tx,STx), \\ d(Sy,Tx)d(y,Tx), \{d(Sy,STx)\}^2 \end{array}\right\}
\end{aligned}$$

for all $x,y \in X$, $Tx \neq y$ and $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8$ are non negative real numbers such that $\alpha_1+\alpha_2+2\alpha_3+\alpha_5+\alpha_6+2\alpha_8 < 1$ then S and T have a common fixed point in X, which is unique whenever, $\alpha_3+\alpha_4+\alpha_7+\alpha_8 < 1$.

Proof. Define a function $\varphi: X \to R^+$ by $\varphi(x) = d(STx,Tx)$ for all $x \in X$. Clearly φ is continuous being the composite of three continuous functions *S*, *T* and *d*. Since *X* is compact, every real valued continuous function over *X* is bounded and attain its bounds. Thus there exists a point $v \in X$ such that $\varphi(v) = \inf\{\varphi(x): x \in X\}$. We now affirm that *v* is a fixed point for *S*. If not, let us suppose that $Sv \neq v$, then using (2.5) we have

$$\begin{aligned}
&\varphi[(Sv)]^2 = \{d(STSv,STv)\}^2 \\
&< \alpha_1[d(TSv,Tv)d(TSv,STSv)]
\end{aligned}$$

$$+\alpha_2\left[\frac{d(TSv,STSv)^2 d(Tv,STv)}{d(TSv,Tv)}\right]$$
$$+\alpha_3\left[d(STSv,Tv)d(STSv,STv)\right]$$
$$+\alpha_4\left[\frac{d(TSv,Tv)d(TSv,STv)}{1+d(Tv,STv)}\right]$$
$$+\alpha_5\left[\frac{(1+d(TSv,Tv))(d(TSv,STSv))d(TSv,Tv)}{1+d(Tv,STv)}\right]$$
$$+\alpha_6\left\{\left(\frac{d(Tv,STv)d(STSv,TSv)}{d(TSv,Tv)}\right)\right\}^2$$
$$+\alpha_7\left\{\left(\frac{d(Tv,STSv)d(TSv,STv)}{d(TSv,Tv)}\right)\right\}^2$$
$$+\alpha_8 \max\left\{\begin{array}{l} d(Tv,STv)d(STv,STSv), \\ d(Tv,STSv)d(TSv,STSv), \\ d(STv,TSv)d(Tv,TSv),\{d(STv,STSv)\}^2 \end{array}\right\}$$

Case(i)-If

$$\max\left\{\begin{array}{l} d(Tv,STv)d(STv,STSv), \\ d(Tv,STSv)d(TSv,STSv), \\ d(STv,TSv)d(Tv,TSv),\{d(STv,STSv)\}^2 \end{array}\right\}$$
$$=\{d(TSv,STSv)\}^2.$$

Then

$$\varphi[(Sv)]^2=\{d(STSv,STv)\}^2$$
$$<\alpha_1\left[d(TSv,Tv)d(TSv,STSv)\right]$$
$$+\alpha_2\left[\frac{d(TSv,STSv)^2 d(Tv,STv)}{d(TSv,Tv)}\right]$$
$$+\alpha_3\left[d(STSv,Tv)d(STSv,STv)\right]$$
$$+\alpha_4\left[\frac{d(TSv,Tv)d(TSv,STv)}{1+d(Tv,STv)}\right]$$
$$+\alpha_5\left[\frac{(1+d(TSv,Tv))(d(TSv,STSv))d(TSv,Tv)}{1+d(Tv,STv)}\right]$$
$$+\alpha_6\left\{\left(\frac{d(Tv,STv)d(STSv,TSv)}{d(TSv,Tv)}\right)\right\}^2$$
$$+\alpha_7\left\{\left(\frac{d(Tv,STSv)d(TSv,STv)}{d(TSv,Tv)}\right)\right\}^2+\alpha_8\{d(TSv,STSv)\}^2$$

$$\varphi[(Sv)]=\{d(STSv,STv)\}$$
$$<\alpha_1 d(TSv,Tv)+\alpha_2 d(TSv,STSv)$$
$$+\alpha_3 d(STSv,Tv)+\alpha_5 d(TSv,Tv)$$
$$+\alpha_6 d(STSv,TSv)+\alpha_8\{d(TSv,STSv)\}$$

or

$$\left[1-(\alpha_2+\alpha_3+\alpha_6+\alpha_8)\right]d(STSv,STv)$$
$$<(\alpha_1+\alpha_3+\alpha_5)d(TSv,Tv)$$

or

$$\left[1-(\alpha_2+\alpha_3+\alpha_6+\alpha_8)\right]\varphi(Sv)$$
$$<(\alpha_1+\alpha_3+\alpha_5)\varphi(v)$$

or

$$\varphi(Sv)<\frac{(\alpha_1+\alpha_3+\alpha_5)}{\left[1-(\alpha_2+\alpha_3+\alpha_6+\alpha_8)\right]}\varphi(v)$$

or

$$\varphi(Sv)<\eta_1\varphi(v)$$

where

$$\eta_1=\frac{(\alpha_1+\alpha_3+\alpha_5)}{\left[1-(\alpha_2+\alpha_3+\alpha_6+\alpha_8)\right]}.$$

But $\alpha_1+\alpha_2+2\alpha_3+\alpha_5+\alpha_6+\alpha_8<1$, which is a contradiction.

Case-(ii) If

$$\max\left\{\begin{array}{l} d(Tv,STv)d(STv,STSv), \\ d(Tv,STSv)d(TSv,STSv), \\ d(STv,TSv)d(Tv,TSv), \\ \{d(STv,STSv)\}^2 \end{array}\right\}$$
$$=d(Tv,STSv)d(TSv,STSv).$$

Then

$$\varphi[(Sv)]^2$$
$$=\{d(STSv,STv)\}^2$$
$$<\alpha_1\left[d(TSv,Tv)d(TSv,STSv)\right]$$
$$+\alpha_2\left[\frac{d(TSv,STSv)^2 d(Tv,STv)}{d(TSv,Tv)}\right]$$
$$+\alpha_3\left[d(STSv,Tv)d(STSv,STv)\right]$$
$$+\alpha_4\left[\frac{d(TSv,Tv)d(TSv,STv)}{1+d(Tv,STv)}\right]$$
$$+\alpha_5\left[\frac{(1+d(TSv,Tv))(d(TSv,STSv))d(TSv,Tv)}{1+d(Tv,STv)}\right]$$
$$+\alpha_6\left\{\left(\frac{d(Tv,STv)d(STSv,TSv)}{d(TSv,Tv)}\right)\right\}^2$$
$$+\alpha_7\left\{\left(\frac{d(Tv,STSv)d(TSv,STv)}{d(TSv,Tv)}\right)\right\}^2$$
$$+\alpha_8\left[d(Tv,STSv)d(TSv,STSv)\right]$$

$$\varphi\left[(Sv)\right]=\left\{d\left(STSv,STv\right)\right\}$$
$$<\alpha_1 d\left(TSv,Tv\right)+\alpha_2 d\left(TSv,STSv\right)$$
$$+\alpha_3 d\left(STSv,Tv\right)+\alpha_5 d\left(TSv,Tv\right)$$
$$+\alpha_6 d\left(STSv,TSv\right)+\alpha_8 d\left(Tv,STSv\right)$$

or

$$\left[1-\left(\alpha_2+\alpha_3+\alpha_6+\alpha_8\right)\right]d\left(STSv,STv\right)$$
$$<\left(\alpha_1+\alpha_3+\alpha_5+\alpha_8\right)d\left(TSv,Tv\right)$$

or

$$\left[1-\left(\alpha_2+\alpha_3+\alpha_6+\alpha_8\right)\right]\left[\varphi\left(Sv\right)\right]$$
$$<\left(\alpha_1+\alpha_3+\alpha_5+\alpha_8\right)\varphi\left(v\right)$$

or

$$\left[\varphi\left(Sv\right)\right]<\eta_2\varphi\left(v\right)$$

where

$$\eta_2=\frac{\left(\alpha_1+\alpha_3+\alpha_5+\alpha_8\right)}{\left[1-\left(\alpha_2+\alpha_3+\alpha_6+\alpha_8\right)\right]}.$$

But $\alpha_1+\alpha_2+2\alpha_3+\alpha_5+\alpha_6+2\alpha_8<1,$ which is a contradiction.

Case-(*iii*) If

$$\max\left\{\begin{array}{l}d\left(Tv,STv\right)d\left(STv,STSv\right),\\ d\left(Tv,STSv\right)d\left(TSv,STSv\right),\\ d\left(STv,TSv\right)d\left(Tv,TSv\right),\left\{d\left(STv,STSv\right)\right\}^2\end{array}\right\}$$
$$=d\left(Tv,STv\right)d\left(STv,STSv\right).$$

Then

$$\varphi\left[(Sv)\right]^2=\left\{d\left(STSv,STv\right)\right\}^2$$
$$<\alpha_1\left[d\left(TSv,Tv\right)d\left(TSv,STSv\right)\right]$$
$$+\alpha_2\left[\frac{d\left(TSv,STSv\right)^2 d\left(Tv,STv\right)}{d\left(TSv,Tv\right)}\right]$$
$$+\alpha_3\left[d\left(STSv,Tv\right)d\left(STSv,STv\right)\right]$$
$$+\alpha_4\left[\frac{d\left(TSv,Tv\right)d\left(TSv,STv\right)}{1+d\left(Tv,STv\right)}\right]$$
$$+\alpha_5\left[\frac{\left(1+d\left(TSv,Tv\right)\right)\left(d\left(TSv,STSv\right)\right)d\left(TSv,Tv\right)}{1+d\left(Tv,STv\right)}\right]$$
$$+\alpha_6\left\{\left(\frac{d\left(Tv,STv\right)d\left(STSv,TSv\right)}{d\left(TSv,Tv\right)}\right)\right\}^2$$
$$+\alpha_7\left\{\left(\frac{d\left(Tv,STSv\right)d\left(TSv,STv\right)}{d\left(TSv,Tv\right)}\right)\right\}^2$$
$$+\alpha_8\left[d\left(Tv,STv\right)d\left(STv,STSv\right)\right]$$

or

$$\varphi\left[(Sv)\right]=\left\{d\left(STSv,STv\right)\right\}$$
$$<\alpha_1 d\left(TSv,Tv\right)+\alpha_2 d\left(TSv,STSv\right)+\alpha_3 d\left(STSv,Tv\right)$$
$$+\alpha_5 d\left(TSv,Tv\right)+\alpha_6 d\left(STSv,TSv\right)+\alpha_8 d\left(Tv,STv\right)$$
$$d\left(STSv,STv\right)<\frac{\left(\alpha_1+\alpha_3+\alpha_5+\alpha_8\right)}{\left[1-\left(\alpha_2+\alpha_3+\alpha_6\right)\right]}d\left(TSv,Tv\right)$$

or

$$\varphi\left(Sv\right)<\eta_3\varphi\left(v\right)$$

where

$$\eta_3=\frac{\left(\alpha_1+\alpha_3+\alpha_5+\alpha_8\right)}{\left[1-\left(\alpha_2+\alpha_3+\alpha_6\right)\right]}.$$

But $\alpha_1+\alpha_2+2\alpha_3+\alpha_5+\alpha_6+\alpha_8<1,$ which is a contradiction.

Here

$$\eta=\max\left\{\eta_1,\eta_2,\eta_3\right\}$$

$$\eta=\max\left\{\begin{array}{l}\dfrac{\left(\alpha_1+\alpha_3+\alpha_5\right)}{\left[1-\left(\alpha_2+\alpha_3+\alpha_6+\alpha_8\right)\right]},\\ \dfrac{\left(\alpha_1+\alpha_3+\alpha_5+\alpha_8\right)}{\left[1-\left(\alpha_2+\alpha_3+\alpha_6+\alpha_8\right)\right]},\\ \dfrac{\left(\alpha_1+\alpha_3+\alpha_5+\alpha_8\right)}{\left[1-\left(\alpha_2+\alpha_3+\alpha_6\right)\right]}\end{array}\right\}.$$

Hence, $v\in X$ is a fixed point of S and so $S\left(v\right)=v.$ Using (2.4), we have

$$ST\left(v\right)=TS\left(v\right)=T\left(v\right). \qquad (2.6)$$

Now we shall prove that $T\left(v\right)=v.$ If possible, let $T\left(v\right)\neq v,$ then by using (2.5) and (2.6) we have

$$\left[d\left(Tv,v\right)\right]^2=\left[d\left(STv,Sv\right)\right]^2$$
$$<\alpha_1\left[d\left(Tv,v\right)d\left(Tv,STv\right)\right]+\alpha_2\left[\frac{\left(d\left(Tv,STv\right)\right)^2 d\left(v,Sv\right)}{d\left(Tv,v\right)}\right]$$
$$+\alpha_3\left[d\left(STv,v\right)d\left(STv,Sv\right)\right]+\alpha_4\left[\frac{\left(d\left(Tv,v\right)\right)d\left(Tv,Sv\right)}{1+d\left(v,Sv\right)}\right]$$
$$+\alpha_5\left[\frac{\left(1+d\left(Tv,v\right)\right)\left(d\left(Tv,STv\right)\right)d\left(Tv,v\right)}{1+d\left(v,Sv\right)}\right]$$
$$+\alpha_6\left\{\left(\frac{d\left(v,Sv\right)d\left(STv,Tv\right)}{d\left(Tv,v\right)}\right)\right\}^2$$
$$+\alpha_7\left\{\left(\frac{d\left(v,STv\right)d\left(Tv,Sv\right)}{d\left(Tv,v\right)}\right)\right\}^2$$
$$+\alpha_8\max\left\{\begin{array}{l}d\left(v,Sv\right)d\left(Sv,STv\right),d\left(v,STv\right)d\left(Tv,STv\right),\\ d\left(Sv,Tv\right)d\left(v,Tv\right),\left\{d\left(Sv,STv\right)\right\}^2\end{array}\right\}$$
$$<\left(\alpha_3+\alpha_4+\alpha_7+\alpha_8\right)\left[d\left(v,Tv\right)\right]^2$$

which is contradiction because $\alpha_3+\alpha_4+\alpha_7+\alpha_8<1$. Hence $v\in X$ is a fixed point of T, that is $Tv=v$.

Uniqueness: Let, if possible $v\neq w$ be another fixed pioint of S and T that is $v=S(v)=T(v)$ and $w=S(w)=T(w)$, then using (2.5), we get

$$
\begin{aligned}
&\left[d(v,w)\right]^2=\left[d(STv,Sw)\right]^2\\
&<\alpha_1\left[d(Tv,w)d(Tv,STv)\right]\\
&+\alpha_2\left[\frac{(d(Tv,STv))^2 d(w,Sw)}{d(Tv,w)}\right]\\
&+\alpha_3\left[d(STv,w)d(STv,Sw)\right]\\
&+\alpha_4\left[\frac{(d(Tv,w))d(Tv,Sw)}{1+d(w,Sw)}\right]\\
&+\alpha_5\left[\frac{(1+d(Tv,w))(d(Tv,STv))d(Tv,w)}{1+d(w,Sw)}\right]\\
&+\alpha_6\left\{\left(\frac{d(w,Sw)d(STv,Tv)}{d(Tv,w)}\right)\right\}^2\\
&+\alpha_7\left\{\left(\frac{d(w,STv)d(Tv,Sw)}{d(Tv,w)}\right)\right\}^2\\
&+\alpha_8\max\begin{Bmatrix}d(w,Sw)d(Sw,STv),d(w,STv)d(Tv,STv),\\ d(Sw,Tv)d(w,Tv),\{d(Sw,STv)\}^2\end{Bmatrix}\\
&<(\alpha_3+\alpha_4+\alpha_7+\alpha_8)\left[d(v,w)\right]^2.
\end{aligned}
$$

This is a contradiction because $\alpha_3+\alpha_4+\alpha_7+\alpha_8<1$. Hence $v\in P$ is unique fixed point of S and T.

Now we give an example to support our results.

Remark: If we $\alpha_8=0$ in Theorem (2.3) then we get Theorem (2.1).

Example 2.1. Let $p=\{1,3,5,7\}$ and let τ is the discrete topology on P. Define $S,T:P\to P$ as

$$S1=1,\ S3=5,\ S5=7,\ S7=1,$$
$$T1=1,\ T3=7,\ T5=3,\ T7=5$$

and let d be a non-negative real valued continuous function over $P\times P$ such that $d(x,y)=|x-y|\,\forall x\neq y\in P$. Then, it is clear that P is a pseudocompact Tichonov space and S and T are continuous self maps of P which satisfy all the conditions of Theorem (2.1) and Theorem (2.3) with 1 as the only common fixed point by setting $x=3,\quad y=5$ for $\alpha_1=0,\quad \alpha_2=\frac{1}{3},\quad \alpha_3=\frac{1}{12},\quad \alpha_4=\frac{1}{2},$ $\alpha_5=0,\ \alpha_6=\alpha_7=\alpha_8=\frac{1}{2}.$

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