Turkish Journal of Analysis and Number Theory, 2017, Vol. 5, No. 5, 153-158
Available online at http://pubs.sciepub.com/tjant/5/5/2

DOI:10.12691/tjant-5-5-2

Sums over Numbers with Restricted Prime Factors

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Received May 12, 2017; Revised June 30, 2107; Accepted July 12, 2017

Abstract Let $q \geq 2$ be a positive integer and $S_q(n)$ be the sum of the digits in basis q of the positive integer n. We prove that the quotient $\frac{\tilde{\Omega}(n)}{\tilde{\omega}(n)}$ has a normal order one, where $\tilde{\omega}(n)$ and $\tilde{\Omega}(n)$ are respectively, the number of distinct prime factors and the number of prime factors p of a positive integer n counted with multiplicity such that $S_q(n) \equiv a \mod b \ (a, b \in \mathbb{Z}, b \geq 2)$. Moreover, we discuss sums of the form $\sum_{n \leq x} f(n)\tilde{\omega}(n)$, where f is a multiplicative function.

Keywords: *sum-of-digits function, uniform distribution modulo1, multiplicative function*

Cite This Article: Khadija Mbarki, and Walid Wannes, "Sums over Numbers with Restricted Prime Factors." *Turkish Journal of Analysis and Number Theory*, vol. 5, no. 5 (2017): 153-158. doi: 10.12691/tjant-5-5-2.

1. Introduction

Let $q \geq 2$ be a fixed integer. Every positive integer n has a unique q-adic expansion of the form

$$n = \sum_{j=0}^{v} n_j q^j, \ n_j \in \{0, 1, \ldots, q-1\},$$

with $n_v \neq 0$. For $n = 0$, we admit to set $v = 0$. The sum of digits function in basis q is defined by

$$S(n) = S_q(n) = n_0 + n_1 + \ldots + n_v.$$

The function S is one of the most natural examples of q-additive functions (i.e. $S(aq^j + b) = S(a) + S(b)$, with $(a, b, j) \in \mathbb{N}^3$, $0 \leq b \leq q^j$) and $\mathbb{N}$ is the set of non negative integers. Such functions were introduced by Gelfond [7] and further studied by Coquet [1], Kátai [9] and others.

For every $a \in \mathbb{N}$ and $b \geq 2$ such that $(b, q-1) = 1$, we define $\tilde{\omega}(n)$ as the number of distinct prime factors p of n such that $S(p) \equiv a \mod b$ and $\tilde{\Omega}(n)$ the number of distinct prime factors of n counted with multiplicity such that $S(p) \equiv a \mod b$. Therefore,

$$\tilde{\omega}(n) = \tilde{\omega}_{a,b,q}(n) = \sum_{\substack{p \mid n \\ S(p) \equiv a \mod b}} 1$$

and $$\tilde{\Omega}(n) = \tilde{\Omega}_{a,b,q}(n) = \sum_{\substack{p^{\alpha} \mid n \\ S(p) \equiv a \mod b}} 1.$$

Both functions have been studied in [12,13], it was proved that their normal order is $\frac{1}{b} \log \log n$. A crucial part in the proof is the following estimation

$$\sum_{\substack{p \leq x \\ S(p) \equiv a \mod b}} \frac{1}{p} = \frac{1}{b} \log \log x + \beta + O\left(\frac{1}{\log x}\right), \quad (1.1)$$

where

$$\beta = \sum_{1 \leq j \leq b-1} e\left(\frac{-aj}{b}\right) \int_2^{+\infty} \left(\sum_{p \leq t} e\left(S(p)\frac{j}{b}\right)\right) \frac{dt}{t^2}.$$

The aim of this paper is to provide asymptotic formulas for sums involving $\tilde{\omega}(n)$ and $\tilde{\Omega}(n)$. We consider firstly the sum

$$\sum_{n \leq x} \frac{\tilde{\Omega}(n)}{\tilde{\omega}(n)}.$$

Then, by using elementary methods, we discuss sums of the form

$$\sum_{n \leq x} f(n)\tilde{\omega}(n)$$

where f is one of the classical arithmetic functions μ, φ and σ, where μ is the Möbius function, while φ and σ

are respectively, the Euler function and the sum of divisors function. This extends the results known through the work of De Koninck [4], De Koninck and Sitaramachandrarao [5] to primes verifying a digital constraint. Finally, we give some related results about the distribution of $\tilde{\Omega}(n)-\tilde{\omega}(n)$ and uniform distribution modulo 1 of sequences involving $\tilde{\Omega}(n)$ and $\tilde{\omega}(n)$.

Throughout this paper, p always denotes a prime number. For any real x, we set $e(x)=e^{2\pi ix}$. The notation (a,b) refers to the greatest common divisor of a and b. $\omega(n)$ is the number of distinct prime factors of n while $\Omega(n)$ is the number of prime factors counted with multiplicity of the integer n. We recall that the notation $U \ll V$ is equivalent to the statement that $U = O(V)$ for positive functions U and V and the implied constants in the symbols "O", "$\ll$" are absolute. We also use the symbol "o" with its usual meaning, the statement $U = o(V)$ is equivalent to $U/V \to 0$.

2. The Sum of Quotients

Since $\omega(n)$ and $\Omega(n)$ have both the normal order $\log\log n$ (see [8]), it is obvious that $\frac{\Omega(n)}{\omega(n)}$ has normal order one which is shown by De Koninck in [4]. A crucial part in his proof is an elementary estimation of $\sum_{2\le n\le x}\frac{1}{\omega(n)}$. In this part, we consider an analogue problem, since $\tilde{\omega}(n)$ and $\tilde{\Omega}(n)$ have both the normal order $\frac{1}{b}\log\log n$ (see [12]). So, we put

$$\Delta(n)=\begin{cases}\frac{\tilde{\Omega}(n)}{\tilde{\omega}(n)}, \text{ if } \tilde{\omega}(n)\neq 0,\\ 1, \text{ otherwise.}\end{cases}$$

Then, the average value of $\Delta(n)$ is given in the following theorem.

Theorem 2.1.

$$\sum_{n\le x}\Delta(n)=x+O\left(\frac{x}{\log\log x}\right). \qquad (2.1)$$

Proof. Our first task is to estimate $\sum_{\substack{n\le x\\ \tilde{\omega}(n)\neq 0}}\frac{1}{\tilde{\omega}(n)}$. Indeed, we have

$$\sum_{\substack{n\le x\\ \tilde{\omega}(n)\neq 0}}\frac{1}{\tilde{\omega}(n)} = \sum_{\substack{n\le x\\ 0<2b\tilde{\omega}(n)<\log\log x}}\frac{1}{\tilde{\omega}(n)}+\sum_{\substack{n\le x\\ 2b\tilde{\omega}(n)\ge\log\log x}}\frac{1}{\tilde{\omega}(n)}$$

$$\le \sum_{\substack{n\le x\\ 2b\tilde{\omega}(n)<\log\log x}} 1+2b\frac{x}{\log\log x}.$$

Proposition 2.2 of [12] implies that

$$\sum_{n\le x}\left(\tilde{\omega}(n)-\frac{1}{b}\log\log x\right)^2\le Cx\log\log x,$$

where C is an absolute constant. So,

$$\sum_{\substack{n\le x\\ 2b\tilde{\omega}(n)<\log\log x}}\left(\tilde{\omega}(n)-\frac{1}{b}\log\log x\right)^2\le Cx\log\log x.$$

Or,

$$\frac{1}{4b^2}(\log\log x)^2\sum_{\substack{n\le x\\ 2b\tilde{\omega}(n)<\log\log x}}1 \le \sum_{\substack{n\le x\\ 2b\tilde{\omega}(n)<\log\log x}}\left(\tilde{\omega}(n)-\frac{1}{b}\log\log x\right)^2.$$

Then,

$$\sum_{\substack{n\le x\\ 2b\tilde{\omega}(n)<\log\log x}}1\le 4b^2C\frac{x}{\log\log x}.$$

So, we obtain

$$\sum_{\substack{n\le x\\ \tilde{\omega}(n)\neq 0}}\frac{1}{\tilde{\omega}(n)}=O\left(\frac{x}{\log\log x}\right). \qquad (2.2)$$

Now, we use (2.2) to estimate the left hand side of (2.1) as follows

$$\sum_{n\le x}\Delta(n)=\sum_{\substack{n\le x\\ \tilde{\omega}(n)=0}}\Delta(n)+\sum_{\substack{n\le x\\ \tilde{\omega}(n)\neq 0}}\Delta(n)$$

$$=\sum_{\substack{n\le x\\ \tilde{\omega}(n)=0}}\Delta(n)+\sum_{\substack{n\le x\\ \tilde{\omega}(n)\neq 0}}\sum_{\substack{p^m|n\\ S(p)\equiv a \bmod b}}\frac{1}{\tilde{\omega}(n)}$$

$$=\sum_{\substack{n\le x\\ \tilde{\omega}(n)=0}}\Delta(n)+\sum_{\substack{n\le x\\ \tilde{\omega}(n)\neq 0}}\sum_{\substack{p|n\\ S(p)\equiv a \bmod b}}\frac{1}{\tilde{\omega}(n)}$$

$$+\sum_{\substack{n\le x\\ \tilde{\omega}(n)\neq 0}}\sum_{\substack{p^m|n, m\ge 2\\ S(p)\equiv a \bmod b}}\frac{1}{\tilde{\omega}(n)}$$

$$=\sum_{\substack{n\le x\\ \tilde{\omega}(n)\neq 0}}1+\sum_{\substack{n\le x\\ \tilde{\omega}(n)\neq 0}}1+\sum_{\substack{p^m\le x, m\ge 2\\ S(p)\equiv a \bmod b}}\sum_{n\le xp^{-m}}\frac{1}{\tilde{\omega}(np^m)}$$

$$=\sum_{n\le x}1+O\left(\sum_{\substack{p^m\le x, m\ge 2\\ S(p)\equiv a \bmod b}}\sum_{\substack{n\le xp^{-m}\\ \tilde{\omega}(n)\neq 0}}\frac{1}{\tilde{\omega}(n)}\right)$$

$$= x + O\left(\sum_{\substack{p^m \le x, m\ge 2}} \sum_{\substack{n\le xp^{-m} \\ \tilde{\omega}(n)\neq 0}} \frac{1}{\tilde{\omega}(n)}\right)$$
$$= x + O\left(\sum_{\substack{n\le x \\ \tilde{\omega}(n)\neq 0}} \sum_{\substack{p^m \le \frac{x}{n} \\ m\ge 2}} \frac{1}{\tilde{\omega}(n)}\right) \quad (2.3)$$

Since $\sum_{\substack{p^m\le t \\ m\ge 2}} 1 = O\left(t^{\frac{1}{2}}\right)$, then by partial summation, we can write

$$\sum_{\substack{n\le x \\ \tilde{\omega}(n)\neq 0}} \sum_{\substack{p^m \le \frac{x}{n} \\ m\ge 2}} \frac{1}{\tilde{\omega}(n)} \le \sum_{\substack{n\le x \\ \tilde{\omega}(n)\neq 0}} \sum_{\substack{p^m \le \frac{x}{n} \\ m\ge 2}} 1$$
$$\ll \sqrt{x} \sum_{\substack{n\le x \\ \tilde{\omega}(n)\neq 0}} \frac{1}{\sqrt{n}\tilde{\omega}(n)} \quad (2.4)$$
$$= \sqrt{x}\left(x^{-\frac{1}{2}} \sum_{\substack{n\le x \\ \tilde{\omega}(n)\neq 0}} \frac{1}{\tilde{\omega}(n)} + \frac{1}{2}\int_1^x t^{-\frac{3}{2}} \sum_{\substack{n\le t \\ \tilde{\omega}(n)\neq 0}} \frac{1}{\tilde{\omega}(n)} dt\right)$$
$$= O\left(\frac{x}{\log\log x}\right),$$

where the last equality is derived from (2.2). Assembling (2.3) and (2.4), we obtain

$$\sum_{n\le x} \Delta(n) = x + O\left(\frac{x}{\log\log x}\right),$$

which completes the proof of the theorem.

3. On the Sum $\sum_{n\le x} f(n)\tilde{\omega}(n)$

In this section, we discuss sums of the form $\sum_{n\le x} f(n)\tilde{\omega}(n)$, where $f(n)$ is either $\mu(n)$, $\varphi(n)$ or $\sigma(n)$. We note that the case $\sum_{n\le x} f(n)\omega(n)$, where f is a multiplicative function, was discussed in [[6], Chapter 9].

Theorem 3.1. *Let q, b be integers ≥ 2 satisfying $(b, q-1) = 1$. Then for any integer $m \ge 1$, there exist constants E_i, $0 \le i \le m-1$ such that*

$$\sum_{n\le x} \mu(n)\tilde{\omega}(n) = \frac{x}{b}\sum_{i=1}^{m} \frac{E_{i-1}}{(\log x)^i} + O\left(\frac{x}{(\log x)^{m+1}}\right).$$

Proof.

$$\sum_{n\le x} \mu(n)\tilde{\omega}(n) = \sum_{n\le x} \mu(n) \sum_{\substack{p|n \\ S(p)\equiv a \bmod b}} 1$$
$$= \sum_{\substack{pn\le x \\ S(p)\equiv a \bmod b}} \mu(pn) = -\sum_{\substack{pn\le x, p\nmid n \\ S(p)\equiv a \bmod b}} \mu(n)$$
$$= -\sum_{\substack{pn\le x \\ S(p)\equiv a \bmod b}} \mu(n) + \sum_{\substack{pn\le x, p|n \\ S(p)\equiv a \bmod b}} \mu(n)$$
$$= -\sum_{\substack{pn\le x \\ S(p)\equiv a \bmod b}} \mu(n) + \sum_{\substack{p^2 n\le x \\ S(p)\equiv a \bmod b}} \mu(pn)$$
$$= -\sum_{\substack{pn\le x \\ S(p)\equiv a \bmod b}} \mu(n) - \sum_{\substack{p^2 n\le x, p\nmid n \\ S(p)\equiv a \bmod b}} \mu(n)$$
$$= -\sum_{\substack{pn\le x \\ S(p)\equiv a \bmod b}} \mu(n) - \sum_{\substack{p^2 n\le x \\ S(p)\equiv a \bmod b}} \mu(n)$$
$$+ \sum_{\substack{p^2 n\le x, p|n \\ S(p)\equiv a \bmod b}} \mu(n). \quad (3.1)$$

By pursuing the same procedure, we get

$$\sum_{n\le x} \mu(n)\tilde{\omega}(n) = -\sum_{\substack{p^\alpha n\le x \\ S(p)\equiv a \bmod b}} \mu(n).$$

Now, we write

$$\sum_{n\le x} \mu(n)\tilde{\omega}(n)$$
$$= -\sum_{\substack{pn\le x \\ S(p)\equiv a \bmod b}} \mu(n) - \sum_{\substack{p^\alpha n\le x, \alpha\ge 2 \\ S(p)\equiv a \bmod b}} \mu(n) \quad (3.2)$$
$$= -\Sigma_1 - \Sigma_2,$$

say.

For the estimation of Σ_1, we use Dirichlet's hyperbola method and we obtain

$$\Sigma_1 = \sum_{\substack{pn\le x \\ S(p)\equiv a \bmod b}} \mu(n)$$
$$= \sum_{\substack{pn\le x, p\le\sqrt{x} \\ S(p)\equiv a \bmod b}} \mu(n) + \sum_{\substack{pn\le x, n\le\sqrt{x} \\ S(p)\equiv a \bmod b}} \mu(n) \quad (3.3)$$
$$- \sum_{n\le\sqrt{x}} \mu(n) \sum_{\substack{p\le\sqrt{x} \\ S(p)\equiv a \bmod b}} 1$$
$$= \Sigma_1^{(1)} + \Sigma_1^{(2)} - \Sigma_1^{(3)}.$$

Now, it is well-known (See [15]) that there exists a constant $A > 0$ such that

$$M(x) = \sum_{n\le\sqrt{x}} \mu(n)$$
$$= O\left(x\exp\left(-A\log^{\frac{3}{5}} x(\log\log x)^{-\frac{1}{5}}\right)\right). \quad (3.4)$$

Hence,

$$\Sigma_1^{(1)} = \sum_{\substack{pn\le x,\, p\le\sqrt{x}\\ S(p)\equiv a \bmod b}} \mu(n) = \sum_{\substack{p\le\sqrt{x}\\ S(p)\equiv a \bmod b}} \sum_{n\le\frac{x}{p}} \mu(n)$$

$$\ll \sum_{\substack{p\le\sqrt{x}\\ S(p)\equiv a \bmod b}} \frac{x}{p}\exp\left(-A\log^{\frac{3}{5}}\left(\frac{x}{p}\right)\log\log^{-\frac{1}{5}}\left(\frac{x}{p}\right)\right) \tag{3.5}$$

$$\ll x\exp\left(-A\log^{\frac{3}{5}} x\right)\sum_{\substack{p\le\sqrt{x}\\ S(p)\equiv a \bmod b}} \frac{1}{p}$$

$$= O\left(x\log\log x\exp\left(-A\log^{\frac{3}{5}} x\right)\right),$$

where the last bound is derived from (1.1). For the estimations of $\Sigma_1^{(2)}$ and $\Sigma_1^{(3)}$, we write

$$\Sigma_1^{(3)} = \sum_{n\le\sqrt{x}} \mu(n) \sum_{\substack{p\le\sqrt{x}\\ S(p)\equiv a \bmod b}} 1$$

$$= \left(\frac{1}{b}Li\left(\sqrt{x}\right)+O\left(\sqrt{x}\left(\log x\right)^{-c}\right)\right)M\left(\sqrt{x}\right) \tag{3.6}$$

$$= \frac{1}{b}Li\left(\sqrt{x}\right)M\left(\sqrt{x}\right)+O\left(x\frac{\exp\left(-A\log^{\frac{3}{5}} x\right)}{\left(\log x\right)^{c}}\right)$$

for any c > 0, by using the prime number theorem and [[11], Théorème 3] in the case $(b,q-1)=1$. Here $Li(x)=\int_2^x \frac{dt}{\log t}$. Also

$$\Sigma_1^{(2)} = \sum_{n\le\sqrt{x}} \mu(n) \sum_{\substack{p\le\frac{x}{n}\\ S(p)\equiv a \bmod b}} 1$$

$$= \sum_{n\le\sqrt{x}} \mu(n)\left\{\frac{1}{b}Li\left(\frac{x}{n}\right)+O\left(\frac{x}{n}\log\left(\frac{x}{n}\right)^{-c}\right)\right\} \tag{3.7}$$

$$= \frac{1}{b}\sum_{n\le\sqrt{x}} \mu(n)Li\left(\frac{x}{n}\right)+O\left(x\frac{\exp\left(-A\log^{\frac{3}{5}} x\right)}{\left(\log x\right)^{c}}\right).$$

However, by partial summation, we derive that

$$\frac{1}{b}\sum_{n\le\sqrt{x}} \mu(n)Li\left(\frac{x}{n}\right)$$

$$= \frac{1}{b}Li\left(\sqrt{x}\right)M\left(\sqrt{x}\right)+\frac{1}{b}\int_1^{\sqrt{x}} \frac{M(t)}{\log(x/t)}\frac{x}{t^2}dt \tag{3.8}$$

$$= \frac{1}{b}Li\left(\sqrt{x}\right)M\left(\sqrt{x}\right)+\frac{x}{b\log x}\int_1^{\sqrt{x}} \frac{M(t)}{t^2}\left[1-\frac{\log t}{\log x}\right]^{-1}dt.$$

Now,

$$\int_1^{\sqrt{x}} \frac{M(t)}{t^2}\left[1-\frac{\log t}{\log x}\right]^{-1}dt$$

$$= \int_1^{\sqrt{x}} \frac{M(t)}{t^2}\left\{\begin{array}{l} 1+\frac{\log t}{\log x}+\ldots+\left[\frac{\log t}{\log x}\right]^{m-1} \\ +O\left(\left(\frac{\log t}{\log x}\right)^{m}\right)\end{array}\right\}dt \tag{3.9}$$

$$= \sum_{i=0}^{m-1} \frac{1}{(\log x)^i}\int_1^{+\infty} \frac{M(t)(\log t)^i}{t^2}dt+O\left(\frac{1}{(\log x)^m}\right).$$

From (3.7), (3.8) and (3.9), we deduce

$$\Sigma_1^{(2)} = \frac{1}{b}Li\left(\sqrt{x}\right)M\left(\sqrt{x}\right)$$

$$+\frac{x}{b}\sum_{i=1}^{m} \frac{1}{(\log x)^i}\int_1^{+\infty} \frac{M(t)(\log t)^{i-1}}{t^2}dt+O\left(\frac{x}{(\log x)^{m+1}}\right).$$

Assembling (3.5), (3.6) and the last equality, one can show that

$$\Sigma_1 = \frac{x}{b}\sum_{i=1}^{m} \frac{E_{i-1}}{(\log x)^i}+O\left(\frac{x}{(\log x)^{m+1}}\right) \tag{3.10}$$

with

$$E_i = \int_1^{+\infty} \frac{M(t)(\log t)^i}{t^2}dt.$$

For $\alpha\ge 2$, by using (3.4), we get

$$\Sigma_2 = \sum_{\substack{p^{\alpha}n\le x\\ S(p)\equiv a \bmod b}} \mu(n) = \sum_{\substack{p\le x^{\frac{1}{\alpha}}\\ S(p)\equiv a \bmod b}} \sum_{n\le\frac{x}{p^{\alpha}}} \mu(n)$$

$$\ll \sum_{\substack{p\le x^{\frac{1}{\alpha}}\\ S(p)\equiv a \bmod b}} \frac{x}{p^{\alpha}}\exp\left(\begin{array}{l} -A\log^{\frac{3}{5}}\left(\frac{x}{p^{\alpha}}\right) \\ \times\left(\log\log\left(\frac{x}{p^{\alpha}}\right)\right)^{-\frac{1}{5}}\end{array}\right) \tag{3.11}$$

$$\ll x\exp\left(-A\log^{\frac{3}{5}} x\right)\sum_{\substack{p\le x^{\frac{1}{\alpha}}\\ S(p)\equiv a \bmod b}} \frac{1}{p^{\alpha}}$$

$$= O\left(x\exp\left(-A\log^{\frac{3}{5}} x\right)\right).$$

Finally, formulas (3.2), (3.10) and (3.11) give the desired estimation.

In order to provide a corresponding result for φ and σ, we need the lemma below, which can be proved by the same method as in the proof provided in [13].

Lemma 3.2. *Let* $i \geq 1$ *and* $a_i = \frac{(i-1)!}{2^i}$. *Then, for any integer* $N \geq 1$ *we have*

$$\sum_{\substack{p \leq x \\ S(p) \equiv a \bmod b}} p = \frac{x^2}{b} \sum_{i=1}^{N} \frac{a_i}{\log^i x} + O_b\left(\frac{x^2}{\log^{N+1} x}\right).$$

Theorem 3.3. *Let* q, b *be integers* ≥ 2 *verifying* $(b, q-1) = 1$. *Then for any integer* $m \geq 1$, *there exist constants* K_1, K_2, F_i, G_i, $0 \leq i \leq m-1$ *such that*

$$\sum_{n \leq x} \varphi(n)\tilde{\omega}(n) = \frac{3}{b\pi^2} x^2 \log\log x + \frac{x^2}{b} \sum_{i=1}^{m-1} \frac{F_i}{(\log x)^i} + K_1 x^2 + O\left(\frac{x^2}{(\log x)^m}\right).$$

$$\sum_{n \leq x} \sigma(n)\tilde{\omega}(n) = \frac{\pi^2}{12b} x^2 \log\log x + \frac{x^2}{b} \sum_{i=1}^{m-1} \frac{G_i}{(\log x)^i} + K_2 x^2 + O\left(\frac{x^2}{(\log x)^m}\right).$$

Proof. For the Euler function, if we proceed as in (3.1), we obtain

$$\sum_{n \leq x} \varphi(n)\tilde{\omega}(n) = \sum_{\substack{p^\alpha n \leq x \\ S(p) \equiv a \bmod b}} (p-1)\varphi(n). \tag{3.12}$$

Since σ is a multiplicative function verifying $\sigma(p^2) + p = \sigma^2(p)$, one can show that

$$\sum_{n \leq x} \sigma(n)\tilde{\omega}(n) = \sum_{\substack{pn \leq x \\ S(p) \equiv a \bmod b}} (p+1)\sigma(n) - \sum_{\substack{p^2 n \leq x \\ S(p) \equiv a \bmod b}} p(p+1)\sigma(n). \tag{3.13}$$

By using the same approach as in Theorem 3.1, Lemma 3.2 and the following well known formulas

$$\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \log x),$$

$$\sum_{n \leq x} \sigma(n) = \frac{\pi^2}{12} x^2 + O(x \log x)$$

combined with (3.12) and (3.13), we get the result.

4. Other Results

4.1. The Distribution of $\tilde{\Omega}(n) - \tilde{\omega}(n)$

Rényi proved that for any positive integer k, the set of numbers n such that $\Omega(n) - \omega(n) = k$ has density d_k, where d_k are the power series coefficients of the meromorphic function

$$F(z) = \sum_{k=0}^{+\infty} d_k z^k = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p-z}\right).$$

Let $N_k(x)$ denote the number of $n \leq x$ for which $\Omega(n) - \omega(n) = k$. In [[14], Theorem 2.16], Montgomery and Vaughan showed a quantitative form of Rényi's theorem which states that for any nonnegative integer k and any $x \geq 2$, one has

$$N_k(x) = d_k x + O\left(\left(\frac{3}{4}\right)^k \sqrt{x} (\log x)^{4/3}\right), \tag{4.1}$$

where

$$d_k = \frac{6}{\pi^2} \sum_{\substack{m \in \mathcal{M} \\ \Omega(m) - \omega(m) = k}} \frac{1}{m} \prod_{p|m} \left(1 + \frac{1}{p}\right)^{-1}$$

and $\mathcal{M}$ is the set of powerful numbers i.e., those m such that $p \mid m \Rightarrow p^2 \mid m$. In this section, we will study the distribution of the function $\tilde{\Omega}(n) - \tilde{\omega}(n)$. An analogous formula to (4.1) holds for the number of $n \leq x$ for which $\tilde{\Omega}(n) - \tilde{\omega}(n) = k$. Denote $\tilde{N}_k(x)$ this number. By following the same steps as in the proof of [[14], Theorem 2.16], we get the following result.

Proposition 4.1. *For any nonnegative integer k and any* $x \geq 2$,

$$\tilde{N}_k(x) = \tilde{d}_k x + O\left(\left(\frac{3}{4}\right)^k \sqrt{x} (\log x)^{4/3}\right),$$

where

$$\tilde{d}_k := \frac{6}{\pi^2} \sum_{\substack{m \in \mathcal{M} \\ \tilde{\Omega}(m) - \tilde{\omega}(m) = k}} \frac{1}{m} \prod_{p|m} \left(1 + \frac{1}{p}\right)^{-1}$$

and $\mathcal{M}$ *is the set defined above.*

4.2. Uniform Distribution Modulo 1

We denote by $\mathcal{M}_1$ the set of multiplicative arithmetical functions f verifying $|f(n)| \leq 1 (n \in \mathbb{N})$.

H. Daboussi proved that for every irrational $(\alpha \in \mathbb{R})$, uniformly for f in $\mathcal{M}_1$, we have

$$\frac{1}{x} \sum_{n \leq x} f(n) e(n\alpha) \to 0.$$

The proof is given in his paper [2] written jointly with H. Delange.

An immediate consequence of Daboussi's result is the following: If α is an irrational number and g is a real valued additive arithmetical function, then the sequence $(g(n) + \alpha n)_{n \in \mathbb{N}}$ is uniformly distributed modulo 1 and this follows from Weyl's criterion (see [[10], Theorem 5.6]). Since $\tilde{\Omega}(n)$ and $\tilde{\omega}(n)$ are both additive real valued arithmetical functions then, for any irrational number α the following sequences $\left(\tilde{\omega}(n)+\alpha n\right)_{n\in\mathbb{N}}$ and $\left(\tilde{\Omega}(n)+\alpha n\right)_{n\in\mathbb{N}}$ are uniformly distributed modulo 1.

References

[1] J. Coquet. *Sur les fonctions Q-multiplicatives et Q-additives*, Thèse 3ème cycle, Orsay, (1975).

[2] H. Daboussi, H. Delange. *On multiplicative arithmetical functions whose module does not exceed one*, J. London Math. Soc. 26, p.245-264, (1982).

[3] M. Drmota, C. Mauduit, J. Rivat. *Primes with an average sum of digits*, Compositio Math. 145, p.271-292, (2009).

[4] J. M. De Koninck. *Sums of quotients of additive functions*, Proc. Amer. Math. Soc, (44), p. 35-38, (1974).

[5] J. M. De Koninck and R. Sitaramachandrarao. *Sums involving the largest prime divisor of an integer II*, Indian J. pure appl. Math., (19), p. 990-1004, (1988).

[6] J. M. De Koninck and A. Ivić, *Topics in Arithmetical functions* Notas de Matematica 72, North Holland, Amsterdam, (1980).

[7] A. O. Gelfond. *Sur les nombres qui ont des propriétés additives et multiplicatives données*, Acta Arith. (13), p. 259-265, (1968).

[8] G.H. Hardy and E. M. Wright. *An introduction to the theory of numbers*, Oxford University Press, (1979).

[9] I. Kàtai. *Distribution of q-additive functions, in* "Probability Theory and Applications" Kluwer Academic, Dordrecht, p. 309-318, (1992).

[10] L. Kuipers, H. Niederreiter. *Uniform distribution of sequences John Willey*, New York, (1974).

[11] C. Mauduit, J.Rivat. *Sur un problème de Gelfond: la somme des chiffres des nombres premiers*, Ann. of Math. 71, p.159-1646, (2010).

[12] M. Mkaouar, W. Wannes. *On the number of restricted prime factors of an integer*, Acta Mathematica Hungarica, (143), p. 88-95, (2014).

[13] M. Mkaouar, W. Wannes. *On the normal number of prime factors of $\varphi(n)$ subject to certain congruence conditions*, J. of Number Theory (160), p. 629-645, (2016).

[14] H.L. Montgomery, R.C. Vaughan. *Multiplicative Number Theory I. Classical Theory* Cambridge Studies in advanced Mathematics 97.

[15] A. Walfisz. *Weylsche Exponentialsummen in der neuren Zahlentheorie*, Berlin, (1963).