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# An Elementary Proof of $\sum_{n\geq 1} 1/n^2 = \pi^2/6$ and a Recurrence Formula for $\zeta(2k)$

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**Abstract** In this note, a series expansion technique introduced recently by Dancs and He for generating Euler-type formulae for odd zeta values $\zeta(2k+1)$, $\zeta(s)$ being the Riemann zeta function and $k$ a positive integer, is modified in a manner to furnish the even zeta values $\zeta(2k)$. As a result, we find an elementary proof of $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$, as well as a recurrence formula for $\zeta(2k)$ from which it follows that the ratio $\zeta(2k)/\pi^{2k}$ is a rational number, without making use of Euler's formula and Bernoulli numbers.

***Keywords:*** *Riemann zeta function, Euler's formula, Euler polynomials*

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## 1. Introduction

For real values of $s$, $s > 1$, the Riemann zeta function is defined as $\zeta(s) := \sum_{n=1}^{\infty} 1/n^s$. In this domain, this series converges according to the integral test.[1] For $s=2k$, $k \in \mathbb{Z}$, $k>0$, Euler (1740) did find that [4]

$$\zeta(2k) = \frac{2^{2k-1}|B_{2k}|}{(2k)!}\pi^{2k}, \tag{1}$$

where $B_k$ is the $k$-th Bernoulli number, i.e. the rational coefficient of $z^k/k!$ in the Taylor series expansion of $z/(e^z-1)$, $0<|z|<2\pi$. As a consequence, since $B_2 = 1/6$ one has $\zeta(2) = \pi^2/6$, which is the Euler solution to the Basel problem (see Ref. [2] and references therein).

In Ref. [3], Dancs and He (2006) introduced a series expansion approach to derive Euler-type formulae for $\zeta(2k+1)$. On noting that their approach could be modified in a manner to furnish similar formulas for $\zeta(2k)$, here in this note we show that the change of $\sin(n\pi)$ by $\cos(n\pi)$ in the Dancs-He initial series in fact yields a series expansion which can be reduced to a finite sum involving only even zeta values. From the first few terms of this sum, we have found an elementary proof of $\zeta(2) = \pi^2/6$ and a recurrence formula for $\zeta(2k)$. The proofs are elementary in the sense they do not involve complex analysis, Fourier series, or multiple integrals.[2]

## 2. Elementary Evaluation of $\zeta(2)$

For any real $\varepsilon > 0$ and $u \in [1, 1+\varepsilon]$, we begin by taking into account the following Taylor series expansion considered by Dancs and He in Ref. [3]:

$$\frac{2\,e^t}{e^t+u} = \sum_{m=0}^{\infty} \phi_m(u)\frac{t^m}{m!}, \tag{2}$$

which converges absolutely for $|t| < \pi$.

From the generating function for the Euler polynomial $E_m(x)$, i.e. $2e^{xt}/(e^t+1) = \sum_{m=0}^{\infty} E_m(x)\frac{t^m}{m!}$, it is clear that $\phi_m(1) = E_m(1)$, for all nonnegative integer values of $m$. For $u > 1$, we have

$$\phi_m(u) = -2\sum_{n=1}^{\infty}\frac{n^m}{(-u)^n}. \tag{3}$$

[1] For $s$=1, one has the harmonic series $\sum_{n\geq 1} 1/n$, which diverges to infinity.

[2] For non-elementary proofs, see, e.g., Refs. [1,5] and references therein.

Let us take this series as our definition of $\phi_{-m}(u)$, $m$ being a positive integer. Therefore

$$\phi_{-m}(1) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^m} = -2\,\zeta^*(m) = 2\,(1-2^{1-m})\,\zeta(m) \quad (4)$$

for all integer $m > 1$.

Now, let

$$f(u) := \sum_{n=1}^{\infty}\frac{(1/u)^n}{n^2}$$

be an auxiliary function, with $u$ belonging to the same domain as above. Since $\cos(n\pi) = (-1)^n$, then $f(u)$ can be written in the form

$$f(u) = \sum_{n=1}^{\infty}(-1)^n\,\frac{\cos(n\pi)}{u^n\,n^2}.$$

On expanding $\cos(n\pi)$ in a Taylor series, one finds

$$f(u) = \sum_{n=1}^{\infty}\left[\frac{(-1)^n}{u^n\,n^2}\cdot\sum_{j=0}^{\infty}(-1)^j\,\frac{(n\pi)^{2j}}{(2j)!}\right] = \sum_{j=0}^{\infty}(-1)^j\,\frac{\pi^{2j}}{(2j)!}\sum_{n=1}^{\infty}(-1)^n\,\frac{n^{2j}}{u^n\,n^2},$$

in which the change of sums justifies by Fubini's theorem. By writing the last series in terms of $\phi_m(u)$, one has

$$f(u) = \sum_{j=0}^{\infty}(-1)^j\,\frac{\pi^{2j}}{(2j)!}\,\frac{\phi_{2j-2}(u)}{(-2)}. \quad (5)$$

This is enough for the derivation of our first result.

**Theorem 1 (Short evaluation of ζ(2)).** $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

Proof. By taking the limit as $u \to 1^+$ on both sides of Eq.(5), one has

$$\lim_{u\to 1^+}\sum_{n=1}^{\infty}\frac{1}{u^n\,n^2} = -\frac{1}{2}\,\phi_{-2}(1) + \frac{1}{2}\frac{\pi^2}{2!}\phi_0(1) - \frac{1}{2}\sum_{j=2}^{\infty}(-1)^j\,\frac{\pi^{2j}}{(2j)!}\phi_{2j-2}(1), \quad (6)$$

which, in face of the value of $\phi_{-2}(1)$ stated in Eq.(4), implies that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{1}{2}\left[2\left(1-2^{1-2}\right)\zeta(2)\right] + \frac{\pi^2}{4}E_0(1) - \frac{1}{2}\sum_{j=2}^{\infty}(-1)^j\,\frac{\pi^{2j}}{(2j)!}E_{2j-2}(1). \quad (7)$$

Since $E_0(1)=1$ and $E_m(1)=0$ for all $m>0$, the right-hand side of this equation reduces to $-\frac{1}{2}\zeta(2) + \frac{\pi^2}{4}$,[3] which implies that

$$\zeta(2) = -\frac{1}{2}\zeta(2) + \frac{\pi^2}{4},$$

and then $\frac{3}{2}\,\zeta(2) = \frac{\pi^2}{4}$.

## 3. Recurrence Formula for $\zeta(2k)$

Interestingly, our approach can be easily adapted to treat higher zeta values by changing the exponent of $n$ from 2 to $2k$. The result is the following *recurrence formula for even zeta values*.

**Theorem 2 (Recurrence for ζ(2k)).** For any positive integer $k$,

$$\left(4-\frac{4}{2^{2k}}\right)\zeta(2k) = \sum_{m=1}^{k-1}\frac{(-1)^{k-m+1}}{(2k-2m)!}\left(2-\frac{4}{2^{2m}}\right)\pi^{2k-2m}\,\zeta(2m) - (-1)^k\,\frac{\pi^{2k}}{(2k)!}.$$

Proof. We begin by defining $f_k(u) := \sum_{n=1}^{\infty}(1/u)^n/n^{2k}$.

Again, since $\cos(n\pi) = (-1)^n$, we may write

$$f_k(u) = \sum_{n=1}^{\infty}(-1)^n\,\frac{\cos(n\pi)}{u^n\,n^{2k}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{u^n\,n^{2k}}\sum_{j=0}^{\infty}(-1)^j\,\frac{(n\pi)^{2j}}{(2j)!} = \sum_{j=0}^{\infty}(-1)^j\,\frac{\pi^{2j}}{(2j)!}\sum_{n=1}^{\infty}(-1)^n\,\frac{n^{2j}}{u^n\,n^{2k}}. \quad (8)$$

On rewriting the last series in terms of $\phi_m(u)$, one has

$$f_k(u) = \sum_{j=0}^{\infty}(-1)^j\,\frac{\pi^{2j}}{(2j)!}\,\frac{\phi_{2j-2k}(u)}{(-2)} = -\frac{1}{2}\sum_{j=0}^{k-1}(-1)^j\,\frac{\pi^{2j}}{(2j)!}\phi_{2j-2k}(u) - \frac{1}{2}\sum_{j=k}^{\infty}(-1)^j\,\frac{\pi^{2j}}{(2j)!}\phi_{2j-2k}(u).$$

Now, on substituting $m = j - k$ in the above series, one finds

$$\begin{aligned} f_k(u) &= -\frac{1}{2}\sum_{m=-k}^{-1}(-1)^{m+k}\,\frac{\pi^{2m+2k}}{(2m+2k)!}\phi_{2m}(u) - \frac{1}{2}\sum_{m=0}^{\infty}(-1)^{m+k}\,\frac{\pi^{2m+2k}}{(2m+2k)!}\phi_{2m}(u) \\ &= -\frac{1}{2}(-1)^k\left[\sum_{\tilde{m}=1}^{k}\frac{(-1)^{\tilde{m}}\,\pi^{2k-2\tilde{m}}}{(2k-2\tilde{m})!}\phi_{-2\tilde{m}}(u) + \sum_{m=0}^{\infty}\frac{(-1)^m\,\pi^{2m+2k}}{(2m+2k)!}\phi_{2m}(u)\right]. \end{aligned} \quad (9)$$

The limit as $u \to 1^+$, taken on both sides of Eq.(9), yields

[3] This occurs because $E_{2m}(1) = 0$ for all positive integer values of $m$.

$$\lim_{u\to 1^+}\sum_{n=1}^{\infty}\frac{1}{u^n n^{2k}}$$
$$=-\frac{1}{2}(-1)^k\left[\sum_{m=1}^{k}\frac{(-1)^m\pi^{2k-2m}}{(2k-2m)!}\phi_{-2m}(1)\right. \quad (10)$$
$$\left.+\sum_{m=0}^{\infty}\frac{(-1)^m\pi^{2m+2k}}{(2m+2k)!}\phi_{2m}(1)\right].$$

From Eq.(4), one knows that

$$\phi_{-2m}(1)=2\left(1-2^{1-2m}\right)\zeta(2m).$$

For nonnegative values of $m$, one has $\phi_{2m}(1)=E_{2m}(1)=0$, the only exception being $\phi_0(1)=E_0(1)=1$. This reduces Eq.(10) to

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=-(-1)^k\sum_{m=1}^{k}\frac{(-1)^m\pi^{2k-2m}}{(2k-2m)!}\left(1-2^{1-2m}\right)\zeta(2m)$$
$$-(-1)^k\frac{\pi^{2k}}{2(2k)!}.$$

By extracting the last term of the sum and isolating $\zeta(2k)$, one finds

$$\left(2-\frac{2}{2^{2k}}\right)\zeta(2k)=(-1)^{k+1}\sum_{m=1}^{k-1}\frac{(-1)^m\pi^{2k-2m}}{(2k-2m)!}$$
$$\times\left(1-2^{1-2m}\right)\zeta(2m)-(-1)^k\frac{\pi^{2k}}{2(2k)!}.$$

A multiplication by 2 on both sides yields the desired result.

The first few even zeta values can be readily obtained from the recurrence formula in Theorem 2. For $k$=1, the sum in the right-hand side is null and our recurrence reduces to

$$3\zeta(2)=-(-1)\frac{\pi^2}{2},$$

which simplifies to $\zeta(2)=\pi^2/6$, in agreement to our Theorem 1. For $k$=2, our recurrence yields

$$\frac{15}{4}\zeta(4)=\frac{\pi^2}{2!}\zeta(2)-\frac{\pi^4}{4!}.$$

By substituting $\zeta(2)=\pi^2/6$ and multiplying both sides by 4, one finds

$$15\,\zeta(4)=\frac{\pi^4}{3}-\frac{\pi^4}{6}=\frac{\pi^4}{6}, \quad (11)$$

which implies that $\zeta(4)=\pi^4/90$.

Note that, by writing the recurrence formula in Theorem 2 in the form

$$\left(1-\frac{1}{2^{2k}}\right)\frac{\zeta(2k)}{\pi^{2k}}$$
$$=\sum_{m=1}^{k-1}\frac{(-1)^{k-m+1}}{(2k-2m)!}\left(\frac{1}{2}-\frac{1}{2^{2m}}\right)\frac{\zeta(2m)}{\pi^{2m}}-\frac{(-1)^k}{4\,(2k)!}, \quad (12)$$

it is straightforward to show, by induction on $k$, that the ratio $\zeta(2k)/\pi^{2k}$ is a rational number for every positive integer $k$, without making use of Euler's formula for $\zeta(2k)$, as stated in Eq.(1), and Bernoulli numbers. In fact, this was the original motivation that has led the author to study the properties of the Dancs-He series expansions. The proofs developed here could well be modified to cover other special functions of interest in analytic number theory.

# References


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