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Relations on the Apostol Type (*p, q*)-Frobenius-Euler Polynomials and Generalizations of the Srivastava-Pintér Addition Theorems

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Abstract In this work, we define and introduce a new kind of the Apostol type Frobenius-Euler polynomials based on the (*p, q*)-calculus and investigate their some properties, recurrence relationships and so on. We give some identities at this polynomial. Moreover, we get (*p, q*)-extension of Carlitz's main result in [1].

Keywords: *Generating function, Frobenius-Euler polynomials and numbers, (p, q)-calculus, (p, q)-Frobenius-Euler polynomials, Apostol-Bernoulli number and polynomials, generalized q-Bernoulli polynomials, generalized q-Euler polynomials.*

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1. Introduction, Definitions and Notations

Throughout this paper, we always make use of the following notation; $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The (*p, q*)-numbers are defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}$$

which is natural generalization of the *q*-number such that

$$\lim_{p\to 1} [n]_{p,q} = [n]_q .$$

Note that (*p, q*)-number is symmetric: that is $[n]_{p,q} = [n]_{q,p}$.

The (*p, q*)-derivative of a function f is defined by

$$D_{p,q:x} f(x) := D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p-q)x}, (x \neq 0).$$

The (*p, q*)-Gauss Binomial formula is defined by

$$(x+y)_{p,q}^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} x^{n-k} y^k$$

where the notations $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ ((*p, q*)-Gauss Binomial coefficients) and $[n]_{p,q}!$ ((*p, q*)-factorial) are defined by

$$[n]_{p,q}! = [n]_{p,q} [n-1]_{p,q} \ldots [2]_{p,q} [1]_{p,q},$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}! [k]_{p,q}!}, (n \geq k).$$

The (*p, q*)-exponential functions, $e_{p,q}(z)$ and $E_{p,q}(z)$, are defined by

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{1}{[n]_{p,q}!} p^{\binom{n}{2}} x^n$$

and

$$E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{1}{[n]_{p,q}!} q^{\binom{n}{2}} x^n.$$

From this form, we easily see that $e_{p,q} E_{p,q}(-z) = 1$.

In this work, we introduce Apostol type (*p, q*)-Frobenius-Euler polynomials. We give some new identities for the Apostol type (*p, q*)-Frobenius-Euler polynomials. Also, we prove some explicit expressions.

Definition 1. *Let* $p, q \in \mathbb{C}$, $\alpha \in \mathbb{N}$. *The (p, q)-Bernoulli numbers* $\mathcal{B}_{n,p,q}$ *and polynomials* $\mathcal{B}_{n,p,q}(x,y)$ *are defined by means of the generating functions in* [6]:

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,p,q} \frac{t^n}{[n]_{p,q}!} = \left(\frac{t}{e_{p,q}(t) - 1} \right), |t| < 2\pi,$$

$$\sum_{n=0}^{\infty}\mathcal{B}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}$$
$$=\left(\frac{t}{e_{p,q}(t)-1}\right)e_{p,q}(tx)E_{p,q}(ty),|t|<2\pi.$$

Definition 2. *Let* $p,q\in\mathbb{C},\quad \alpha\in\mathbb{N}$. *The (p, q)-Euler numbers* $\mathcal{E}_{n,p,q}$ *and polynomials* $\mathcal{E}_{n,p,q}(x,y)$ *are defined by means of the generating functions in* [6]:

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,p,q}\frac{t^n}{[n]_{p,q}!}=\left(\frac{[2]_{p,q}}{e_{p,q}(t)+1}\right),|t|<\pi,$$

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}$$
$$=\left(\frac{[2]_{p,q}}{e_{p,q}(t)+1}\right)e_{p,q}(tx)E_{p,q}(ty),|t|<\pi.$$

Definition 3. *Let* $p,q\in\mathbb{C},\quad \alpha\in\mathbb{N}$. *The (p, q)-Bernoulli numbers* $\mathcal{B}_{n,p,q}^{(\alpha)}$ *and polynomials* $\mathcal{B}_{n,p,q}^{(\alpha)}(x,y)$ *in x, y of order* α *are defined by means of the generating functions in* [6]:

$$\sum_{n=0}^{\infty}\mathcal{B}_{n,p,q}^{(\alpha)}\frac{t^n}{[n]_{p,q}!}=\left(\frac{t}{e_{p,q}(t)-1}\right)^{\alpha},|t|<2\pi, \quad (1.1)$$

$$\sum_{n=0}^{\infty}\mathcal{B}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}$$
$$=\left(\frac{t}{e_{p,q}(t)-1}\right)^{\alpha}e_{p,q}(tx)E_{p,q}(ty),|t|<2\pi. \quad (1.2)$$

Definition 4. *Let* $p,q\in\mathbb{C},\quad \alpha\in\mathbb{N}$. *The (p, q)-Euler numbers* $\mathcal{E}_{n,p,q}^{(\alpha)}$ *and polynomials* $\mathcal{E}_{n,p,q}^{(\alpha)}(x,y)$ *in x, y of order* α *are defined by means of the generating functions in* [6]:

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,p,q}^{(\alpha)}\frac{t^n}{[n]_{p,q}!}=\left(\frac{[2]_{p,q}}{e_{p,q}(t)+1}\right)^{\alpha},|t|<\pi \quad (1.3)$$

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,p,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_{p,q}!}$$
$$=\left(\frac{[2]_{p,q}}{e_{p,q}(t)+1}\right)^{\alpha}e_{p,q}(tx)E_{p,q}(ty),|t|<\pi. \quad (1.4)$$

Classical Frobenius-Euler polynomials $\mathcal{H}_n^{(\alpha)}(x;u)$ of order α is defined by the following relation [1,7,10,11].

$$\sum_{n=0}^{\infty}\mathcal{H}_n^{(\alpha)}(x;u)\frac{t^n}{n!}=\left(\frac{1-u}{e^t-u}\right)^{\alpha}e^{xt} \quad (1.5)$$

where $\alpha\in\mathbb{N}$, u algebraic number.

Similarly Frobenius-Euler polynomials $\mathcal{H}_n^{(\alpha)}(x;u;\lambda)$ of order α is defined by the following relation ([17])

$$\sum_{n=0}^{\infty}\mathcal{H}_n^{(\alpha)}(x;u;\lambda)\frac{t^n}{n!}=\left(\frac{1-u}{\lambda e^t-u}\right)^{\alpha}e^{xt}. \quad (1.6)$$

Definition 5. *The Apostol type q-Frobenius-Euler polynomials* $\mathcal{H}_{n,q}^{(\alpha)}(x,y;u;\lambda)$ *of order* α *in x, y and Apostol type q-Frobenius-Euler number* $\mathcal{H}_{n,q}^{(\alpha)}(0,0;u;\lambda)$ *of order* α, *in* [9] *respectively*

$$\sum_{n=0}^{\infty}\mathcal{H}_{n,q}^{(\alpha)}(x,y;u;\lambda)\frac{t^n}{n!}=\left(\frac{1-u}{\lambda e_q(t)-u}\right)^{\alpha}e_q(tx)E_q(ty),$$

$$\sum_{n=0}^{\infty}\mathcal{H}_{n,q}^{(\alpha)}(0,0;u;\lambda)\frac{t^n}{n!}=\left(\frac{1-u}{\lambda e_q(t)-u}\right)^{\alpha}.$$

Definition 6. *Let* $p,q\in\mathbb{C},\quad \alpha\in\mathbb{N}$ *and* $0<\left|\frac{p}{q}\right|<1$. *We define the Apostol type (p, q)-Bernoulli polynomials* $\mathcal{B}_{n,p,q}^{(\alpha)}(x,y;u;\lambda)$ *of order* α *in x, y and the Apostol type (p, q)-Bernoulli numbers* $\mathcal{B}_{n,p,q}^{(\alpha)}(0,0;u;\lambda)$ *of order* α *in x, y respectively*

$$\sum_{n=0}^{\infty}\mathcal{B}_{n,p,q}^{(\alpha)}(x,y;u;\lambda)\frac{t^n}{[n]_{p,q}!}$$
$$=\left(\frac{t}{\lambda e_{p,q}(t)-1}\right)^{\alpha}e_q(tx)E_q(ty),$$

$$\sum_{n=0}^{\infty}\mathcal{B}_{n,p,q}^{(\alpha)}(0,0;u;\lambda)\frac{t^n}{[n]_{p,q}!}=\left(\frac{t}{\lambda e_{p,q}(t)-1}\right)^{\alpha}.$$

Definition 7. *Let* $p,q\in\mathbb{C},\quad \alpha\in\mathbb{N}$ *and* $0<\left|\frac{p}{q}\right|<1$. *We define the Apostol type (p, q)-Euler polynomials* $\mathcal{E}_{n,p,q}^{(\alpha)}(x,y;u;\lambda)$ *of order* α *in x, y and the Apostol type (p, q)-Euler numbers* $\mathcal{E}_{n,p,q}^{(\alpha)}(0,0;u;\lambda)$ *of order* α *in x, y respectively*

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,p,q}^{(\alpha)}(x,y;u;\lambda)\frac{t^n}{[n]_{p,q}!}$$
$$=\left(\frac{[2]_{p,q}}{\lambda e_{p,q}(t)+1}\right)^{\alpha}e_q(tx)E_q(ty),$$

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,p,q}^{(\alpha)}(0,0;u;\lambda)\frac{t^n}{[n]_{p,q}!}=\left(\frac{[2]_{p,q}}{\lambda e_{p,q}(t)+1}\right)^{\alpha}.$$

Definition 8. *We de.ne Apostol type (p, q)-Frobenius-Euler polynomials* $\mathcal{H}_{n,p,q}^{(\alpha)}(x,y;u;\lambda)$ *of order* α *in x, y* *and Apostol type (p, q)-Frobenius-Euler numbers* $\mathcal{H}_{n,p,q}^{(\alpha)}(0,0;u;\lambda)$ *of order* α, *respectively*

$$\sum_{n=0}^{\infty}\mathcal{H}_{n,p,q}^{(\alpha)}(x,y;u;\lambda)\frac{t^n}{[n]_{p,q}!}=\left(\frac{1-u}{\lambda e_{p,q}(t)-u}\right)^{\alpha}e_{p,q}(tx)E_{p,q}(ty), \tag{1.7}$$

$$\sum_{n=0}^{\infty}\mathcal{H}_{n,p,q}^{(\alpha)}(0,0;u;\lambda)\frac{t^n}{[n]_{p,q}!}=\left(\frac{1-u}{\lambda e_{p,q}(t)-u}\right)^{\alpha}. \tag{1.8}$$

Letting $\lim_{p\to 1}$ in (1.7), we have

$$\lim_{p\to 1}\sum_{n=0}^{\infty}\mathcal{H}_{n,p,q}^{(\alpha)}(x,y;u;\lambda)\frac{t^n}{[n]_{p,q}!}=\lim_{p\to 1}\left(\frac{1-u}{\lambda e_{p,q}(t)-u}\right)^{\alpha}e_{p,q}(tx)E_{p,q}(ty)$$

$$\sum_{n=0}^{\infty}\mathcal{H}_{n,q}^{(\alpha)}(x,y;u;\lambda)\frac{t^n}{[n]_{q}!}=\left(\frac{1-u}{\lambda e_{q}(t)-u}\right)^{\alpha}e_{q}(tx)E_{q}(ty)$$

[9].

Putting $\lambda=1$ and $u=-1$ in (1.7), we have

$$\mathcal{H}_{n,p,q}^{(\alpha)}(x,y;-1;1)=\mathcal{E}_{n,p,q}^{(\alpha)}(x,y),$$

where $\mathcal{E}_{n,p,q}^{(\alpha)}(x,y)$ is (p,q)-Euler polynomials of order α.

Using $\lim_{p\to 1}$ in last equation, we have

$$\mathcal{H}_{n,q}^{(\alpha)}(x,y;-1;1)=\mathcal{E}_{n,q}^{(\alpha)}(x,y),$$

where $\mathcal{E}_{n,q}^{(\alpha)}(x,y)$ is q-Euler polynomials of order α.

Letting $\lim_{q\to 1^-}$ in last equation, we have

$$\mathcal{H}_{n}^{(\alpha)}(x,y;-1;1)=\mathcal{E}_{n}^{(\alpha)}(x,y),$$

where $\mathcal{E}_{n}^{(\alpha)}(x,y)$ is Hermite based Euler polynomials of order α.

2. Some Basic Properties for the Apostol Type *q*-Frobenius-Euler Polynomials

Proposition 1. *Apostol type Frobenius-Euler polynomials satisfy the following relations*

$$\mathcal{H}_{n,p,q}^{(\alpha+\beta)}(x,y;u;\lambda)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\mathcal{H}_{k,p,q}^{(\alpha)}(x,y;u;\lambda)\mathcal{H}_{n-k,p,q}^{(\beta)}(0,0;u;\lambda), \tag{2.1}$$

$$\lambda\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\mathcal{H}_{k,p,q}(x,y;u;\lambda)-\mathcal{H}_{n,p,q}(x,y;u;\lambda)=(1-u)(x+y)_{p,q}^{n}, \tag{2.2}$$

$$\mathcal{H}_{n,p,q}^{(\alpha-\beta)}(x,y;u;\lambda)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\mathcal{H}_{k,p,q}^{(\alpha)}(x,0;u;\lambda)\mathcal{H}_{n-k,p,q}^{(-\beta)}(0,y;u;\lambda). \tag{2.3}$$

Theorem 1. *For* $n\in\mathbb{N}_0$ *and* $x,y,a\in\mathbb{C}$, *the following relationships hold true*:

$$\mathcal{H}_{n,p,q}^{(\alpha)}(x+a,y;u;\lambda)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\mathcal{H}_{n-k,p,q}^{(\alpha)}(0,y;u;\lambda)p^{\binom{k}{2}}\sum_{s=0}^{k}\binom{k}{s}x^s a^{k-s}, \tag{2.4}$$

$$\mathcal{H}_{n,p,q}^{(\alpha)}(x,y+a;u;\lambda)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\mathcal{H}_{n-k,p,q}^{(\alpha)}(x,0;u;\lambda)p^{\binom{k}{2}}\sum_{s=0}^{k}\binom{k}{s}y^s a^{k-s}.$$

Proof. Using Definition

$$\sum_{n=0}^{\infty}\mathcal{H}_{n,p,q}^{(\alpha)}(x+a,y;u;\lambda)\frac{t^n}{[n]_{p,q}!}$$

$$=\left(\frac{1-u}{\lambda e_{p,q}(t)-u}\right)^{\alpha}E_{p,q}(ty)e_{p,q}((x+a)t)$$

$$=\left(\sum_{l=0}^{\infty}\mathcal{H}_{l,p,q}^{(\alpha)}(0,y;u;\lambda)\frac{l^n}{[l]_{p,q}!}\right)\times\left(\sum_{k=0}^{\infty}p^{\binom{k}{2}}(x+a)^k\frac{t^k}{[k]_{p,q}!}\right)$$

$$=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\left(\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\mathcal{H}_{n-k,p,q}^{(\alpha)}(0,y;u;\lambda)\times p^{\binom{k}{2}}(x+a)^k\right)\right\}\frac{t^n}{[n]_{p,q}!}$$

$$=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\mathcal{H}_{n-k,p,q}^{(\alpha)}(0,y;u;\lambda)p^{\binom{k}{2}}\times\sum_{s=0}^{k}\binom{k}{s}y^s a^{k-s}\right\}\frac{t^n}{[n]_{p,q}!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$, we have (2.4).

Similarly the other equation is been calculation.

Theorem 2. *There is the following relation for the generalized Apostol type q-Frobenius-Euler polynomials*

$$(2u-1)\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\mathcal{H}_{k,p,q}(0,0;u;\lambda)\mathcal{H}_{n-k,p,q}(x,y;1-u;\lambda)=u\mathcal{H}_{n,p,q}(x,y;u;\lambda)-\mathcal{H}_{n,p,q}(x,y;1-u;\lambda). \tag{2.5}$$

Proof. By using the identity

$$\frac{2u-1}{\left(\lambda e_{p,q}(t)-u\right)\left(\lambda e_{p,q}(t)-(1-u)\right)} = \frac{1}{\lambda e_{p,q}(t)-u}-\frac{1}{\lambda e_{p,q}(t)-(1-u)},$$

$$(2u-1)\frac{(1-u)e_{p,q}(xt)(1-(1-u))E_{p,q}(ty)}{\left(\lambda e_{p,q}(t)-u\right)\left(\lambda e_{p,q}(t)-(1-u)\right)} = \frac{(1-u)e_{p,q}(xt)uE_{p,q}(ty)}{\lambda e_{p,q}(t)-u} - \frac{(1-u)e_{p,q}(xt)(1-(1-u))E_{p,q}(ty)}{\lambda e_{p,q}(t)-(1-u)},$$

$$(2u-1)\sum_{n=0}^{\infty}\mathcal{H}_{n,p,q}(0,0;u;\lambda)\frac{t^n}{[n]_{p,q}!} \times\sum_{n=0}^{\infty}\mathcal{H}_{n,p,q}(0,0;1-u;\lambda)\frac{t^n}{[n]_{p,q}!} = u\sum_{n=0}^{\infty}\mathcal{H}_{n,p,q}(x,y;u;\lambda)\frac{t^n}{[n]_{p,q}!} -(1-u)\sum_{n=0}^{\infty}\mathcal{H}_{n,p,q}(x,y;1-u;\lambda)\frac{t^n}{[n]_{p,q}!}.$$

Comparing the coefficient of $\frac{t^n}{[n]_{p,q}!}$, we prove (2.5).

Remark 1. *For* $\lim_{\substack{q\to 1^-\\ p\to 1}}\mathcal{H}_{n,p,q}(x,y;u;\lambda)$. *Substituting* $\lambda=1$, $y=0$ *in (2.5). We have Carlitz result ([1], equation 2.19).*

Theorem 3. *There is the following relation for the generalized Apostol type (p, q)-Frobenius-Euler polynomial*

$$u\mathcal{H}_{n,p,q}(x,y;u;\lambda) = \lambda\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}\mathcal{H}_{k,p,q}(x,y;u;\lambda)-\lambda(1-u)(x+y)_{p,q}^{n}. \tag{2.6}$$

Proof. By using the identity $e_{p,q}(t)E_{p,q}(-t)=1$,

$$\frac{u}{\lambda^2\left(e_{p,q}(t)-u\right)e_{p,q}(t)}=\frac{1}{\lambda\left(e_{p,q}(t)-u\right)}-\frac{1}{\lambda e_{p,q}(t)}.$$

We write as

$$\frac{u(1-u)e_{p,q}(tx)E_{p,q}(yt)}{\left(\lambda e_{p,q}(t)-u\right)\lambda e_{p,q}(t)} = \frac{(1-u)e_{p,q}(tx)E_{p,q}(yt)}{\lambda e_{p,q}(t)-u}-\frac{(1-u)e_{p,q}(tx)E_{p,q}(yt)}{\lambda e_{p,q}(t)},$$

$$\frac{u}{\lambda}\sum_{n=0}^{\infty}\mathcal{H}_{n,p,q}(x,y;u;\lambda)\frac{t^n}{[n]_{p,q}!}\frac{1}{e_{p,q}(t)} = \sum_{n=0}^{\infty}\mathcal{H}_{n,p,q}(x,y;u;\lambda)\frac{t^n}{[n]_{p,q}!} -\frac{1-u}{\lambda e_{p,q}(t)}e_{p,q}(tx)E_{p,q}(yt),$$

$$\frac{u}{\lambda}\sum_{n=0}^{\infty}\mathcal{H}_{n,p,q}(x,y;u;\lambda)\frac{t^n}{[n]_{p,q}!} = \sum_{n=0}^{\infty}\mathcal{H}_{n,p,q}(x,y;u;\lambda)\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}\frac{t^n}{[n]_{p,q}!} -(1-u)\sum_{n=0}^{\infty}(x+y)_{p,q}^{n}\frac{t^n}{[n]_{p,q}!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$, we have

$$u\mathcal{H}_{n,p,q}(x,y;u;\lambda) = \lambda\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}\mathcal{H}_{k,q}(x,y;u;\lambda)-(1-u)(x+y)_{p,q}^{n}.$$

3. Explicit Relation for the Apostol Type (*p, q*)-Frobenius-Euler Polynomials

Theorem 4. *There is the following relation for the Apostol type (p, q)-Frobenius-Euler poly-nomials*

$$\mathcal{H}_{n,p,q}^{(\alpha)}(x,y;u;\lambda) = \frac{1}{1-u}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}\begin{Bmatrix}\mathcal{H}_{n-k,p,q}(1,y;u;\lambda)\\ -u\mathcal{H}_{k,p,q}(0,y;u;\lambda)\end{Bmatrix} \times\mathcal{H}_{n-k,p,q}^{(\alpha)}(x,0;u;\lambda). \tag{3.1}$$

Proof. Since (1.7);

$$\sum_{n=0}^{\infty}\mathcal{H}_{n,p,q}^{(\alpha)}(x,y;u;\lambda)\frac{t^n}{[n]_{p,q}!} = \left(\frac{1-u}{\lambda e_{p,q}(t)-u}\right)^{\alpha}e_{p,q}(tx)E_{p,q}(ty) = \frac{1-u}{\lambda e_{p,q}(t)-u}E_{p,q}(ty)\frac{\lambda e_{p,q}(t)-u}{1-u} \times\left(\frac{1-u}{\lambda e_{p,q}(t)-u}\right)^{\alpha}e_{p,q}(tx)$$

$$=\frac{1}{1-u}\left\{\frac{1-u}{\lambda e_{p,q}(t)-u}E_{p,q}(ty)\lambda e_{p,q}(t)\right.$$
$$\times\left(\frac{1-u}{\lambda e_{p,q}(t)-u}\right)^{\alpha}e_{p,q}(tx)$$
$$\left.-u\left(\frac{1-u}{\lambda e_{p,q}(t)-u}\right)E_{p,q}(ty)\left(\frac{1-u}{\lambda e_{p,q}(t)-u}\right)^{\alpha}e_{p,q}(tx)\right\}$$
$$=\frac{1}{1-u}\left\{\lambda\sum_{k=0}^{\infty}\mathcal{H}_{k,p,q}(1,y;u;\lambda)\frac{t^k}{[k]_{p,q}!}\right.$$
$$\times\sum_{l=0}^{\infty}\mathcal{H}_{l,p,q}^{(\alpha)}(x,0;u;\lambda)\frac{t^l}{[l]_{p,q}!}$$
$$-u\sum_{k=0}^{\infty}\mathcal{H}_{k,p,q}(0,y;u;\lambda)\frac{t^k}{[k]_{p,q}!}$$
$$\left.\times\sum_{l=0}^{\infty}\mathcal{H}_{l,p,q}^{(\alpha)}(x,0;u;\lambda)\frac{t^l}{[l]_{p,q}!}\right\}.$$

Comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$, we have (3.1).

Theorem 5. *There is the following relation between Apostol type (p, q)-Frobenius-Euler polynomials and the generalized Apostol (p, q)-Bernoulli polynomials*

$$\mathcal{H}_{n,p,q}^{(\alpha)}(x,y;u;\lambda)$$
$$=\frac{1}{[n+1]_{p,q}}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix}n+1\\r\end{bmatrix}_{p,q}\right.$$
$$\times\sum_{k=0}^{r}\begin{bmatrix}r\\k\end{bmatrix}_{p,q}\mathcal{B}_{n+1-r,p,q}(x,0;\lambda) \quad (3.2)$$
$$\left.-\sum_{k=0}^{n+1}\begin{bmatrix}n+1\\k\end{bmatrix}_{p,q}\mathcal{B}_{n+1-k,p,q}(x,0;\lambda)\right\}$$
$$\times\mathcal{H}_{k,p,q}^{(\alpha)}(0,y;u;\lambda).$$

Proof.

$$\left(\frac{1-u}{\lambda e_{p,q}(t)-u}\right)^{\alpha}e_{p,q}(tx)E_{p,q}(ty)$$
$$=\left(\frac{1-u}{\lambda e_{p,q}(t)-u}\right)^{\alpha}E_{p,q}(ty)\frac{t}{\lambda e_{p,q}(t)-1}\frac{\lambda e_{p,q}(t)-1}{t}e_{p,q}(tx),$$
$$=\frac{1}{t}\left\{\lambda\sum_{n=0}^{\infty}\sum_{k=0}^{r}\begin{bmatrix}r\\k\end{bmatrix}_{p,q}\right.$$
$$\times\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_{p,q}\mathcal{H}_{k,p,q}^{(\alpha)}(0,y;u;\lambda)\mathcal{B}_{n-r,p,q}(x,0;\lambda)$$
$$\left.-\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\mathcal{H}_{k,p,q}^{(\alpha)}(0,y;u;\lambda)\mathcal{B}_{n-k,p,q}(x,0;\lambda)\right\}\frac{t^n}{[n]_{p,q}!},$$

$$=\sum_{n=0}^{\infty}\frac{1}{[n+1]_{p,q}}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix}n+1\\r\end{bmatrix}_{p,q}\right.$$
$$\times\sum_{k=0}^{r}\begin{bmatrix}r\\k\end{bmatrix}_{p,q}\mathcal{B}_{n+1-r,p,q}(x,0;\lambda)$$
$$\left.-\sum_{k=0}^{n+1}\begin{bmatrix}n+1\\k\end{bmatrix}_{p,q}\mathcal{B}_{n+1-k,p,q}(x,0;\lambda)\right\}$$
$$\times\mathcal{H}_{k,p,q}^{(\alpha)}(0,y;u;\lambda)\frac{t^n}{[n]_{p,q}!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$, we have (3.2).

Corollary 1. *There is the following relation between Apostol type (p, q)-Frobenius-Euler polynomials and the generalized Apostol (p, q)-Euler polynomials*

$$\mathcal{H}_{n,p,q}^{(\alpha)}(x,y;u;\lambda)$$
$$=\frac{1}{2}\left\{\lambda\sum_{k=0}^{r}\begin{bmatrix}r\\k\end{bmatrix}_{p,q}\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_{p,q}\mathcal{E}_{n-r,p,q}(x,0;\lambda)\right.$$
$$\left.-\sum_{k=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}_{p,q}\mathcal{E}_{n-k,p,q}(x,0;\lambda)\right\}\mathcal{H}_{k,p,q}^{(\alpha)}(0,y;u;\lambda).$$

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