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Some Integral Inequalities of Hermite-Hadamard Type for ε-convex Functions

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Abstract In the paper, the authors establish a new integral identity. By this integral identity and Hölder's inequality, the authors obtain some new inequalities of the Hermite-Hadamard type for ε-convex functions.

Keywords: *Hermite-Hadamard type type inequality, Integral identity, ε-convex function*

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1. Introduction

We first list some definitions concerning various convex functions.

Definition 1.1. A function $f : I \subseteq R = (-\infty,\infty) \to R$ is said to be convex if

$$f(\lambda x+(1-\lambda)y) \le \lambda f(x)+(1-\lambda)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

In [1] the concept of ε-convex functions was introduced as follows.

Definition 1.2. [1] A function $f : I \subseteq R = (-\infty,\infty) \to R$ and $\varepsilon \ge 0$, if

$$f(\lambda x+(1-\lambda)y) \le \lambda f(x)+(1-\lambda)f(y)+\varepsilon, \tag{1.2}$$

is valid for all $x, y \in I$ and $\lambda \in [0,1]$, then we say that $f(x)$ is a ε-convex function on I.

The following inequalities of Hermite-Hadamard type were established for some of the above convex functions.

Theorem 1.1. [2] Let $f : I^\circ \subseteq R \to R$ be differentiable on I°, $a,b \in I^\circ$ with $a<b$.

(i) If $|f'|$ is convex function on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{1.3}$$

(ii) If $|f'|^{p/(p-1)}$ is convex function on $[a,b]$, $p>1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2(p+1)^{1/p}}\left(\frac{|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}}{2}\right)^{(p-1)/p}. \tag{1.4}$$

Theorem 1.2. [3] Let $f : I^\circ \subseteq R \to R$ be differentiable on I°, $a,b \in I^\circ$ with $a<b$ and $q \ge 1$.

(i) If $|f|^q$ is convex function on $[a,b]$, then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q}. \tag{1.5}$$

(ii) If $|f|^q$ is convex function on $[a,b]$, $p>1$, then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|, \tag{1.6}$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{1.7}$$

In this paper, we establish a new integral identity. By this identity and Hölder's inequality, some new Hermite-Hadamard type for the product of ε-convex function and discussed, some results are obtained.

2. Some Lemmas

Lemma 2.1 Let $f: I \subseteq R \to R$ be differentiable on I° and where $a,b \in I$ with $a<b$, $n \in N_+$. If $f' \in L_1([a,b])$, then the following identity holds:

$$\frac{1}{n+1}\left[f(a)+\sum_{k=2}^{n} f\left(a+\frac{(k-1)(b-a)}{n}\right)+f(b)\right] -\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \tag{2.1}$$
$$=\frac{b-a}{n^2}\sum_{k=1}^{n}\int_0^1 \left[\left(t-\frac{n-(k-1)}{n+1}\right) f'\left((1-t)\left(a+\frac{(k-1)(b-a)}{n}\right)+t\left(a+\frac{k(b-a)}{n}\right)\right)\right]\mathrm{d}t.$$

Proof.

$$\frac{b-a}{n^2}\sum_{k=1}^{n}\int_0^1 \left[\left(t-\frac{n-(k-1)}{n+1}\right) f'\left((1-t)\left(a+\frac{(k-1)(b-a)}{n}\right)+t\left(a+\frac{k(b-a)}{n}\right)\right)\right]\mathrm{d}t$$
$$=\frac{1}{n}\sum_{k=1}^{n}\left[\frac{n-(k-1)}{n+1} f\left(a+\frac{(k-1)(b-a)}{n}\right)+\frac{k}{n+1} f\left(a+\frac{k(b-a)}{n}\right) -\int_0^1 f\left((1-t)\left(a+\frac{(k-1)(b-a)}{n}\right)+t\left(a+\frac{k(b-a)}{n}\right)\right)\mathrm{d}t\right]. \tag{2.2}$$

Since

$$\frac{1}{n}\sum_{k=1}^{n}\left[\frac{n-(k-1)}{n+1} f\left(a+\frac{(k-1)(b-a)}{n}\right)+\frac{k}{n+1} f\left(a+\frac{k(b-a)}{n}\right)\right]$$
$$=\frac{1}{n}\left[\frac{n}{n+1} f(a)+\frac{1}{n+1} f\left(a+\frac{b-a}{n}\right)\right.$$
$$+\frac{n-1}{n+1} f\left(a+\frac{b-a}{n}\right)+\frac{2}{n+1} f\left(a+\frac{2(b-a)}{n}\right)$$
$$+\frac{n-2}{n+1} f\left(a+\frac{2(b-a)}{n}\right)+\frac{3}{n+1} f\left(a+\frac{3(b-a)}{n}\right)$$
$$+\frac{n-3}{n+1} f\left(a+\frac{3(b-a)}{n}\right)+\frac{4}{n+1} f\left(a+\frac{4(b-a)}{n}\right)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+\frac{2}{n+1} f\left(a+\frac{(n-2)(b-a)}{n}\right)+\frac{n-1}{n+1} f\left(a+\frac{(n-1)(b-a)}{n}\right)$$
$$\left.+\frac{1}{n+1} f\left(a+\frac{(n-1)(b-a)}{n}\right)+\frac{n}{n+1} f(b)\right]$$
$$=\frac{1}{n+1}\left[f(a)+\sum_{k=2}^{n} f\left(a+\frac{(k-1)(b-a)}{n}\right)+f(b)\right]. \tag{2.3}$$

Changing variables with

$$x=(1-t)\left(a+\frac{(k-1)(b-a)}{n}\right)+t\left(a+\frac{k(b-a)}{n}\right)$$

for $0 \le t \le 1$, we get

$$\frac{1}{n}\sum_{k=1}^{n}\int_0^1 f\left((1-t)\left(a+\frac{(k-1)(b-a)}{n}\right)+t\left(a+\frac{k(b-a)}{n}\right)\right)\mathrm{d}t$$
$$=\frac{1}{n}\sum_{k=1}^{n}\frac{n}{b-a}\int_{a+\frac{(k-1)(b-a)}{n}}^{a+\frac{k(b-a)}{n}} f(x)\mathrm{d}x \tag{2.4}$$
$$=\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x.$$

Put the equalities (2.3) to (2.4) into the equality (2.2), the inequality (2.1) is thus proved. This completes of the proof.

By taking n=1 in Lemma 2.1, we have the following identities.

Lemma 2.2 [2] Let $f: I \subseteq R \to R$ be differentiable on I° and where $a,b \in I$ with $a<b$, if $f' \in L_1([a,b])$, then

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{f(a)+f(b)}{2} =\frac{b-a}{2}\int_0^1 (1-2t) f'(ta+(1-t)b)\mathrm{d}t.$$

3. Some Integral Inequalities of Hermite-Hadamard Type

Now we are in a position to establish some new integral inequalities of Hermite-Hadamard type involving the ε-convex functions.

Theorem 3.1 Let $f:[a,b] \subseteq R \to R$ be differentiable function on $[a,b]$ and $\varepsilon \ge 0$, $n \in N_+$, $q \ge 1$. If $f' \in L_1([a,b])$, then

$$\left|\frac{1}{n+1}\left[f(a)+\sum_{k=2}^{n} f\left(a+\frac{(k-1)(b-a)}{n}\right)+f(b)\right] -\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \tag{3.1}$$
$$\le \frac{(b-a)(2n+1)}{6n(n+1)}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}+\varepsilon\right]^{1/q}.$$

Proof. Since $|f'|^q$ is ε-convex function on $[a,b]$, using the Lemma 2.1 and by the Hölder's inequality, we have

$$\left|\begin{array}{l}\frac{1}{n+1}\left[f(a)+\sum_{k=2}^{n}f\left(a+\frac{(k-1)(b-a)}{n}\right)+f(b)\right]\\ -\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\end{array}\right|$$

$$\le\frac{b-a}{n^2}\sum_{k=1}^{n}\int_0^1\left|t-\frac{n-k+1}{n+1}\right|$$

$$\times\left|f'\left(\begin{array}{l}(1-t)\left(a+\frac{(k-1)(b-a)}{n}\right)\\ +t\left(a+\frac{k(b-a)}{n}\right)\end{array}\right)\right|\mathrm{d}t$$

$$\le\frac{b-a}{n^2}\left(\sum_{k=1}^{n}\int_0^1\left|t-\frac{n-k+1}{n+1}\right|\mathrm{d}t\right)^{1-1/q}$$

$$\left(\sum_{k=1}^{n}\int_0^1\left|t-\frac{n-k+1}{n+1}\right|\times\left|f'\left(\begin{array}{l}(1-t)\left(a+\frac{(k-1)(b-a)}{n}\right)\\ +t\left(a+\frac{k(b-a)}{n}\right)\end{array}\right)\right|^q\mathrm{d}t\right)^{1/q}$$

$$\le\frac{b-a}{n^2}\left(\sum_{k=1}^{n}\int_0^1\left|t-\frac{n-k+1}{n+1}\right|\mathrm{d}t\right)^{1-1/q}$$

$$\left[\sum_{k=1}^{n}\int_0^1\left|t-\frac{n-k+1}{n+1}\right|\times\left(\begin{array}{l}\frac{n-k+1-t}{n}|f'(a)|^q\\ +\frac{k-1+t}{n}|f'(b)|^q+\varepsilon\end{array}\right)\mathrm{d}t\right]^{1/q}$$

$$=\frac{b-a}{n^2}\left(\frac{n(2n+1)}{6(n+1)}\right)^{1-1/q}$$

$$\times\left[\begin{array}{l}\frac{n(2n+1)\left[|f'(a)|^q+|f'(b)|^q\right]}{12(n+1)}\\ +\varepsilon\frac{n(2n+1)}{6(n+1)}\end{array}\right]^{1/q}$$

$$=\frac{(b-a)(2n+1)}{6n(n+1)}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}+\varepsilon\right]^{1/q}.$$

Theorem 3.1 is proved.

Crollary3.1. Under the conditions of Theorem 3.1, then

(1) if $n=1$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le\frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}+\varepsilon\right]^{1/q}.$$

(2) if $q=1$, we have

$$\left|\begin{array}{l}\frac{1}{n+1}\left[f(a)+\sum_{k=2}^{n}f\left(a+\frac{(k-1)(b-a)}{n}\right)+f(b)\right]\\ -\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\end{array}\right|$$

$$\le\frac{(b-a)(2n+1)}{6n(n+1)}\left[\frac{|f'(a)|+|f'(b)|}{2}+\varepsilon\right].$$

Theorem 3.2 Let $f:[a,b]\subseteq R\to R$ be differentiable function on $[a,b]$ and $\varepsilon\ge 0$, $n\in N_+$, $q>1$ and $0\le r\le q$. If $f'\in L_1([a,b])$, then

$$\left|\begin{array}{l}\frac{1}{n+1}\left[f(a)+\sum_{k=2}^{n}f\left(a+\frac{(k-1)(b-a)}{n}\right)+f(b)\right]\\ -\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\end{array}\right|$$

$$\le\frac{b-a}{n^2}\left[\frac{q-1}{2q-r-1}\sum_{k=1}^{n}\left[\begin{array}{l}\left(\frac{k}{n+1}\right)^{\frac{2q-r-1}{q-1}}\\ +\left(1-\frac{k}{n+1}\right)^{\frac{2q-r-1}{q-1}}\end{array}\right]\right]^{1-1/q}$$

$$\times\left\{\sum_{k=1}^{n}\left[\left(\frac{k}{n+1}\right)^r n[n(r+2)+r+1]k^2|f'(a)|^q\right.\right.\qquad(3.2)$$

$$+\left(1-\frac{k}{n+1}\right)^r\left[(n(r+2)+r+1)k\right.$$

$$\left.\left.\left.-(n+1)(r+1)\right]|f'(b)|^q\right]\right\}^{1/q}.$$

Proof. Since $|f'|^q$ is ε-convex function on $[a,b]$, using the Lemma 2.1 and by the Hölder's inequality, we have

$$\left|\begin{array}{l}\frac{1}{n+1}\left[f(a)+\sum_{k=2}^{n}f\left(a+\frac{(k-1)(b-a)}{n}\right)+f(b)\right]\\ -\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\end{array}\right|$$

$$\le\frac{b-a}{n^2}\sum_{k=1}^{n}\int_0^1\left|t-\frac{n-k+1}{n+1}\right|$$

$$\times\left|f'\left((1-t)\left(a+\frac{(k-1)(b-a)}{n}\right)+t\left(a+\frac{k(b-a)}{n}\right)\right)\right|\mathrm{d}t$$

$$\le\frac{b-a}{n^2}\left(\sum_{k=1}^{n}\int_0^1\left|t-\frac{n-k+1}{n+1}\right|^{(q-r)/(q-1)}\mathrm{d}t\right)^{1-1/q}$$

$$\times\left(\sum_{k=1}^{n}\int_0^1\left|t-\frac{n-k+1}{n+1}\right|^r\times\left|f'\left(\begin{array}{l}(1-t)\left(a+\frac{(k-1)(b-a)}{n}\right)\\ +t\left(a+\frac{k(b-a)}{n}\right)\end{array}\right)\right|^q\mathrm{d}t\right)^{1/q}$$

$$\leq \frac{b-a}{n^2}\left(\sum_{k=1}^{n}\int_0^1\left|t-\frac{n-k+1}{n+1}\right|^{(q-r)/(q-1)}\mathrm{d}t\right)^{1-1/q}$$
$$\times\left[\sum_{k=1}^{n}\int_0^1\left|t-\frac{n-k+1}{n+1}\right|^{r}\right.$$
$$\left.\times\left(\frac{n-k+1-t}{n}|f'(a)|^q+\frac{k-1+t}{n}|f'(b)|^q+\varepsilon\right)\mathrm{d}t\right]^{1/q}$$
$$\leq \frac{b-a}{n^2}\left[\frac{q-1}{2q-r-1}\sum_{k=1}^{n}\left(\begin{array}{l}\left(\frac{k}{n+1}\right)^{\frac{2q-r-1}{q-1}}\\ +\left(1-\frac{k}{n+1}\right)^{\frac{2q-r-1}{q-1}}\end{array}\right)\right]^{1-1/q}$$
$$\times\left\{\sum_{k=1}^{n}\left[\left(\frac{k}{n+1}\right)^{r}n[n(r+2)+r+1]k^2|f'(a)|^q\right.\right.$$
$$+\left(1-\frac{k}{n+1}\right)^{r}[(n(r+2)+r+1)k$$
$$\left.\left.-(n+1)(r+1)]|f'(b)|^q\right]\right\}^{1/q}. \qquad (3.3)$$

Theorem 3.2 is proved.

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