Turkish Journal of Analysis and Number Theory, 2017, Vol. 5, No. 3, 101-116
Available online at http://pubs.sciepub.com/tjant/5/3/4

DOI:10.12691/tjant-5-3-4

Avoidance of Type (1,2) Patterns by Catalan Words

Toufik Mansour[1,*], Mark Shattuck[2]

[1]Department of Mathematics, University of Haifa, 31905 Haifa, Israel
[2]Department of Mathematics, University of Tennessee, Knoxville, TN 37996
*Corresponding author: tmansour@univ.haifa.ac.il

Received April 02, 2017; Revised May 10, 2017; Accepted May 19, 2017

Abstract A certain subset of the multiset permutations of length n satisfying two restrictions has been recently shown to be enumerated by the Catalan number C_{n-1}. These sequences have been termed *Catalan words* and are closely related to the 321-avoiding permutations. Here, we consider the problem of avoidance of patterns of type (1,2) wherein the second and third letters within an occurrence of a pattern are required to be adjacent. We derive in several cases functional equations satisfied by the generating functions enumerating members of the avoidance class which we solve by various methods. In one case, the generating function can be expressed in terms of a sum of reciprocals of Chebyshev polynomials, while in another, in terms of a previously studied q-Bell number. Among the sequences arising as enumerators of avoidance classes are the Motzkin and Fibonacci numbers. In several cases, it is more convenient to consider first the problem of avoidance on the subset of Catalan words whose members have no adjacent letters the same before moving to the larger problem on all Catalan words.

Keywords: *vincular patterns, functional equation, kernel method, pattern avoidance*

Cite This Article: Toufik Mansour, and Mark Shattuck, “Avoidance of Type (1,2) Patterns by Catalan Words.” *Turkish Journal of Analysis and Number Theory*, vol. 5, no. 3 (2017): 101-116. doi: 10.12691/tjant-5-3-4.

1. Introduction

By a Catalan word $w=w_1w_2...w_n$, we will mean a word over the alphabet of non-negative integers satisfying

(i) $w_{i+1}\geq w_i-1$ for $1\leq i<n$, and

(ii) if $w_i=k>0$ with i minimal, then there exist $i_1<i<i_2$ such that $w_{i_1}=w_{i_2}=k-1$.

If $n\geq 1$, then let $W(n)$ denote the set of Catalan words of length n. For example, there are 5 members of $W(4)$, namely,

$$0000, 0100, 0010, 0110, 0101.$$

Property (i) states that within members of $W(n)$, there are no drops of size greater than one, while (ii) says that the left-most occurrence of each $k>0$ has $k-1$ somewhere to its left and somewhere to its right. It is well-known (see, e.g., [7,12]) that $W(n)$ is equinumerous with the set of Dyck paths of semilength $n-1$.

Let $[k]=\{1,2,...,k\}$. Given $w=w_1w_2...w_n\in[k]^n$, define the *reduction* of w, denoted red(w), to be the word obtained by replacing all occurrences of the i-th smallest letter of w with i for each i. For example, red(7931379) = 3421234. The words v and w are said to be *order-isomorphic* if red(v) = red(w) and is denoted $v\sim w$. By a *pattern*, we will mean a word that contains every letter in $[j]$ for some $j\geq 1$. A word $\pi=\pi_1\pi_2\cdots\pi_n$ *contains* $\sigma=\sigma_1\sigma_2\cdots\sigma_\ell$ as a *classical* pattern if there exists a subsequence $\pi_{i_1}\pi_{i_2}...\pi_{i_\ell}$ for $1\leq i_1<i_2<\cdots<i_\ell\leq n$ such that $\pi_{i_1}\pi_{i_2}...\pi_{i_\ell}\sim\sigma$, and is said to *avoid* σ otherwise.

Similar terminology applies to *vincular*, or *dashed*, patterns, which resemble classical patterns except that some of the indices i_j must be consecutive (see, e.g., [2]). If $(t_1,t_2,...,t_r)$ is a vector of positive integers, then a vincular pattern is said to be of type $(t_1,t_2,...,t_r)$ if the first t_1 letters within an occurrence are adjacent, the next t_2 letters are adjacent, and so on. A classical pattern is then one of type (1, 1, . . . , 1). Vincular patterns are often represented by inserting dashes between consecutive letters where there is no adjacency requirement. Here we will consider patterns x-yz of type (1,2), that is, subsequences of the form $\pi_i\pi_j\pi_{j+1}$ where $i<j$. For example, the word π = 135432414 contains two occurrences of the (1, 2) pattern 3-12 (as witnessed by the subsequences 524 and 514), but avoids the pattern 2-11 (note that the 4’s are not adjacent within the 544 subsequences).

In this paper, we consider the pattern avoidance problem on Catalan words and enumerate members of $W(n)$ avoiding a single pattern of type (1,2). The analogous question has been considered earlier on permutations [2,3], compositions [4], and set partitions [6]. Given $n\geq 1$ and a pattern τ, let $W_\tau(n)$ denote the subset of $W(n)$ whose members avoid τ and let $a_\tau(n)=|W_\tau(n)|$. To determine the $a_\tau(n)$, we start by writing recurrences for refinements or slight variations of these numbers, which are obtained by considering suitable statistics on $W_\tau(n)$. The recurrences may be expressed equivalently as functional equations satisfied by the corresponding generating functions, which we take steps to solve.

We now describe the general kinds of functional equations encountered in our study. Some had singularities at the particular value for which we sought a solution, while others did not. The first type of equation was of the form

$$u(x,y)F(x,y)=v(x,y)+w(x,y)F(x,1),$$

and we are interested in finding $F(x,1)$, where $u(x,y)$ is not defined at $y=1$. In this case, we employ the *kernel method [5]* to ascertain $F(x,1)$, and hence the generating function for the numbers $a_\tau(n)$. For four of the patterns, the generating function satisfied functional equations that were special cases of the equation

$$F(x,y)=u(x,y)+v(x,y)F(x,1)$$
$$+xyw(x,y)F(x,g(x,y)).$$

where all functions are defined at $y=1$. In the case when $g(x,y)=1+xy$ (which occurred for two patterns), one is able to solve the general equation for F by iteration if x is sufficiently close to zero and y is restricted to a given bounded interval. For two other patterns, we encountered particular cases of the equation above when $g(x,y)=\frac{1}{1-x^2y}$ and $g(x,y)=\frac{1+xy}{1-x^2y}$. To solve the equation in these cases, we iterate it for special values of y expressed in terms of quotients of linear combinations of Chebyshev polynomials, the properties of which bring about a significant simplification in the resulting series. This allows one to determine explicit expressions for $F(x,1)$ and $F(x,y)$.

The paper is divided as follows. In the next section, we introduce notation and provide some preliminary results relating the number of Catalan words in an avoidance class to the number of members in the class having no two adjacent letters the same. In the next three sections, we determine generating function formulas in several cases by solving various types of functional equations. Among our results, we find that $a_{1-22}(n)$ is given by the (n−1)-st Motzkin number, which provides an apparently new combinatorial interpretation of this sequence. We also find that the generating function in the case 1-23 can be expressed in terms of q-Bell numbers, while in the cases 2-13 and 1-11, it can be expressed in terms of Chebyshev polynomials of the second kind. In the final section, we complete the enumeration of all (1,2) avoidance classes, up to one case. Our results are summarized in Table 1.

2. Preliminaries

In our study, it will be useful to consider the members of an avoidance class that contain a fixed number of zeros and/or that have no two adjacent letters the same. Here, we provide some algebraic results which will be needed in later sections. If $1\le m\le n$ and τ is a pattern, then let $W(n,m)$ and $W_\tau(n,m)$ denote, respectively, the subsets of $W(n)$ and $W_\tau(n)$ whose members contain exactly m zeros. Let $a(n,m)=|W(n,m)|$ and $a_\tau(n,m)=|W_\tau(n,m)|$.

Members of an avoidance class having no adjacent letters the same will be termed as *primitive*. Let $W^*(n)$ and $W_\tau^*(n)$ denote the subsets of $W(n)$ and $W_\tau(n)$ consisting of the primitive members, with the obvious comparable meanings for $W^*(n,m)$ and $W_\tau^*(n,m)$. Let $a^*(n)=|W^*(n)|$, $a_\tau^*(n)==|W_\tau^*(n)|$, $a^*(n,m)=|W^*(n,m)|$ and $a_\tau^*(n,m)=|W_\tau^*(n,m)|$.

A *run* of a letter ℓ within a word w is a maximal subsequence of consecutive letters in w each of which is ℓ.

Table 1. Sequences enumerating the (1,2) avoidance classes for Catalan words

τ	Sequence	Reference
1-11	1,1,1,3,6,15,39,102,275,754	Theorem 5.15
1-12	A000045 in [11]	Observation 6.2
1-21	1,1,1,1,1,1,1,1,1,1	Observation 6.1
1-22	A001006	Theorem 3.3
1-23	1,1,2,5,13,33,82,202,497,1224	Proposition 2.2 and Theorem 5.4
1-32	A000325	Observation 6.2
2-11	1,1,2,4,10,26,71,199,570,1659	Open
2-12	1,1,2,4,9,22,56,146,388,1047	Proposition 2.2 and Theorem 4.3
2-13	1,1,2,5,13,34,91,248,685,1911	Proposition 2.2 and Theorem 5.8
2-21	A011782	Observation 6.2
2-31	A000108	Observation 6.1
3-12	1,1,2,5,14,41,123,376,1167,3665	Proposition 2.2 and Theorem 4.5
3-21	A001519	Observation 6.2

Definition 2.1. *By the skeleton of a word w, denoted by skel(w), we will mean the word obtained from w by removing all but one of the letters from each of its runs.*

Note that $w \in W(n)$ implies skel(w) is a (primitive) Catalan word. Furthermore, if $w \in W(n,i)$ with $i < n$, then $\operatorname{skel}(w) \in W^*(r,s)$ for some $2 \le s \le i$ and $2s-1 \le r \le n+s-i$. For example, if $w = 010002233221 \in W(12,4)$, then $\operatorname{skel}(w) = 0102321 \in W^*(7,2)$. Also, if τ is a pattern in which no two consecutive letters are equal, then w avoids τ if and only if skel(w) avoids τ.

Let $g(x) = \sum_{n\ge1} a(n)x^n$ and $h(x) = \sum_{n\ge1} a^*(n)x^n$, with similar meanings for $g_\tau(x)$ and $h_\tau(x)$.

The following formulas relate $a(n)$ to $a^*(n)$ (and $a_\tau(n)$ to $a_\tau^*(n)$).

Proposition 2.2. *If* $n \ge 1$, *then*

$$a(n) = \sum_{i=1}^{n} \binom{n-1}{i-1} a^*(i), \qquad (2.1)$$

and thus $g(x) = h\left(\frac{x}{1-x}\right)$. *Comparable relations hold for* $a_\tau(n)$ *and* $h_\tau(x)$ *whenever no two consecutive letters are equal within the pattern* τ.

Proof. We count the members $\pi \in W(n)$ according to the number i of runs. Observe that deleting all letters within each run of π except for one leaves a member of $W^*(i)$ for some $1 \le i \le n$. Conversely, given any $\lambda \in W^*(i)$, there are $\binom{n-1}{i-1}$ members of $W(n)$ that have skeleton λ, upon adding $n-i$ letters across the i runs of λ to generate members of $W(n)$. Grouping together members of $W(n)$ having the same skeleton of length i and summing over i gives (2.1). For the second statement, note that

$$g(x) = \sum_{n\ge1}\sum_{i=1}^{n} \binom{n-1}{i-1} a^*(i) = \sum_{i\ge1} a^*(i) \sum_{n\ge i} \binom{n-1}{i-1} x^n$$
$$= \sum_{i\ge1} a^*(i) \left(\frac{x}{1-x}\right)^i.$$

The same relations are seen to hold if no two consecutive letters of τ are equal for then at most one letter out of each run within a Catalan word can appear as part of an occurrence of τ.

Let $g(x,y) = \sum_{n\ge1}\sum_{m=1}^{n} a(n,m)x^n y^m$ and $h(x,y) = \sum_{n\ge1}\sum_{m=1}^{t} a^*(n,m)x^n y^m$, where $t = \left\lfloor \frac{n+1}{2} \right\rfloor$, with similar meanings for $g_\tau(x,y)$ and $h_\tau(x,y)$.

Similar, though more complicated, relations can be given when the number of zeros in a Catalan word is prescribed.

Proposition 2.3. *If* $n > i \ge 2$, *then*

$$a(n,i) = \sum_{s=2}^{i} \sum_{r=2s-1}^{n+s-i} \binom{i-1}{s-1}\binom{n-i-1}{r-s-1} a^*(r,s), \qquad (2.2)$$

with $a(n,1) = \delta_{n,1}$ *and* $a(n,n) = 1$ *for all* $n \ge 1$. *In terms of generating functions, we have*

$$g(x,y) = h\left(\frac{x}{1-x}, \frac{y(1-x)}{1-xy}\right). \qquad (2.3)$$

Comparable relations hold for $a_\tau(n,i)$ *and* $h_\tau(x,y)$ *whenever no two consecutive letters are equal within the pattern* τ.

Proof. We first show the recurrence (2.2), its boundary values being clear. For each $u \in W^*(r,s)$, we claim that there are $\binom{i-1}{s-1}\binom{n-i-1}{r-s-1}$ words $w \in W(n,i)$ such that $\operatorname{skel}(w) = u$. To see this, note first that we must add $i-s$ zeros to u (in order to obtain a member of $W(n,i)$), which can be achieved only by increasing the lengths of its zero runs since otherwise the skeleton would change. Thus, there are $\binom{i-s+s-1}{s-1} = \binom{i-1}{s-1}$ ways of adding zeros. We must also add $n-r-(i-s)$ non-zero letters to u by increasing the lengths of its non-zero runs, which can be done in $\binom{n-r-(i-s)+r-s-1}{r-s-1} = \binom{n-i-1}{r-s-1}$ ways. Putting together the prior two observations establishes the claim. Grouping together members of $W(n,i)$ having skeletons of the same length and number of zeros then gives (2.2).

Multiplying (2.2) by x^n and summing over all $n \ge i+1$ implies

$$\sum_{n\ge i+1} a(n,i)x^n$$
$$= \sum_{s=2}^{i} \sum_{r\ge 2s-1} a^*(r,s)\binom{i-1}{s-1} \sum_{n\ge i+r-s} \binom{n-i-1}{r-s-1} x^n \qquad (2.4)$$
$$= x^i \sum_{s=2}^{i} \binom{i-1}{s-1}\left(\frac{1-x}{x}\right)^s \sum_{r\ge2s-1} a^*(r,s)\left(\frac{x}{1-x}\right)^r, \quad i \ge 2.$$

Multiplying (2.4) by y^i and summing over $i \ge 2$ yields

$$g(x,y) - \frac{xy}{1-xy} = \sum_{i\ge2}\sum_{n\ge i+1} a(n,i)x^n y^i$$
$$= \sum_{s\ge2}\left(\frac{1-x}{x}\right)^s \sum_{i\ge s}\binom{i-1}{s-1}(xy)^i \sum_{r\ge2s-1} a^*(r,s)\left(\frac{x}{1-x}\right)^r$$
$$= \sum_{s\ge2}\left(\frac{y(1-x)}{1-xy}\right)^s \sum_{r\ge2s-1} a^*(r,s)\left(\frac{x}{1-x}\right)^r$$
$$= \sum_{s\ge2}\sum_{r\ge2s-1} a^*(r,s)\left(\frac{x}{1-x}\right)^r\left(\frac{y(1-x)}{1-xy}\right)^s$$
$$= h\left(\frac{x}{1-x}, \frac{y(1-x)}{1-xy}\right) - \frac{xy}{1-xy},$$

which gives (2.3).

3. The Case 1-22

Given $n \geq 3$ and $1 \leq j \leq i \leq \lfloor n/2 \rfloor$, let $a_{n,i,j}$ denote the number of 1-22 avoiding Catalan words of length *n* having *i* ones in which the right-most 1 that is followed by at least one zero is the *j*-th 1 from the left. Let $A_{n,i,j}$ denote the subset of $W(n)$ enumerated by $a_{n,i,j}$. Let $b_{n,i} = a(n,i)$ and $b^*_{n,i} = a^*(n,i)$, where $a(n,i)$ and $a^*(n,i)$ are as previously defined. Note that $b_{n,1} = 0$ if $n > 1$ and that $b^*_{n,i} = 0$ if $n < 2i-1$. We have the following recurrence relation for the numbers $a_{n,i,j}$.

Lemma 3.1. *The array* $a_{n,i,j}$ *may assume non-zero values only when* $n \geq 3$ *and* $1 \leq j \leq i \leq \lfloor n/2 \rfloor$, *with* $a_{3,1,1} = 1$, $a_{4,1,1} = 2$, $a_{4,2,1} = 1$, *and* $a_{4,2,2} = 0$. *If* $n \geq 5$, *then*

$$a_{n,i,j} = a_{n-1,i,j} + \sum_{\ell=1}^{j-1}\left(a_{n-1,i,\ell} + a_{n-2,i-1,\ell}\right) + \sum_{\ell=0}^{n-i-2}\left(b^*_{n-\ell-2,i} + b^*_{n-\ell-3,i-1}\right), \quad 1 \leq j < i \leq \lfloor n/2 \rfloor, \tag{3.1}$$

with

$$a_{n,i,i} = a_{n-1,i,i} + \sum_{\ell=1}^{i-1} a_{n-1,i,\ell} + \sum_{\ell=0}^{n-i-2} b^*_{n-\ell-2,i}, \quad 1 \leq i \leq \lfloor n/2 \rfloor. \tag{3.2}$$

Proof. The boundary conditions follow from the definitions; note that $i \leq \lfloor n/2 \rfloor$, for otherwise there would be two adjacent ones, which is not allowed. To prove the recurrences, note first that there are $a_{n-1,i,j}$ members of $A_{n,i,j}$ in which the final run of zeros has length greater than one, for adding a zero directly after the *j*-th one from the left within a member of $A_{n-1,i,j}$ defines a bijection. On the other hand, there are $\sum_{\ell=1}^{j-1} a_{n-1,i,\ell}$ members of $A_{n,i,j}$ where the final run of zeros consists of a single zero, with the letter (if it exists) following this zero greater than one, and where there are at least two zeros outside of the initial run of zeros. This follows from adding a single zero directly after the *j*-th one from the left within some member of $\bigcup_{\ell=1}^{j-1} A_{n-1,i,\ell}$, which defines a bijection between this set and the subset of $A_{n,i,j}$ in question. In the case when $j < i$, it is also possible for the letter following the right-most zero to equal one. Then there are $\sum_{\ell=1}^{j-1} a_{n-2,i-1,\ell}$ additional members of $A_{n,i,j}$ in this case, upon adding 01 directly after the *j*-th one from the left within some member of $\bigcup_{\ell=1}^{j-1} A_{n-2,i-1,\ell}$.

Combining the cases in the previous paragraph accounts for all members of $A_{n,i,j}$ in which there are at least two zeros outside of zeros found within the initial run. To complete the proof, we then need to show that there are

$$\sum_{\ell=0}^{n-i-2}\left(b^*_{n-\ell-2,i} + b^*_{n-\ell-3,i-1}\right)$$

members of $A_{n,i,j}$ containing only a single zero outside of the initial run of zeros when $j < i$ and $\sum_{\ell=0}^{n-i-2} b^*_{n-\ell-2,i}$ such members of $A_{n,i,j}$ when $j = i$. In both cases, we may form possible words as follows. Suppose $w \in W^*(n-\ell-2,i)$ for some $0 \leq \ell \leq n-i-2$ is written using positive instead of non-negative integers and is of the form $w = x1y$, where the 1 represents the *j*-th one from the left within *w*. Then $w' = 0^{\ell+1}x10y \in A_{n,i,j}$ is of the desired form. Allowing ℓ and *w* to vary gives all words in question in the case $j = i$ and thus completes the proof of (3.2). If $j < $ i, then it is also possible for a one to follow the right-most zero. Such words may be created by starting with $w \in W^*(n-\ell-3,i-1)$ for some $0 \leq \ell \leq n-i-2$, expressed as $w = x1y$ as before, and letting $w' = 0^{\ell+1}x101y$. This gives $\sum_{\ell=0}^{n-i-2} b^*_{n-\ell-3,i-1}$ additional members of $A_{n,i,j}$ of the desired form when $j < i$, which completes the proof of (3.1).

Let

$$g(x,y) = \sum_{n\geq 1}\sum_{i=1}^{n} b_{n,i}x^n y^i \ \textit{and}\ h(x,y) = \sum_{n\geq 1}\sum_{i=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} b^*_{n,i}x^n y^i.$$

Given $n \geq 3$ and $1 \leq i \leq \lfloor n/2 \rfloor$, let

$$A_{n,i}(v) = \sum_{j=1}^{i} a_{n,i,j}v^j \ \textit{and}\ A_n(u,v) = \sum_{i=1}^{\lfloor n/2 \rfloor} A_{n,i}(v)u^i.$$

Define the generating function $f(x;u,v)$ by

$$f(x;u,v) = \sum_{n\geq 3} A_n(u,v)x^n.$$

Lemma 3.2. *We have*

$$\left(1 - \frac{x(1+xuv)}{1-v}\right)f(x;u,v) + \frac{xv(1+xu)}{1-v}f(x;uv,1) = \frac{x^2v(1+xu)}{(1-x)(1-v)}\left(h(x,u) - h(x,uv)\right). \tag{3.3}$$

Proof. Multiplying both sides of (3.1) by v^j and summing over $1 \leq j \leq i-1$ and adding v^i times (3.2) gives

$$\begin{aligned}&A_{n,i}(v)\\&=\sum_{j=1}^{i}v^j\sum_{\ell=1}^{j}a_{n-1,i,\ell}+\sum_{j=2}^{i}v^j\sum_{\ell=1}^{j-1}a_{n-2,i-1,\ell}\\&\quad-v^i\sum_{\ell=1}^{i-1}a_{n-2,i-1,\ell}+\sum_{j=1}^{i}v^j\sum_{\ell=0}^{n-i-2}\left(b^*_{n-\ell-2,i}+b^*_{n-\ell-3,i-1}\right)\\&\quad-v^i\sum_{\ell=0}^{n-i-2}b^*_{n-\ell-3,i-1}\\&=\sum_{\ell=1}^{i}\frac{v^\ell\left(1-v^{i-\ell+1}\right)}{1-v}a_{n-1,i,\ell}+\sum_{\ell=1}^{i-1}\frac{v^{\ell+1}\left(1-v^{i-\ell}\right)}{1-v}a_{n-2,i-1,\ell}\\&\quad-v^i\sum_{\ell=1}^{i-1}a_{n-2,i-1,\ell}\\&\quad+\frac{v\left(1-v^i\right)}{1-v}\sum_{\ell=0}^{n-i-2}\left(b^*_{n-\ell-2,i}+b^*_{n-\ell-3,i-1}\right)\\&\quad-v^i\sum_{\ell=0}^{n-i-2}b^*_{n-\ell-3,i-1},\end{aligned}$$

which for $n \geq 5$ implies

$$\begin{aligned}A_{n,i}(v)&=\frac{1}{1-v}\left(A_{n-1,i}(v)-v^{i+1}A_{n-1,i}(1)\right)\\&\quad+\frac{v}{1-v}\left(A_{n-2,i-1}(v)-v^{i-1}A_{n-2,i-1}(1)\right)\\&\quad+\frac{v\left(1-v^i\right)}{1-v}\sum_{\ell=0}^{n-i-2}\left(b^*_{n-\ell-2,i}+b^*_{n-\ell-3,i-1}\right)\\&\quad-v^i\sum_{\ell=0}^{n-i-2}b^*_{n-\ell-3,i-1},\quad 1\leq i\leq\lfloor n/2\rfloor.\end{aligned}\tag{3.4}$$

Note that (3.4) is also seen to hold for $n = 3$ and $n = 4$ since $A_{3,1}(v)=A_{4,2}(v)=v$ and $A_{4,1}(v)=2v$, where we take $A_{n,i}(v)=0$ if $n<3$ or $i=0$ and $b^*_{n,0}=0$ for all n.

Multiplying (3.4) by u^i and summing over $1\leq i\leq\lfloor n/2\rfloor$ gives

$$\begin{aligned}&A_n(u,v)\\&=\frac{1}{1-v}\left(A_{n-1}(u,v)-vA_{n-1}(uv,1)\right)\\&\quad+\frac{uv}{1-v}\left(A_{n-2}(u,v)-A_{n-2}(uv,1)\right)\\&\quad+\frac{v}{1-v}\sum_{\ell=0}^{n-3}\sum_{i=1}^{n-\ell-2}\left(b^*_{n-\ell-2,i}+b^*_{n-\ell-3,i-1}\right)\left(u^i-(uv)^i\right)\\&\quad-\sum_{\ell=0}^{n-3}\sum_{i=1}^{n-\ell-2}b^*_{n-\ell-3,i-1}(uv)^i,\qquad n\geq 3,\end{aligned}\tag{3.5}$$

with $A_1(u,v)=A_2(u,v)=0$. Multiplying both sides of (3.5) by x^n and summing over $n\geq 3$ yields

$$\begin{aligned}f(x;u,v)&=\frac{x}{1-v}\left(f(x;u,v)-vf(x;uv,1)\right)\\&\quad+\frac{x^2uv}{1-v}\left(f(x;u,v)-f(x;uv,1)\right)\\&\quad+\frac{v}{1-v}\sum_{\ell\geq 0}x^{\ell+2}\sum_{i\geq 1}\left(u^i-(uv)^i\right)\sum_{n\geq i}b^*_{n,i}x^n\\&\quad+\frac{v}{1-v}\sum_{\ell\geq 0}x^{\ell+3}\sum_{i\geq 1}\left(u^i-(uv)^i\right)\sum_{n\geq i-1}b^*_{n,i-1}x^n\\&\quad-\sum_{\ell\geq 0}x^{\ell+3}\sum_{i\geq 1}(uv)^i\sum_{n\geq i-1}b^*_{n,i-1}x^n\\&=\frac{x}{1-v}\left(f(x;u,v)-vf(x;uv,1)\right)\\&\quad+\frac{x^2uv}{1-v}\left(f(x;u,v)-f(x;uv,1)\right)\\&\quad+\frac{x^2v}{(1-x)(1-v)}\left(h(x,u)-h(x,uv)\right)\\&\quad+\frac{x^3v}{(1-x)(1-v)}\left(uh(x,u)-uvh(x,uv)\right)\\&\quad-\frac{x^3uv}{1-x}h(x,uv).\end{aligned}$$

Rearranging the last equation gives (3.3).

Let C_n and M_n denote the Catalan and Motzkin number sequences, respectively (see A000108 and A001006 in [11]). Recall that

$$C(x)=\sum_{n\geq 0}C_nx^n=\frac{1-\sqrt{1-4x}}{2x}$$

$$\text{and } M(x)=\sum_{n\geq 0}M_nx^n=\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$$

and the relation $xC(x)^2=C(x)-1$. Let $a_n=a_{1-22}(n)$. By the definitions, we have

$$a_n=1+\sum_{i=1}^{\lfloor n/2\rfloor}\sum_{j=1}^{i}a_{n,i,j},\quad n\geq 3,$$

with $a_1=a_2=1$ since we must also include the Catalan word consisting of all zeros.

Theorem 3.3. *We have*

$$\sum_{n\geq 1}a_nx^n=\frac{1-x-\sqrt{1-2x-3x^2}}{2x},\tag{3.6}$$

and hence $a_n = M_{n-1}$ *for all* $n \geq 1$.

Proof. Taking $v = 1/u$ in (3.3) implies

$$\begin{aligned}&\left(1+\frac{xu(1+x)}{1-u}\right)f(x;u,1/u)+\frac{x(1+xu)}{u-1}f(x;1,1)\\&=\frac{x^2(1+xu)}{(1-x)(u-1)}\left(h(x,u)-h(x,1)\right).\end{aligned}\tag{3.7}$$

To solve (3.7), we use the *kernel method* [5] and let $u=\frac{1}{1-x-x^2}$ to cancel out the $f(x;u,1/u)$ term. This yields

$$f(x;1,1)=\frac{x}{1-x}\left(h\left(x,\frac{1}{1-x-x^2}\right)-h(x,1)\right)=\frac{x}{1-x}\left(g\left(\frac{x}{1+x},\frac{1}{1-x}\right)-g\left(\frac{x}{1+x},1\right)\right),$$

where we have used (2.3).

Recall from [[7], Theorem 2.8] that $g(x,y)=\frac{xy}{1-xyC(x)}$, and, in particular, that $g(x,1)=\frac{x}{1-xC(x)}=xC(x)$. Thus, we have

$$\begin{aligned}f(x;1,1)&=\frac{x}{1-x}g\left(\frac{x}{1+x},\frac{1}{1-x}\right)-\frac{x}{1-x}g\left(\frac{x}{1+x},1\right)\\&=\frac{x^2M}{\left(1-x^2\right)\left((1-x)M-\frac{x}{1+x}M^2\right)}-\frac{x^2M}{1-x^2}\\&=\frac{x^2M}{1-x^2}\left(\frac{1}{1-xM}-1\right)=\frac{x^3M^2}{\left(1-x^2\right)(1-xM)},\end{aligned}$$

where $M=C\left(\frac{x}{1+x}\right)$. It follows that

$$\begin{aligned}\sum_{n\ge1}a_nx^n&=\frac{x}{1-x}+f(x;1,1)=\frac{x(1+x)(1-xM)+x^3M^2}{\left(1-x^2\right)(1-xM)}\\&=\frac{x(1+x)(1-xM)+x^2(1+x)(M-1)}{\left(1-x^2\right)(1-xM)}\\&=\frac{x}{1-xM}=\frac{x+1-xM}{1-xM}-1=M-1\\&=\frac{1-x-\sqrt{1-2x-3x^2}}{2x}=\sum_{n\ge1}M_{n-1}x^n,\end{aligned}$$

which completes the proof.

Remark: Theorem 3.3 provides an apparently new combinatorial interpretation for the Motzkin number sequence.

4. The Cases 2-12 and 3-12

We first recall some notation. If i is a positive integer and x is an indeterminate, then let $[i]_x=1+x+\ldots+x^{i-1}$, with $[0]_x=0$. Define $[n]_x!=\prod_{i=1}^{n}[i]_x$ for n positive, with $[0]_x!=1$. We will need the following preliminary result.

Theorem 4.1. *Suppose F(x, y) satisfies the functional equation for* $-1<x<1$ *and y real*:

$$\begin{aligned}F(x,y)=a(x,y)+b(x,y)F(x,1)\\+xyc(x,y)F(x,1+xy),\end{aligned}\qquad(4.1)$$

where c(x, y) is continuous. Then we have

$$F(x,1)=\frac{\sum_{j\ge0}x^ja\left(x,[j+1]_x\right)[j]_x!\prod_{i=1}^{j}c\left(x,[i]_x\right)}{1-\sum_{j\ge0}x^jb\left(x,[j+1]_x\right)[j]_x!\prod_{i=1}^{j}c\left(x,[i]_x\right)},\qquad(4.2)$$

for all x sufficiently close to zero.

Proof. Note first that since c is continuous, we have for all positive integers j,

$$\left|x\left([j]_x+x^jy\right)c\left(x,[j]_x+x^jy\right)\right|<k,$$

for some constant $k<1$ whenever y lies on any given bounded interval and x is sufficiently close to zero. Thus, iteration of (4.1) leads to

$$\begin{aligned}F(x,y)&=\sum_{j\ge0}x^ja\left(x,[j]_x+x^jy\right)\prod_{i=0}^{j-1}\left([i]_x+x^iy\right)\\&\quad\times\prod_{i=0}^{j-1}c\left(x,[i]_x+x^iy\right)\\&\quad+F(x,1)\sum_{j\ge0}x^jb\left(x,[j]_x+x^jy\right)\prod_{i=0}^{j-1}\left([i]_x+x^iy\right)\\&\quad\times\prod_{i=0}^{j-1}c\left(x,[i]_x+x^iy\right)\end{aligned}\qquad(4.3)$$

for all such x and y. Note that both infinite series in (4.3) are seen to converge, upon comparison with a geometric series. Substituting $y=1$ in (4.3), we have

$$\begin{aligned}F(x,1)&=\sum_{j\ge0}x^ja\left(x,[j+1]_x\right)[j]_x!\prod_{i=1}^{j}c\left(x,[i]_x\right)\\&\quad+F(x,1)\sum_{j\ge0}x^jb\left(x,[j+1]_x\right)[j]_x!\prod_{i=1}^{j}c\left(x,[i]_x\right),\end{aligned}$$

and solving for $F(x, 1)$ gives (4.2).

Note that substituting the expression for $F(x, 1)$ from (4.2) back into (4.3) yields a formula for $F(x, y)$.

If $\tau\in$ {2-12, 3-12}, then let $a^*_{n,m}=\left|W^*_\tau(n,m)\right|$ for $1\le m\le n$. Note that $a^*_{n,m}$ may assume non-zero values only when $m\ge1$ and $n\ge2m-1$. Define the generating functions $A^*_m(x)=\sum_{n\ge m}a^*_{n,m}x^n$ for $m\ge1$ and $A^*(x,y)=\sum_{m\ge1}A^*_m(x)y^m$.

We now determine formulas for the generating functions in the cases 2-12 and 3-12.

4.1. The case 2-12. It will be more convenient to consider the primitive members of $W_{2\text{-}12}(n)$. The numbers $a^*_{n,m}$ in this case are determined recursively as follows.

Lemma 4.2. *We have*

$$a^*_{n,m} = \sum_{k=m-1}^{n-m} \binom{k}{m-1} a^*_{n-m,k},\ 2 \le m \le n-1, \qquad (4.4)$$

with $a^*_{n,1} = a^*_{n,n} = \delta_{n,1}$ *for all* $n \ge 1$.

Proof. We show recurrence (4.4), the boundary conditions being clear. Let $\mathcal{A}^*_{n,m} = W^*_{2-12}(n,m)$. Note that members of $\mathcal{A}^*_{n,m}$ containing k ones, where $m-1 \le k \le n-m$, may be formed by adding single zeros at the beginning and following exactly $m-1$ of the ones within a member of $\mathcal{A}^*_{n-m,k}$, expressed using positive letters. Note that no occurrence of 2-12 is introduced by inserting zeros as described since a letter succeeding any added zero (except for the zero at the very beginning) is strictly larger than one. Conversely, removing the zeros from any member of $\mathcal{A}^*_{n,m}$ containing exactly k ones, and decreasing each remaining letter by one, is seen to result in a member of $\mathcal{A}^*_{n-m,k}$. Thus, there are $\binom{k}{m-1} a_{n-m,k}$ members of $\mathcal{A}^*_{n,m}$ that contain k ones. Summing over k gives (4.4).

Multiplying (4.4) by $x^n y^m$, summing over $2 \le m < n$, and noting $A^*_1(x) = x$, we obtain

$$A^*(x,y) = xy\left(1 - A^*(x,1)\right) + xyA^*(x, 1+xy). \qquad (4.5)$$

Letting $a(x, y) = xy$, $b(x, y) = -xy$, and $c(x, y) = 1$ in Theorem 4.1 gives

$$A^*(x,1) = \frac{\sum_{j\ge1} x^j [j]_x!}{1+\sum_{j\ge1} x^j [j]_x!}.$$

Furthermore, by the proof of Theorem 4.1, we have

$$A^*(x,y) = xy\left(1-A^*(x,1)\right)\sum_{j\ge1} x^j \prod_{i=1}^{j}\left(1+x+\ldots+x^{i-1}+x^i y\right),$$

which implies

$$A^*(x,y) = \frac{xy\sum_{j\ge0} x^j [j]_x! \prod_{i=1}^{j}\left(1+\frac{x^i}{[i]_x}y\right)}{1+\sum_{j\ge1} x^j [j]_x!} = \frac{xy\sum_{j\ge0} x^j [j]_x! \sum_{i=0}^{j} e_i\left(\frac{x^1}{[1]_x},\frac{x^2}{[2]_x},\ldots,\frac{x^j}{[j]_x}\right)y^i}{1+\sum_{j\ge1} x^j [j]_x!},$$

where e_i denotes the i-th elementary symmetric function. Comparing the coefficients of y^m on both sides of the last equation leads to the following result.

Theorem 4.3. *The generating function* $\sum_{n\ge1} a^*_n x^n$ *for the sequence* $a^*_n = \sum_{m=1}^{n} a^*_{n,m}$ *is given by*

$$A^*(x,1) = \frac{\sum_{j\ge1} x^j [j]_x!}{1+\sum_{j\ge1} x^j [j]_x!}. \qquad (4.6)$$

Moreover, the generating function $A^*_m(x) = \sum_{n\ge m} a^*_{n,m} x^n$ *is given by*

$$A^*_m(x) = \frac{x\sum_{j\ge0} x^j [j]_x! e_{m-1}\left(\frac{x^1}{[1]_x},\frac{x^2}{[2]_x},\ldots,\frac{x^j}{[j]_x},\right)}{1+\sum_{j\ge1} x^j [j]_x!}. \qquad (4.7)$$

Replacing x by $x/(1 - x)$ in (4.6) gives the generating function that counts all members of $W_{2\text{-}12}(n)$ for $n \ge 1$, by Proposition 2.2.

4.2. The case 3-12. Again, we consider the primitive members of the avoidance class in question. The numbers $a^*_{n,m}$ in this case are determined recursively as follows.

Lemma 4.4. *If* $2 \le m \le n - 1$, *then*

$$a^*_{n,m} = \sum_{k=m}^{n-m} \binom{k-1}{m-1} a^*_{n-m,k} + \sum_{i=0}^{m-2} \sum_{k=m-i-1}^{n-m-i} \binom{k-1}{m-i-2}\left(a^*_{n-m-i,k} + a^*_{n-m-i-1,k}\right), \qquad (4.8)$$

with $a^*_{n,1} = a^*_{n,n} = \delta_{n,1}$ *for all* $n \ge 1$.

Proof. To show (4.8), let $\mathcal{A}^*_{n,m} = W_{3-12}(n,m)$, where $1 < m < n$. Then the first sum on the right side of (4.8) counts all members of $\mathcal{A}^*_{n,m}$ starting 012 according to the number k of ones. To see this, note that the letter following any zero, other than the first, within such members of $\mathcal{A}^*_{n,m}$ must be greater than one in order to avoid an occurrence of 3-12. Thus, these members of $\mathcal{A}^*_{n,m}$ may be obtained by inserting single 0's at the beginning and after any of the 1's, except the first, within a member of $\mathcal{A}^*_{n-m,k}$ expressed using positive letters, which can be done in $\binom{k-1}{m-1}$ ways.

To finish the proof, we must show that the double sum on the right-hand side of (4.8) counts all members of $\mathcal{A}^*_{n,m}$ starting 010. To do so, it is enough to show for each $0 \le i \le m-2$, that there are

$$\sum_{k=m-i-1}^{n-m-i} \binom{k-1}{m-i-2}\left(a^*_{n-m-i,k} + a^*_{n-m-i-1,k}\right)$$

members $\pi \in \mathcal{A}^*_{n,m}$ of the form $\pi = 010(10)^i \rho$ or $\pi = 010(10)^i 1\rho$, where ρ starts with 2 if non-empty. Let us denote this subset by $\mathcal{A}^*_{n,m,i}$. Suppose that $\pi \in \mathcal{A}^*_{n,m,i}$ is such that the word 1ρ contains exactly k ones. Note that $m-i-1 \le k \le n-m-i$. By subtraction, 1ρ has length $n-2i-2$ or $n-2i-3$ and contains $m-i-2$ zeros.

Deleting the zeros from 1ρ (and decreasing the remaining letters by one) results in a member $\alpha \in \mathcal{A}^*_{n-m-i,k} \cup \mathcal{A}^*_{n-m-i-1,k}$. Since these deleted zeros occurred singly within ρ, for each α produced, there are $\binom{k-1}{m-i-2}$ possible words 1ρ that could have given rise to it (note that a zero cannot follow the initial 1 in 1ρ). Thus, there are $\binom{k-1}{m-i-2}\left(a^*_{n-m-i,k}+a^*_{n-m-i-1,k}\right)$ members of $\mathcal{A}^*_{n,m,i}$ such that 1ρ contains k ones. Summing over all k gives the desired formula for $\left|\mathcal{A}^*_{n,m,i}\right|$ and completes the proof.

Multiplying both sides of (4.8) by y^m and summing over $m \ge 2$ gives

$$A^*_m(x) = x^m \sum_{j\ge m}\binom{j-1}{m-1}A^*_j(x) + (1+x)\sum_{i=0}^{m-2} x^{m+i} \sum_{j\ge m-i-1}\binom{j-1}{m-i-2}A^*_j(x),\ m\ge 2, \tag{4.9}$$

with $A^*_1(x) = x$. Multiplying (4.9) by x^n and summing over $n \ge m$ yields

$$\begin{aligned} A^*(x,y) &= xy + xy\sum_{j\ge 2}\left((1+xy)^{j-1}-1\right)A^*_j(x) + \frac{x^2y^2(1+x)}{1-x^2y}\sum_{j\ge 0}A^*_j(x)\sum_{i=0}^{j}\binom{j}{i}x^jy^j \\ &= xy\left(1-A^*(x,1)\right) + \frac{xy}{1+xy}A^*(x,1+xy) + \frac{x^2y^2(1+x)}{1-x^2y}\sum_{j\ge 0}(1+xy)^j A^*_{j+1}(x) \\ &= xy\left(1-A^*(x,1)\right) + \frac{xy}{1+xy}A^*(x,1+xy) + \frac{x^2y^2(1+x)}{\left(1-x^2y\right)(1+xy)}A^*(x,1+xy) \\ &= xy\left(1-A^*(x,1)\right) + \frac{xy}{1-x^2y}A^*(x,1+xy). \end{aligned}$$

Let $a(x,y) = xy$, $b(x,y) = -xy$, and $c(x,y) = \frac{1}{1-x^2y}$ in (4.1) above. Note that even though $c(x,y)$ fails to be continuous for all x and y, it is continuous for all y on a given bounded interval if x is sufficiently close to zero. Thus, the proof of Theorem 4.1 can be slightly modified to accommodate this case and gives the following result.

Theorem 4.5. *The generating function* $\sum_{n\ge 1} a^*_n x^n$ *for the sequence* $a^*_n = \sum_{m=1}^{n} a^*_{n,m}$ *is given by*

$$A^*(x,1) = \frac{\sum_{j\ge 1} x^j\,[j]_x!\prod_{i=0}^{j-1}\left(1-x^2[i]_x\right)^{-1}}{1+\sum_{j\ge 1} x^j\,[j]_x!\prod_{i=0}^{j-1}\left(1-x^2[i]_x\right)^{-1}}. \tag{4.10}$$

5. The Cases 1-23, 2-13, and 1-11

We will need the following additional definition for the combinatorial proofs in this section.

Definition 5.1. *By an alternating binary string within* $\pi = \pi_1\pi_2...\pi_n \in W(n)$, *we will mean a subsequence of the form* $\pi_i\pi_{i+1}...\pi_{i+r} = 1010...$ *for some* $r \ge 0$ *and contained in no other subsequences of this form (i.e., it is maximal in this respect).*

For example, if π = 01010210101032101 $\in W(17)$, then the alternating binary strings are 1010, 101010 and 101. Note that zeros belonging to the initial run of zeros are not part of any binary string.

For $\tau \in$ {1-23, 2-13} and $1 \le m \le n$, let $p_{n,m}$ and $q_{n,m}$ denote the number of members of $W^*_\tau(n,m)$ either not ending in zero or ending in zero, respectively. For $m \ge 1$, note that $p_{n,m}$ and $q_{n,m}$ may assume non-zero values only when $n \ge 2m$ or $n \ge 2m-1$, respectively. Let $\mathcal{P}_{n,m}$ and $\mathcal{Q}_{n,m}$ denote the subsets of $W^*_\tau(n,m)$ enumerated by $p_{n,m}$ and $q_{n,m}$, respectively. Given $m \ge 1$, let $P_m(x) = \sum_{n\ge m} p_{n,m}x^n$ and $Q_m(x) = \sum_{n\ge m} q_{n,m}x^n$ for either pattern. Finally, let $P(x,y) = \sum_{m\ge 1} P_m(x)y^m$ and $Q(x,y) = \sum_{m\ge 1} Q_m(x)y^m$.

5.1. The case 1-23. We determine a formula for the generating function in the case 1-23. The numbers $p_{n,m}$ and $q_{n,m}$ may be determined recursively as follows in this case.

Lemma 5.2. *If* $n \ge 3$ *and* $2 \le m \le n-1$, *then*

$$p_{n,m} = \sum_{k=1}^{m-1}\binom{m-2}{k-1}p_{n-2m+k+1,k} + \sum_{k=1}^{m}\binom{m-1}{k-1}q_{n-2m+k,k} \tag{5.1}$$

and

$$q_{n,m} = \sum_{k=1}^{m-1}\binom{m-2}{k-1}q_{n-2m+k+1,k}, \tag{5.2}$$

with $p_{n,1} = p_{n,n} = 0$ *and* $q_{n,1} = q_{n,n} = \delta_{n,1}$ *for all* $n \ge 1$.

Proof. The boundary conditions follow easily from the definitions. We first show recurrence (5.1). Suppose $\alpha = \alpha_1\alpha_2...\alpha_n \in \mathcal{P}_{n,m}$ contains k (alternating) binary strings. First assume that $\alpha_n > 1$. Note that within a binary string $\alpha_i\alpha_{i+1}...\alpha_{i+r}$ of α, we must have $\alpha_{i+r} = 0$ (whence r is odd) in order to avoid an occurrence of 1-23. We remove the initial zero of α as well as all letters except for the first within each binary string. Then concatenate the remaining letters and decrease each letter by one to obtain α'. Let s denote the number of ones that are removed from α in the process of obtaining α'. Note that for each one that is removed from α, the zero directly following it is removed as well. Taking also into account the removal from α of the initial zero as well as the second letter (a zero) of each binary string, we have that $k+1+s=m$. Thus, there are

$$k+1+2s=k+1+2(m-k-1)=2m-k-1$$

letters removed in all. This implies $\alpha' \in \mathcal{P}_{n-2m+k+1,k}$. For example, if

$$\alpha = 01010210210321010432 \in \mathcal{P}_{20,7},$$

which contains 4 binary strings, then $\alpha' \in 01010210321 \in \mathcal{P}_{11,4}$.

Conversely, starting with $\lambda \in \mathcal{P}_{n-2m+k+1,k}$, first increase each letter by one and then insert single zeros following each 1 and at the beginning. Let λ' denote the resulting word; note that $\lambda' \in \mathcal{P}_{n-2m+2k+2,k+1}$ and contains k binary strings. We then increase the lengths of the binary strings of λ' by adding sequences of the form $(10)^j$ for some $j \geq 0$ to them, with a total of $m-k-1$ strings of 10 to be added. Note that this may be achieved in $\binom{m-k-1+k-1}{k-1}=\binom{m-2}{k-1}$ ways and hence each $\lambda \in \mathcal{P}_{n-2m+k+1,k}$ gives rise to $\binom{m-2}{k-1}$ members of $\mathcal{P}_{n,m}$ ending in a letter greater than one. Thus, the first sum on the right-hand side of (5.1) counts such members of $\mathcal{P}_{n,m}$ according to the number k of binary strings; note that in this case, we have $1 \leq k \leq m-1$, from the definitions.

Similar reasoning shows that there are $\binom{m-1}{k-1} q_{n-2m+k,k}$ members of $\mathcal{P}_{n,m}$ containing k binary strings and ending in a 1. In this case, one would first increase by one each letter of $\lambda \in \mathcal{Q}_{n-2m+k,k}$ and then add single 0's at the beginning and after each 1, except for the last, to obtain $\lambda' \in \mathcal{P}_{n-2m+k,k}$ ending in 1. To λ', one can then add a sequence of the form $(10)^j$ following any 0, except after the initial 0, as well as a sequence of the form $(01)^j$ following the terminal 1. As the total number of 10 or 01 strings to be added is $m-k$, this may be achieved in $\binom{m-1}{k-1}$ ways. Summing over all k gives the second sum on the right-hand side of (5.1) and completes the proof of it. Similar reasoning as in the first case above gives (5.2).

By the above recurrences, we obtain for $m \geq 2$,

$$P_m(x)=\sum_{k=1}^{m-1}\binom{m-2}{k-1}x^{2m-k-1}P_k(x)+\sum_{k=1}^{m}\binom{m-1}{k-1}x^{2m-k}Q_k(x) \quad (5.3)$$

and

$$Q_m(x)=\sum_{k=1}^{m-1}\binom{m-2}{k-1}x^{2m-k-1}Q_k(x), \quad (5.4)$$

with $P_1(x)=0$ and $Q_1(x)=x$. Thus,

$$P_m(x) = \sum_{k=1}^{m-1}\binom{m-2}{k-1}x^{2m-k-1}P_k(x)+\frac{1}{x}Q_{m+1}(x), m \geq 2. \quad (5.5)$$

Recall from [[13], Eq. 24] the sequence $B_m(q)$ defined by the recurrence

$$B_{m+1}(q)=\sum_{j=0}^{m}\binom{m}{j}q^j B_j(q), \; m \geq 0, \quad (5.6)$$

with $B_0(q)=1$. Note that $B_m(q)$ is a polynomial generalization of the Bell numbers that reduces to them when $q=1$. Comparison of (5.4) with (5.6) reveals

$$Q_m(x)=x^{2m-1}B_{m-1}(x), m \geq 1. \quad (5.7)$$

At the conclusion of this subsection, we provide a bijective proof of (5.7). We have the following explicit formula for the exponential generating function of the sequence $B_n(q)$.

Lemma 5.3. *We have*

$$B(q,y):=\sum_{n\geq 0}B_n(q)\frac{y^n}{n!}=\frac{1}{e_q(1)}\sum_{n\geq 0}\frac{e^{[n]_q y}}{[n]_q!}, \quad (5.8)$$

where $e_q(1)=\sum_{n\geq 0}\frac{1}{[n]_q!}$. *Moreover,*

$$B_n(q)=\frac{1}{e_q(1)}\sum_{j\geq 0}\frac{[j]_q^n}{[j]_q!}, n \geq 0. \quad (5.9)$$

Proof. The recurrence relation $B_{d+1}(q)=\sum_{j=0}^{d}\binom{d}{j}q^j B_j(q)$ with $B_0(q)=1$ can be written as

$$\frac{d}{dy}B(q,y)=e^y B(q,qy) \text{ with } B(q,0)=1.$$

Let $B(q,y)=\sum_{n\geq 0}\beta_n(q)\frac{e^{[n]_q y}}{[n]_q!}$. Then $\beta_n(q)=\beta_{n-1}(q)$ for $n \geq 1$. Hence, $\beta_n(q)=\beta_0(q)$ for all n, which implies that $B(q,y)=\beta_0(q)\sum_{n\geq 0}\frac{e^{[n]_q y}}{[n]_q!}$. Upon noting $B(q,0)=1$, it follows that $\beta_0(q)=\frac{1}{e_q(1)}$, as required.

Remark: Formula (5.9) is seen to be a q-generalization of the Dobinski formula, reducing to it when $q = 1$.

Note that by the definitions, we have $a^{*}_{1-23}(n,m)=\sum_{m=1}^{n}(p_{n,m}+q_{n,m})$ By Proposition 2.3, to determine the generating function for the sequence $a_{1-23}(n,m)$, we only need to do so for $a^{*}_{1-23}(n,m)$.

Theorem 5.4. *If $m \geq 2$, then*

$$\sum_{n\geq m} a^{*}_{1-23}(n,m)x^{n} = P_m(x)+Q_m(x), \qquad (5.10)$$

where

$$P_m(x) = \frac{x^{2m}}{e_x(1)}\sum_{n\geq 1}\left(\sum_{i=2}^{n}[i]_x - \frac{1}{(e_x(1)-1)}\sum_{j\geq 2}\frac{\sum_{i=2}^{j}[i]_x}{[j]_x!}\right)\frac{[n]_x^{m-1}}{[n]_x!}$$

and

$$Q_m(x)=\frac{x^{2m-1}}{e_x(1)}\sum_{n\geq 1}\frac{[n]_x^{m-1}}{[n]_x!}.$$

Proof. Let $\tilde{P}_m(x)=x^{-2m}P_m(x)$. By (5.5) and (5.7), we have

$$\tilde{P}_m(x)=\sum_{k=1}^{m-1}\binom{m-2}{k-1}x^{k-1}\tilde{P}_k(x)+B_m(x),\ m\geq 2,$$

with $\tilde{P}_1(x)=0$.

Define $\tilde{P}(x,y)=\sum_{m\geq 1}\tilde{P}_m(x)\frac{y^m}{m!}$. Then the above recurrence can be written as

$$\frac{d^2}{dy^2}\tilde{P}(x,y)=\frac{1}{x}e^{y}\frac{d}{dy}\tilde{P}(x,xy)+\frac{d^2}{dy^2}B(x,y).$$

Assume that $\tilde{P}(x,y)=\sum_{n\geq 0}\alpha_n(x)\frac{e^{[n]_x y}}{[n]_x!}$. Then

$$\alpha_n(x)=\frac{[n-1]_x}{[n]_x}\alpha_{n-1}(x)+\frac{1}{e_x(1)},\ n\geq 2,$$

which implies that $\alpha_n(x)=\frac{\alpha_1(x)}{[n]_x}+\frac{1}{[n]_x e_x(1)}\sum_{i=2}^{n}[i]_x$ or all $n\geq 1$. Hence,

$$\tilde{P}(x,y)=\alpha_0(x)+\sum_{n\geq 1}\left(\frac{\alpha_1(x)}{[n]_x}+\frac{1}{[n]_x e_x(1)}\sum_{i=2}^{n}[i]_x\right)\frac{e^{[n]_x y}}{[n]_x!}.$$

Since $\tilde{P}(x,0)=0$ and $\frac{d}{dy}\tilde{P}(x,y)|_{y=0}=0$, we obtain

$$\alpha_1(x)=\frac{-1}{e_x(1)(e_x(1)-1)}\sum_{n\geq 2}\frac{\sum_{i=2}^{n}[i]_x}{[n]_x!},$$

$$\alpha_0(x)=\frac{1}{e_x(1)(e_x(1)-1)}\sum_{n\geq 2}\frac{\sum_{i=2}^{n}[i]_x}{[n]_x!}\sum_{n\geq 1}\frac{1}{[n]_x[n]_x!} - \frac{1}{e_x(1)}\sum_{n\geq 2}\frac{\sum_{i=2}^{n}[i]_x}{[n]_x[n]_x!}.$$

Hence, for all $m\geq 2$,

$$\tilde{P}_m(x) = \frac{1}{e_x(1)}\sum_{n\geq 1}\left(\sum_{i=2}^{n}[i]_x - \frac{1}{(e_x(1)-1)}\sum_{j\geq 2}\frac{\sum_{i=2}^{j}[i]_x}{[j]_x!}\right)\frac{[n]_x^{m-1}}{[n]_x!},$$

which implies that

$$P_m(x) = \frac{x^{2m}}{e_x(1)}\sum_{n\geq 1}\left(\sum_{i=2}^{n}[i]_x - \frac{1}{(e_x(1)-1)}\sum_{j\geq 2}\frac{\sum_{i=2}^{j}[i]_x}{[j]_x!}\right)\frac{[n]_x^{m-1}}{[n]_x!}.$$

By (5.7) and (5.9), we have

$$Q_m(x)=\frac{x^{2m-1}}{e_x(1)}\sum_{n\geq 0}\frac{[n]_x^{m-1}}{[n]_x!},\ m\geq 1.$$

Combinatorial proof of formula (5.7).

Let $\prod(n)$ denote the set of all partitions of $[n]$. Let $\pi = B_1B_2... \in \prod(n)$ with its blocks satisfying $\min B_1 < \min B_2 < ...$. Define the statistic w on $\prod(n)$ by setting $w(\pi)=\sum_{i\geq 1}(i-1)|B_i|$. It is well known (see, e.g., [13]) that

$$B_n(x)=\sum_{\pi\in\prod(n)}x^{w(\pi)},\ n\geq 0.$$

Thus, to show (5.7), we need to show for all $m\geq 1$ and $r\geq 0$ that the number of members of $\mathcal{Q}_{r+2m-1,m}$ equals the number of partitions of $[m-1]$ having w statistic value r. To do so, suppose $\rho\in\mathcal{Q}_{r+2m-1,m}$. Since ρ avoids 1-23, all (maximal) subsequences of consecutive non-zero letters of ρ must be of the form $i(i-1)...1$ for various i. Since ρ is primitive, contains exactly m zeros, and ends in a zero, it follows that there are exactly $m-1$ such subsequences. Furthermore, the first subsequence of the form $i(i-1)...1$ must occur to the left of the first of the form $j(j-1)...1$ if $1\leq i<j$, for otherwise the second defining property of a Catalan sequence would be violated.

Let τ be the subsequence of ρ of length $m-1$ obtained by taking the first letter of each (maximal) string of non-zero letters. Let us represent τ sequentially as $\tau=\tau_1\tau_2...\tau_{m-1}$. Then the word τ possesses the restricted growth property (see, e.g., [9]) and thus represents a partition π of $[m-1]$. Furthermore, equating expressions for the length of ρ implies $r+2m-1=m+\sum_{i=1}^{m-1}\tau_i$, and thus $r=\sum_{i=1}^{m-1}(\tau_i-1)=w(\pi)$, the second equality following from the definition of the w statistic. Hence, it is seen that the mapping $\rho \mapsto \pi$ defines a bijection from the set $\mathcal{Q}_{r+2m-1,m}$ to the subset of $\prod(m-1)$ whose members have w statistic value r, as desired.

5.2. The case 2-13. We determine a formula for the generating function. The numbers $p_{n,m}$ and $q_{n,m}$ are defined recursively in this case as follows.

Lemma 5.5. *If $n \geq 3$ and $2 \leq m \leq n-1$, then*

$$p_{n,m} = \sum_{k=1}^{n-2m+1} \binom{m+k-2}{k-1} \left(p_{n-2m+1,k} + q_{n-2m+1,k}\right) \quad (5.11)$$

and

$$q_{n,m} = \sum_{k=1}^{n-2m+2} \binom{m+k-3}{k-1} q_{n-2m+2,k}, \quad (5.12)$$

with $p_{n,1} = p_{n,n} = 0$ and $q_{n,1} = q_{n,n} = \delta_{n,1}$ for all $n \geq 1$.

Proof. We apply similar reasoning as in the case of 1-23 above and show that the right-hand sides of (5.11) and (5.12) count the members of $\mathcal{P}_{n,m}$ and $\mathcal{Q}_{n,m}$, respectively, according to the number k of binary strings. Note that a binary string not ending a member of either $\mathcal{P}_{n,m}$ or $\mathcal{Q}_{n,m}$ must have odd length (i.e., end in a 1) in order to avoid an occurrence of 2-13.

For (5.11), note first that members of $\mathcal{P}_{n,m}$ having k binary strings may be obtained by adding a single zero to some $\alpha \in \mathcal{P}_{n-2m+1,k} \cup \mathcal{Q}_{n-2m+1,k}$, expressed using positive letters, and then inserting $m-1$ strings of 01 in positions directly following the ones of α, which can be achieved in $\binom{m+k-2}{k-1}$ ways. This implies that there are $\binom{m+k-2}{k-1}\left(p_{n-2m+1,k} + q_{n-2m+1,k}\right)$ members of $\mathcal{P}_{n,m}$ that have k binary strings and summing over k gives (5.11).

For (5.12), we show that there are $\binom{m+k-3}{k-1} q_{n-2m+2,k}$ members of $\mathcal{Q}_{n,m}$ having k binary strings. Such members may be obtained by adding a single zero to the beginning and end of some $\beta \in \mathcal{Q}_{n-2m+2,k}$ on positive letters and then adding sequences of the form $(01)^j$ for some $j \geq 0$ directly following the ones and a sequence of the form $(10)^j$ following the terminal zero. Since the total number of 01 or 10 strings to be added is $m-2$, this can be achieved in $\binom{m+k-3}{k-1}$ ways, as desired.

By (5.12) and (5.11), we have

$$Q_m(x) = x^{2m-2} \sum_{k \geq 1} \binom{m+k-3}{k-1} Q_k(x), \ m \geq 2,$$

and

$$\begin{aligned} P_m(x) &= x^{2m-1} \sum_{k \geq 1} \binom{m+k-2}{k-1} P_k(x) \\ &\quad + x^{2m-1} \sum_{k \geq 1} \binom{m+k-2}{k-1} Q_k(x) \\ &= x^{2m-1} \sum_{k \geq 1} \binom{m+k-2}{k-1} P_k(x) + \frac{1}{x} Q_{m+1}(x), \ m \geq 2, \end{aligned}$$

with $Q_1(x) = x$ and $P_1(x) = 0$.

Then the above recurrence relations can be expressed as

$$Q(x,y) = xy + x^2y^2 Q\left(x, \frac{1}{1-x^2y}\right), \quad (5.13)$$

$$\begin{aligned} P(x,y) &= xyP\left(x, \frac{1}{1-x^2y}\right) - xyP(x,1) \\ &\quad + \frac{1}{xy}\left(Q(x,y) - xy - x^2y^2Q(x,1)\right). \end{aligned} \quad (5.14)$$

Let U_j denote the j-th Chebyshev polynomial of the second kind (see, e.g., [10]) defined by the recurrence $U_j(x) = 2xU_{j-1}(x) - U_{j-2}(x)$ for $j \geq 2$, with $U_0(x) = 1$ and $U_1(x) = 2x$. Define $a_j(y)$ by

$$a_j(y) = \frac{U_{j-1}(1/(2x)) - xyU_{j-2}(1/(2x))}{x\left(U_j(1/(2x)) - xyU_{j-1}(1/(2x))\right)}, \ j \geq 0.$$

It is possible to express the generating function $Q(x,y)$ in terms of Chebyshev polynomials as follows.

Lemma 5.6. *We have*

$$Q(x,y) = xy + x^2y^2 \sum_{j \geq 1} \frac{1}{\left(U_{j-1}(1/(2x)) - xyU_{j-2}(1/(2x))\right)\left(U_j(1/(2x)) - xyU_{j-1}(1/(2x))\right)}$$

and, moreover, $Q(x,1) = xC\left(x^2\right)$.

Proof. By (5.13) and the recurrence for the Chebyshev polynomials, we have

$$Q(x, a_i(y)) = xa_i(y) + x^2a_i^2(y)Q(x, a_{i+1}(y)).$$

Iterating this equation for $i \geq 0$, we obtain

$$Q(x,y) = xy + xy^2 \sum_{j \geq 1} x^{2j} \left(a_1(y)a_2(y)\ldots a_{j-1}(y)\right)^2 a_j(y),$$

which implies

$$Q(x,y) = xy + x^2y^2 \sum_{j \geq 1} \frac{1}{\left(U_{j-1}(1/(2x)) - xyU_{j-2}(1/(2x))\right)\left(U_j(1/(2x)) - xyU_{j-1}(1/(2x))\right)}.$$

Letting $y = 1$ gives

$$Q(x,1) = x + x^2\sum_{j\ge1}\frac{1}{\begin{pmatrix}U_{j-1}(1/(2x))\\ -xU_{j-2}(1/(2x))\end{pmatrix}\begin{pmatrix}U_j(1/(2x))\\ -xU_{j-1}(1/(2x))\end{pmatrix}},$$

which is equivalent to

$$Q(x,1) = x + \sum_{j\ge1}\frac{1}{U_j(1/(2x))U_{j+1}(1/(2x))}.$$

By [[7], Corollary 2.7], we have $Q(x,1) = x + x^3C^2(x^2) = xC(x^2)$, as required.

By (5.14) and the recurrence for the Chebyshev polynomials, we have

$$P(x,a_i(y)) = \frac{1}{xa_i(y)}\left(Q(x,a_i(y)) - xa_i(y) - x^2a_i^2(y)Q(x,1)\right) - xa_i(y)P(x,1) + xa_i(y)P(x,a_{i+1}(y)),$$

for all $i \ge 0$. Iterating this equation for $i \ge 0$, we obtain

$$P(x,y) = \sum_{j\ge0}x^ja_0(y)a_1(y)\dots a_{j-1}(y) \times\frac{\begin{bmatrix}Q(x,a_j(y)) - xa_j(y)\\ -x^2a_j^2(y)Q(x,1) - x^2a_j^2(y)P(x,1)\end{bmatrix}}{xa_j(y)},$$

which is equivalent to

$$P(x,y) = \sum_{j\ge0}x^{j-1}a_0(y)a_1(y)\dots a_{j-1}(y)\frac{Q(x,a_j(y))}{a_j(y)} - \sum_{j\ge0}x^ja_0(y)a_1(y)\dots a_{j-1}(y) - (P(x,1)+Q(x,1))\sum_{j\ge0}x^{j+1}a_0(y)a_1(y)\dots a_j(y).$$

Setting $y = 1$ and noting that $a_i(1) = \frac{U_i(1/(2x))}{xU_{i+1}(1/(2x))}$ and $a_0(1)a_1(1)\dots a_j(1) = \frac{1}{x^{j+1}U_{j+1}(1/(2x))}$, we find

$$P(x,1) = \frac{\begin{bmatrix}\sum_{j\ge0}\frac{Q(x,a_j(1))}{xa_j(1)U_j(1/(2x))}\\ -\sum_{j\ge0}\frac{1}{U_j(1/(2x))}\\ -Q(x,1)\sum_{j\ge1}\frac{1}{U_j(1/(2x))}\end{bmatrix}}{1+\sum_{j\ge1}\frac{1}{U_j(1/(2x))}},$$

which is equivalent to

$$P(x,1) = \frac{\begin{bmatrix}\sum_{j\ge0}\frac{U_{j+1}(1/(2x))Q(x,a_j(1))}{U_j^2(1/(2x))}\\ -1-(1+Q(x,1))\sum_{j\ge1}\frac{1}{U_j(1/(2x))}\end{bmatrix}}{1+\sum_{j\ge1}\frac{1}{U_j(1/(2x))}}.$$

Since $a_j(a_i(y)) = a_{i+j}(y)$ and $a_i(1) = \frac{U_i(1/(2x))}{xU_{i+1}(1/(2x))}$, it follows that

$$\begin{aligned}Q(x,a_i(1)) &= xa_i(1) + xa_i^2(1)\\ &\times\sum_{j\ge1}x^{2j}\left(a_1(a_i(1))a_2(a_i(1))\dots a_{j-1}(a_i(1))\right)^2a_j(a_i(1))\\ &= xa_i(1) + \sum_{j\ge1}x^{2j+1}a_i^2(1)a_{i+1}^2(1)\dots a_{i+j-1}^2(1)a_{i+j}(1)\\ &= \frac{U_i(1/(2x))}{U_{i+1}(1/(2x))} + \sum_{j\ge1}\frac{U_i^2(1/(2x))}{U_{i+j}(1/(2x))U_{i+j+1}(1/(2x))}.\end{aligned}$$

Hence,

$$P(x,1) = \frac{\begin{bmatrix}\sum_{i\ge0}\frac{1}{U_i(1/(2x))}\\ +\sum_{i\ge0}\sum_{j\ge1}\frac{U_{i+1}(1/(2x))}{U_{i+j}(1/(2x))U_{i+j+1}(1/(2x))}\\ -1-(1+Q(x,1))\sum_{j\ge1}\frac{1}{U_j(1/(2x))}\end{bmatrix}}{1+\sum_{j\ge1}\frac{1}{U_j(1/(2x))}},$$

which is equivalent to

$$P(x,1) = \frac{\sum_{i\ge0}\sum_{j\ge1}\frac{U_{i+1}(1/(2x))}{U_{i+j}(1/(2x))U_{i+j+1}(1/(2x))}+Q(x,1)}{1+\sum_{j\ge1}\frac{1}{U_j(1/(2x))}} - Q(x,1).$$

Since $Q(x,1) = xC(x^2)$, we have

$$P(x,1) = \frac{\sum_{i\ge1}\frac{\sum_{j=1}^{i}U_j(1/(2x))}{U_i(1/(2x))U_{i+1}(1/(2x))}+xC(x^2)}{1+\sum_{j\ge1}\frac{1}{U_j(1/(2x))}} - xC(x^2).$$

In order to simplify the expression of $P(x,1)$, we need the following lemma.

Lemma 5.7. *We have*

$$\sum_{i\geq 1}\frac{\sum_{j=1}^{i}U_j\left(1/(2x)\right)}{U_i\left(1/(2x)\right)U_{i+1}\left(1/(2x)\right)}=\frac{x}{1-2x}\left(1-(1-x)C\left(x^2\right)\right).$$

Proof. Let $m=\sum_{i\geq 1}\frac{\sum_{j=1}^{i}U_j\left(1/(2x)\right)}{U_i\left(1/(2x)\right)U_{i+1}\left(1/(2x)\right)}$. Define $\alpha=\frac{1+\sqrt{1-4x^2}}{2x}$ and $\beta=\frac{1-\sqrt{1-4x^2}}{2x}$. Using the fact that $U_j\left(1/(2x)\right)=\frac{\alpha^{j+1}-\beta^{j+1}}{\alpha-\beta}$, we obtain that

$$\begin{aligned}m&=\frac{1}{\alpha-\beta}\sum_{i\geq 1}\frac{\frac{\alpha^{i+2}}{\alpha-1}-\frac{\beta^{i+2}}{\beta-1}-\frac{\alpha^2}{\alpha-1}+\frac{\beta^2}{\beta-1}}{U_i\left(1/(2x)\right)U_{i+1}\left(1/(2x)\right)}\\
&=(\alpha-\beta)\sum_{i\geq 1}\frac{\frac{\alpha^{i+2}}{\alpha-1}-\frac{\beta^{i+2}}{\beta-1}}{\left(\alpha^{i+1}-\beta^{i+1}\right)\left(\alpha^{i+2}-\beta^{i+2}\right)}\\
&\quad+\frac{1}{\alpha-\beta}\sum_{i\geq 1}\frac{\frac{\beta^2}{\beta-1}-\frac{\alpha^2}{\alpha-1}}{U_i\left(1/(2x)\right)U_{i+1}\left(1/(2x)\right)}\\
&=\frac{x(\alpha-\beta)}{1-2x}\sum_{i\geq 1}\left(\frac{1}{\alpha^{i+1}-\beta^{i+1}}-\frac{1}{\alpha^{i+2}-\beta^{i+2}}\right)\\
&\quad-\frac{1-x}{1-2x}\sum_{i\geq 1}\frac{1}{U_i\left(1/(2x)\right)U_{i+1}\left(1/(2x)\right)}\\
&=\frac{x}{1-2x}\sum_{i\geq 1}\left(\frac{1}{U_i\left(1/(2x)\right)}-\frac{1}{U_{i+1}\left(1/(2x)\right)}\right)\\
&\quad-\frac{1-x}{1-2x}\sum_{i\geq 1}\frac{1}{U_i\left(1/(2x)\right)U_{i+1}\left(1/(2x)\right)}\\
&=\frac{x^2}{1-2x}-\frac{1-x}{1-2x}\sum_{i\geq 1}\frac{1}{U_i\left(1/(2x)\right)U_{i+1}\left(1/(2x)\right)}.\end{aligned}$$

By [[7], Corollary 2.7], we have

$$\begin{aligned}m&=\frac{x^2}{1-2x}-\frac{1-x}{1-2x}\left(-x+xC\left(x^2\right)\right)\\
&=\frac{x}{1-2x}\left(1-(1-x)C\left(x^2\right)\right),\end{aligned}$$

which completes the proof.

Hence, by Lemma 5.7, one can write $P(x,1)$ as

$$P(x,1)=\frac{\frac{x}{1-2x}\left(1-(1-x)C\left(x^2\right)\right)+xC\left(x^2\right)}{1+\sum_{j\geq 1}\frac{1}{U_j\left(1/(2x)\right)}}-xC\left(x^2\right),$$

which is equivalent to

$$P(x,1)=\frac{x\left(1-xC\left(x^2\right)\right)}{(1-2x)\left(1+\sum_{j\geq 1}\frac{1}{U_j\left(1/(2x)\right)}\right)}-xC\left(x^2\right).$$

The last equality, combined with Lemma 5.6, gives the following result.

Theorem 5.8. *We have*

$$\begin{aligned}\sum_{n\geq 1}a^{*}_{2-13}(n)x^n&=P(x,1)+Q(x,1)\\
&=\frac{x\left(1-xC\left(x^2\right)\right)}{(1-2x)\left(1+\sum_{j\geq 1}\frac{1}{U_j\left(1/(2x)\right)}\right)},\end{aligned}\qquad(5.15)$$

where U_m *is the m-th Chebyshev polynomial of the second kind.*

We conclude this section with a couple of combinatorial results. We first provide a combinatorial proof of the relation $Q(x,1)=xC\left(x^2\right)$ from Lemma 5.6.

Let $q_n=\sum_{m=1}^{n}q_{n,m}$.

Proposition 5.9. *If* $n\geq 1$, *then*

$$q_n=\begin{cases}0, & \text{if } n \text{ is even;}\\ C_{\frac{n-1}{2}}, & \text{if } n \text{ is odd.}\end{cases}\qquad(5.16)$$

Proof. First note that a Catalan word avoids 2-13 if and only if it avoids 13 (i.e., ascents of size more than one). For if $\pi=\pi_1\pi_2\ldots\pi_n$ is a Catalan word and i is an index such that $\pi_{i+1}-\pi_i>1$, then there exists an index $i_1<i$ such that $\pi_{i_1}=\pi_{i+1}-1$ and thus $\pi_{i_1}\pi_i\pi_{i+1}$ would be an occurrence of 2-13. Let $\mathcal{Q}_n=\bigcup_{m=1}^{n}\mathcal{Q}_{n,m}$ and $\sigma=\sigma_1\sigma_2\ldots\sigma_n\in\mathcal{Q}_n$. Then $\sigma_{i+1}-\sigma_i=\pm 1$ for all i, by the previous observation, the first condition for Catalan words, and the primitiveness of σ. Thus $\sigma_1=\sigma_n=0$ implies $\mathcal{Q}_n=\varnothing$ if n is even, upon observing $\sigma_n-\sigma_1=\sum_{i=1}^{n-1}(\sigma_{i+1}-\sigma_i)$. If n is odd, then recording the sequence of differences $\sigma_{i+1}-\sigma_i$ for $1\leq i\leq n-1$ and replacing +1's by up steps and −1's by down steps defines a bijection between members of $\mathcal{Q}_n$ and Catalan paths of length $n-1$. This implies the odd case of (5.16) and completes the proof.

By a *first quadrant path*, we will mean one consisting of up (1,1) steps and down (1,-1) steps starting at the origin and never dipping below the *x*-axis. The *height* of a point will refer to the *y*-coordinate of the point. Points on the *x*-axis have height zero, including the initial point.

Definition 5.10. *We will say that a first quadrant path P satisfies the height condition if for each i > 0 for which there is at least one point of P of height i, there is at least one point of height i – 1 to the right of the left-most point of height i.*

By the $i=1$ case, paths satisfying the height condition must have at least one return to the *x*-axis. Upon recording the sequence of differences, one sees that first quadrant paths of length $n-1$ satisfying the height condition are in bijection with members of $W^{*}_{2-13}(n)$, with the number of returns to the *x*-axis corresponding to the number of zeros (excluding the initial zero).

First quadrant paths where horizontal (1,0) steps are also allowed are known as *Motzkin left factors* (see, e.g., [[1], p. 111]). The set of Motzkin left factors having $n-1$ steps is enumerated by the sequence L_n, which occurs as A005773 in [11]. Upon recording the sequence of differences, one obtains the following description of the members of $W_{2\text{-}13}(n)$ in terms of lattice paths.

Proposition 5.11. *Members of $W_{2\text{-}13}(n)$ are in one-to-correspondence with Motzkin left factors of length $n-1$ satisfying the height condition.*

By Proposition 2.2, replacing x by $\frac{x}{1-x}$ in equation (5.15) gives the generating function for the sequence $a_{2-13}(n)$ and hence for the sequence that counts Motzkin left factors of length $n-1$ satisfying the height condition.

5.3. The case 1-11. For this pattern, we refine the enumerating sequence in a slightly different way. First observe that if a Catalan word avoids 1-11, then all runs of any letter have length one, except for possibly the first run, which can have length two. Let $r_{n,m}$ denote the number of members of $W_{1-11}(n,m)$ in which all runs of the letter zero have length one. Note that $a_{1-11}(n,m)=r_{n,m}+r_{n-1,m-1}$, since the initial run of zeros can have length one or two. The numbers $r_{n,m}$ are determined recursively as follows.

Lemma 5.12. *If $n \geq 3$ and $2 \leq m \leq n-1$, then*

$$r_{n,m} = \sum_{k=1}^{n-m}\sum_{j=0}^{m-1}\binom{j+k-1}{k-1}\binom{k}{m-j-1}\left(r_{n-m-j,k}+r_{n-m-j-1,k}\right), \tag{5.17}$$

with $r_{n,1}=r_{n,n}=\delta_{n,1}$ for all $n \geq 1$.

Proof. Let $\mathcal{R}_{n,m}$ denote the subset of $W_{1-11}(n,m)$ enumerated by $r_{n,m}$, where $2 \leq m \leq n-1$. Let $\mathcal{R}'_{n,m}$ be the subset of $\mathcal{R}_{n,m}$ all of whose runs of ones have length one as well and $\mathcal{R}'_{n,m,k}$ denote those members of $\mathcal{R}'_{n,m}$ having exactly k binary strings for $1 \leq k \leq n-m.$ We will show that

$$\left|\mathcal{R}'_{n,m,k}\right| = \sum_{j=0}^{m-1}\binom{j+k-1}{k-1}\binom{k}{m-j-1}r_{n-m-j,k}, \quad 1 \leq k \leq n-m, \tag{5.18}$$

whence the first part of the sum on the right-hand side of (5.17) gives the cardinality of $\mathcal{R}'_{n,m}$.

To do so, we show that there are $\binom{j+k-1}{k-1}\binom{k}{m-j-1}r_{n-m-j,k}$ members of $\mathcal{R}'_{n,m,k}$ that contain $j+k$ ones for $0 \leq j \leq m-1$, whence (5.18) follows from summing over j. Observe first that members of $\mathcal{R}'_{n,m,k}$ containing $j+k$ ones may be formed from members of $\mathcal{R}_{n-m-j,k}$, expressed using positive letters, by adding a zero at the beginning and then inserting in positions following the ones j strings of 01 together with $m-j-1$ single zeros. Note that a single zero can follow a (possibly empty) sequence of the form $(01)^{\ell}$ after a one, but that a binary string cannot contain more than one single zero. Given these restrictions, we see for each $\lambda \in \mathcal{R}_{n-m-j,k}$, that there are $\binom{j+k-1}{k-1}\binom{k}{m-j-1}$ members of $\mathcal{R}'_{n,m,k}$ containing $j+k$ ones which give rise to λ when the initial zero and all letters except the first within each binary string are removed, from which the claim follows.

Upon inserting an additional one within the initial run of ones, we see that $\left|\mathcal{R}_{n,m}-\mathcal{R}'_{n,m}\right|$ equals $\left|\mathcal{R}'_{n-1,m}\right|$ and is thus given by the second part of the sum on the right-hand side of (5.17), which completes the proof.

Define $R_m(x)=\sum_{n\geq m} r_{n,m}x^n$. Multiplying (5.17) by x^n and summing over $n \geq m+1$ yields

$$R_m(x) = (1+x)\sum_{j=0}^{m-1}\left(x^{m+j}\sum_{k\geq 1}\binom{j+k-1}{j}\binom{k}{m-j-1}R_k(x)\right), \quad m \geq 2,$$

with $R_1(x)=x.$

Define $R(x,y)=\sum_{m\geq 1} R_m(x)y^m$. Multiplying the last recurrence by y^m and summing over $m \geq 2$ yields

$$R(x,y)-xy = -xy(1+x)\sum_{k\geq 1}R_k(x) + xy(1+x)\sum_{k\geq 1}R_k(x)(1+xy)^k\sum_{j\geq 0}\binom{j+k-1}{k-1}x^{2j}y^j,$$

which is equivalent to

$$R(x,y) = xy\left(1-(1+x)R(x,1)\right) + xy(1+x)R\left(x,\frac{1+xy}{1-x^2y}\right). \tag{5.19}$$

In order to solve this equation, we define

$$\rho_m(x,y)=\frac{1+x\rho_{m-1}(x,y)}{1-x^2\rho_{m-1}(x,y)}, \quad m \geq 1,$$

with $\rho_0(x,y)=y.$

Lemma 5.13. *For all $m \geq 0$,*

$$\rho_m(x,y)=\frac{\frac{1+xy}{\sqrt{x(1+x)}}U_{m-1}(t)-yU_{m-2}(t)}{\left[\begin{array}{l}\sqrt{x(1+x)}\left(\frac{1+xy}{\sqrt{x(1+x)}}U_m(t)-yU_{m-1}(t)\right)\\ -x\left(\frac{1+xy}{\sqrt{x(1+x)}}U_{m-1}(t)-yU_{m-2}(t)\right)\end{array}\right]},$$

where $t=\sqrt{\frac{1+x}{4x}}$.

Proof. Let $\rho_m(x,y) = f_m / g_m$. Then, by the definitions, we have $f_m = g_{m-1} + xf_{m-1}$ and $g_m = g_{m-1} - x^2 f_{m-1}$, with $f_0 = y$ and $g_0 = 1$. This implies $f_m - f_{m-1} = -x^2 f_{m-2} + xf_{m-1} - xf_{m-2}$, or

$$f_m = (1+x) f_{m-1} - x(1+x) f_{m-2}, \ m \geq 2,$$

with $f_0 = y$ and $f_1 = 1 + xy$. By induction and the defining recurrence of the Chebyshev polynomials U_m, we obtain

$$f_m = \left(\sqrt{x(1+x)}\right)^m \left(\frac{1+xy}{\sqrt{x(1+x)}} U_{m-1}(t) - yU_{m-2}(t) \right).$$

Noting $\rho_m(x,y) = f_m / g_m = f_m / (f_{m+1} - xf_m)$ gives the desired result.

By (5.19) and the definition of $\rho_j(x,y)$, we have

$$R(x, \rho_j(x,y)) = x\rho_j(x,y)(1-(1+x)R(x,1)) + x\rho_j(x,y)(1+x)R(x, \rho_{j+1}(x,y)), \ j \geq 0.$$

Iteration of this last equation yields

$$R(x,y) = (1-(1+x)R(x,1)) \sum_{j\geq 1} x^j (1+x)^{j-1} \prod_{i=0}^{j-1} \rho_i(x,y),$$

which implies

$$R(x,1) = \frac{\sum_{j\geq 1} x^j (1+x)^{j-1} \prod_{i=0}^{j-1} \rho_i(x,1)}{1 + \sum_{j\geq 1} x^j (1+x)^j \prod_{i=0}^{j-1} \rho_i(x,1)}.$$

Note that $\rho_i(x,1) = \frac{U_i(t)}{xU_{i+1}(t)}$, where $t = \sqrt{\frac{1+x}{4x}}$, which gives the following result.

Theorem 5.14. *We have*

$$R(x,1) = \frac{\sum_{j\geq 1} \frac{\sqrt{1+x}^{2j-1}}{\sqrt{x} U_j(t) U_{j+1}(t)}}{1 + \sum_{j\geq 1} \frac{\sqrt{1+x}^{2j+1}}{\sqrt{x} U_j(t) U_{j+1}(t)}}, \quad (5.20)$$

where $t = \sqrt{\frac{1+x}{4x}}$ *and* U_m *is the m-th Chebyshev polynomial of the second kind.*

Since $a_{1-11}(n) = \sum_{m=1}^{n} (r_{n,m} + r_{n-1,m-1})$ for $n \geq 1$, the generating function $\sum_{n\geq 1} a_{1-11}(n) x^n$ is given by $(1+x)R(x,1)$. Hence, we can state the following result.

Theorem 5.15. *We have*

$$\sum_{n\geq 1} a_{1-11}(n) x^n = \frac{\sum_{j\geq 1} \frac{\sqrt{1+x}^{2j+1}}{\sqrt{x} U_j(t) U_{j+1}(t)}}{1 + \sum_{j\geq 1} \frac{\sqrt{1+x}^{2j+1}}{\sqrt{x} U_j(t) U_{j+1}(t)}}, \quad (5.21)$$

where $t = \sqrt{\frac{1+x}{4x}}$ *and* U_m *is the m-th Chebyshev polynomial of the second kind.*

6. The Remaining Cases

We consider the remaining patterns of type (1,2). Our work is reduced by the following observations.

Observation 6.1. *We have* $a_{2-31}(n) = C_{n-1}$ *for all* $n \geq 1$, *i.e., there is no restriction introduced by 2-31, by the first defining property of Catalan words. We have* $a_{1-21}(n) = 1$ *for all* $n \geq 1$ *since only the word consisting of all zeros is possible, by the second property of Catalan words.*

Let F_m denote the Fibonacci sequence defined by $F_m = F_{m-1} + F_{m-2}$ if $m \geq 3$, with $F_1 = F_2 = 1$.

Observation 6.2. If $\tau \in$ {1-12, 1-32, 2-21, 3-21}, then it is seen that a Catalan word avoids τ if and only if it avoids the corresponding classical pattern of length three (obtained by removing the adjacency requirement concerning the second and third letters). Furthermore, in [8], it was shown that $a_{1\text{-}1\text{-}2}(n) = F_n$, $a_{1\text{-}3\text{-}2}(n) = 2^{n-1} - (n-1)$, $a_{2\text{-}2\text{-}1}(n) = 2^{n-2}$ if $n \geq 2$ with $a_{2\text{-}2\text{-}1}(1) = 1$, and $a_{3\text{-}2\text{-}1}(n) = F_{2n-3}$.

Finally, the only pattern of type (1,2) that remains is 2-11. Note, in this case, that it is not useful to first consider the primitive members of the avoidance class due to the equal consecutive letters within the pattern. We have not been able to find a recurrence satisfied by the numbers $a_{2-11}(n)$ nor have we determined a formula for the generating function $\sum_{n\geq 1} a_{2-11}(n) x^n$. Based on numerical evidence, the sequence $a_{2-11}(n)$ does not seem to occur in OEIS [11]. We leave the question of determining a formula for $a_{2-11}(n)$ as an open problem.

References

[1] L. Alonso and R. Schott, *Random Generation of Trees*, Kluwer Academic Publishers, Dordrecht, The Netherlands 1995.

[2] A. Claesson, Generalized pattern avoidance, *European J. Combin.* 22:7 (2001), 961-971.

[3] A. Claesson and T. Mansour, Counting occurrences of a pattern of type (1,2) or (2,1) in permutations, *Adv. in Appl. Math.* 29:2 (2002), 293-310.

[4] S. Heubach, T. Mansour, and A. O. Munagi, Avoiding permutations of type (2,1) in compositions, *Online J. Anal. Comb.* 4 (2009), Article 3.

[5] Q. H. Hou and T. Mansour, Kernel method and systems of functional equations with several conditions, *J. Comput.Appl. Math.* 235:5 (2011), 1205-1212.

[6] T. Mansour and M. Shattuck, Avoiding type (1,2) or (2,1) patterns in a partition of a set, *Integers* 12 (2012), #A20.

[7] T. Mansour and M. Shattuck, Chebyshev polynomials and statistics on a new collection of words in the Catalan family, *J. Difference Equ. Appl.* 20:11 (2014), 1568-1582.

[8] T. Mansour and M. Shattuck, Avoidance of a pattern of length three by Catalan words, *Filomat* 31:3 (2017), 543-558.

[9] S. Milne, A *q*-analogue of restricted growth functions, Dobinski's equality, and Charlier polynomials, *Trans. Amer. Math. Soc.* 245 (1978), 89-118.

[10] T. Rivlin, *Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory*, John Wiley, New York, 1990.

[11] N. J. A. Sloane, On-line Encyclopedia of Integer Sequences, http://oeis.org, 2015.

[12] C. Stump, On a new collection of words in the Catalan family, *J. Integer Seq.* 17 (2014), Art. 14.7.1.

[13] C. G. Wagner, Partition statistics and q-Bell numbers ($q = -1$), *J. Integer Seq.* 7 (2004), Art. 04.1.1.