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The Multi Rough Ideal Convergence of Difference Strongly of χ^2 In *p*-Metric Spaces Defined by Orlicz Functions

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Abstract The aim of this paper is to introduce multi rough and study a new concept of the χ^2 space via ideal convergence of difference operator defined by Orlicz. Some topological properties of the resulting sequence spaces are also discussed.

Keywords: *Orlicz function, double sequences, χ^2 space, p-metric space, multi rough ideal*

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1. Introduction

The idea of rough convergence was introduced by Phu [3], who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar [1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal et al. [2] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence. Statistical convergence and ideal convergence was introduced by Mursaleen [7] and Tripathy et al. [8-13] and many others. In this paper we have introduced some Orlicz sequence spaces of fuzzy number using the notion of rough I- convergence and studied some algabraic and topological properties of these spaces.

Throughout ω, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences respectively. We write ω^2 for the set of all complex double sequences (x_{mn}), where $m,n \in \mathbb{N}$, the set of positive integers. Then, ω^2 is a linear space under the coordinate wise addition and scalar multiplication.

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ give one space is said to be convergent if and only if the double sequence (S_{mn}) is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} \ (m,n = 1,2,3,\ldots).$$

A double sequence $x = (x_{mn})$ is said to be double analytic if

$$\sup_{m,n} |x_{mn}|^{\frac{1}{m+n}} < \infty.$$

The vector space of all double analytic sequences are usually denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if

$$|x_{mn}|^{\frac{1}{m+n}} \to 0 \text{ as } m,n \to \infty.$$

The vector space of all double entire sequences are usually denoted by Γ^2. Let the set of sequences with this property be denoted by Λ^2 and Γ^2 is a metric space with the metric

$$d(x,y) = \sup_{m,n} \left\{ |x_{mn} - y_{mn}|^{\frac{1}{m+n}} : m,n : 1,2,3,\ldots \right\}, \quad (1.1)$$

for all $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in Γ^2. Let $\phi = \{\textit{finite sequences}\}$

Consider a double sequence $x = (x_{mn})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\delta_{ij}$ for all $m,n \in \mathbb{N}$,

$$\delta_{mn}=\begin{pmatrix}0 & 0 & ...0 & 0 & ...\\ 0 & 0 & ...0 & 0 & ...\\ \vdots & & & & \\ 0 & 0 & ...1 & 0 & ...\\ 0 & 0 & ...0 & 0 & ...\end{pmatrix}$$

with 1 in the $(m,n)^{th}$ position and zero otherwise.

An Orlicz function is a function $f:[0,\infty)\to[0,\infty)$ which is continuous, non-decreasing and convex with $f(0)=0,\quad f(x)>0,$ for $x>0$ and $f(x)\to\infty$ as $x\to\infty$. If convexity of Orlicz function f is replaced by $f(x+y)\le f(x)+f(y),$ then this function is called modulus function. An Orlicz function f is said to satisfy Δ^2-condition for all values u if there exists $K>0$ such that $f(2u)\le Kf(u),\quad u\ge 0.$

Remark 1: An Orlicz function satis.es the inequality $f(\lambda x)\le\lambda f(x)$ for all λ with $0<\lambda<1$.

1.1. Lemma. Let f be an Orlicz function which satisfies Δ^2- condition and let $0<\delta<1$. Then for each $t\ge\delta$, we have $f(t)<K\delta^{-1}f(2)$ for some constant $K>0$.

Let M and Φ be mutually complementary Orlicz functions. Then, we have

(i) For all $u,y\ge 0$,

$$uy\le M(u)+\Phi(y),(Young's\ inequality),[4]\qquad(1.2)$$

(ii) For all $u\ge 0$,

$$u\eta(u)=M(u)+\Phi(\eta(u)).\qquad(1.3)$$

(iii) For all $u\ge 0$, and $0<\lambda<1$,

$$M(\lambda u)\le\lambda M(u).\qquad(1.4)$$

[5] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M=\left\{x\in w:\sum_{k=1}^{\infty}M\left(\frac{|x_k|}{\rho}\right)<\infty,\ for\ some\ \rho>0\right\},$$

The space ℓ_M with the norm

$$\|x\|=\inf\left\{\rho>0:\sum_{k=1}^{\infty}M\left(\frac{|x_k|}{\rho}\right)\le 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t)=t^p\ (1\le p<\infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f=(f_{mn})$ of Orlicz function is called a Musielak-Orlicz function. A sequence $g=(g_{mn})$ defined by

$$g_{mn}(v)=\sup\{|v|u-f_{mn}(u):u\ge 0\},\ m,n=1,2,...$$

is called the complementary function of a Musielak-Orlicz function *f*. For a given Musielak Orlicz function *f*, the Musielak-Orlicz sequence space t_f is defined by

$$t_f=\left\{x\in w^3:M_f\left(|x_{mnk}|\right)^{1/m+n+k}\to 0\ as\ m,n,k\to\infty\right\},$$

where M_f is a convex modular defined by

$$M_f(x)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}f_{mnk}\left(|x_{mnk}|\right)^{1/m+n+k},\quad x=(x_{mnk})\in t_f.$$

We consider t_f equipped with the Luxemburg metric space.

Let $(X_i,d_i),\ i\in I$ be a family of metric spaces such that each two elements of the family are disjoint. Denote $X:\bigcup_{i\in I}X_i$. If we define

$$d(x,y)=\begin{cases}d_i(x,y), & if\ x,y\in X_i\\ +\infty, & if\ x\in X_i,\ y\in X_j, i\ne j\end{cases}$$

then the pair (X,d) is a Luxemburg metric space. The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [6] as follows

$$Z(\Delta)=\{x=(x_k)\in w:(\Delta x_k)\in Z\},$$

for $Z=c,c_0$ and ℓ_∞, where $\Delta x_k=x_k-x_{k+1}$ for all $k\in\mathbb{N}$.

Here c,c_0 and ℓ_∞ denote the classes of convergent, null and bounded scalar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1\le p\le\infty$ by Başar and Altay and in the case $0<p<1$. The spaces $c(\Delta),\ c_0(\Delta),\ \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\|=|x_1|+\sup_{k\ge 1}|\Delta x_k|\ and$$

$$\|x\|_{bv_p}=\left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p},(1\le p<\infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta)=\left\{x=(x_{mn})\in w^2:(\Delta x_{mn})\in Z\right\},$$

where $Z=\Lambda^2,\quad\chi^2$ and $\Delta x_{mn}-(x_{mn}-x_{mn+1})-(x_{m+1n}-x_{m+1n+1})\quad=x_{mn}-x_{mn+1}+x_{m+1n+1}$ for all $m,n\in\mathbb{N}$. The generalized difference double notion has the following representation: $\Delta^m x_{mn}=\Delta^{m-1}x_{mn}-\Delta^{m-1}x_{mn+1}-\Delta^{m-1}x_{m+1n}+\Delta^{m-1}x_{m+1n+1}$, and also this generalized difference double notion has the following binomial representation:

$$\Delta^m x_{mn}=\sum_{i=0}^{m}\sum_{j=0}^{m}(-1)^{i+j}\binom{m}{i}\binom{m}{j}x_{m+i,n+j}.$$

2. Definitions and Preliminaries

Let X be a non empty set. A non-void class $I \subseteq 2^X$ (power set, of X) is called an ideal if I is additive (i.e $A, B \in I \Rightarrow A \bigcup B \in I$) and hereditary (i.e $A \in I$ and $B \subseteq A \Rightarrow B \in I$). A non-empty family of sets $F \subseteq 2^{\Delta^m X}$ is said to be a filter on $\Delta^m X$ if $\phi \notin F : A, B \in F \Rightarrow A \bigcup B \in F$ and $A \in F$, $A \subseteq B \Rightarrow B \in F$. For each ideal I there is a filter $F(I)$ given by $F(I) = \{K \subseteq N : N \setminus K \in I\}$. A non-trivial ideal $I \subseteq 2^{\Delta^m X}$ is called admissible if and only if $\{\{x\} : x \in \Delta^m X\} \subset I$.

A double sequence space E is said to be solid or normal if $(\alpha_{mn} x_{mn}) \in E$, whenever $(x_{mn}) \in E$ and for all double sequences $\alpha = (\alpha_{mn})$ of scalars with $|\alpha_{mn}| \leq 1$ for all $m, n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and X be a real vector space of dimension w, where $n \leq w$. A real valued function $d_p(x_1, ..., x_n) = \|(d_1(x_1, 0), ..., d_n(x_n, 0))\|_p$ on X satisfying the following four conditions:

(i) $\|(d_1(x_1, 0), ..., d_n(x_n, 0))\|_p = 0$ if and only if $d_1(x_1, 0), ..., d_n(x_n, 0)$ are linearly dependent,

(ii) $\|(d_1(x_1, 0), ..., d_n(x_n, 0))\|_p$ is invariant under permutation,

(iii)
$$\|(\alpha d_1(x_1, 0), ..., d_n(x_n, 0))\|_p = |\alpha| \|(d_1(x_1, 0), ..., d_n(x_n, 0))\|_p, \alpha \in \mathbb{R}$$

(iv)
$$d_p((x_1, y_1), (x_2, y_2) ... (x_n, y_n)) = \left(d_X(x_1, x_2, ... x_n)^p + d_Y(y_1, y_2, ... y_n)^p\right)^{1/p}$$

for $1 \leq p \leq \infty$; (or)

(v)
$$d((x_1, y_1), (x_2, y_2), ... (x_n, y_n)) := \sup\{d_X(x_1, x_2, ... x_n), d_Y(y_1, y_2, ... y_n)\},$$

for $x_1, x_2, ... x_n \in X$, $y_1, y_2, ... y_n \in Y$ is called the p-product metric of the Cartesian product of n-metric spaces is the p-norm of the n-vector of the norms of the n-sub spaces.

A trivial example of p-product metric of n-metric space is the p-norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p-norm:

$$\|(d_1(x_1, 0), ..., d_n(x_n, 0))\|_E = \sup\left(\left|\det\left(d_{mn}(x_{mn}, 0)\right)\right|\right)$$
$$= \sup\left(\begin{vmatrix} d_{11}(x_{11}, 0) & d_{12}(x_{12}, 0) & ... & d_{1n}(x_{1n}, 0) \\ d_{21}(x_{21}, 0) & d_{22}(x_{22}, 0) & ... & d_{2n}(x_{2n}, 0) \\ \vdots & & ... & \\ d_{n1}(x_{n1}, 0) & d_{n2}(x_{n2}, 0) & ... & d_{nn}(x_{nn}, 0) \end{vmatrix}\right)$$

where $x_i = (x_{i1}, ... x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, ... n$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the p-metric. Any complete p-metric space is said to be p-Banach metric space.

2.1. Definition. A sequence (x_{mn}) is said to be rough convergent (r-convergent) to $\bar{0}$ if for every $\epsilon > 0$ there exists a positive integer n0 such that $\|x_{mn}, \bar{0}\| < r + \epsilon$ for all $n > n_0$, where r is a non negative real number called the convergence degree.

2.2. Definition. A sequence (x_{mn}) is said to be rough I- convergent to $\bar{0}$ if for each $\epsilon > 0$,

$$\{m, n \in N : \|x_{mn}, \bar{0}\| > r + \epsilon\} \in I.$$

Here $\bar{0}$ is called the rough I-limit of the sequence (x_{mn}) and we write $I_r - \lim x_{mn} = \bar{0}$.

3. Main Results

In this section we introduce the notion of different types of I-convergent double sequences. This generalizes and unifies different notions of convergence for χ^2. We shall denote the ideal of $2^{N \times N}$ by I_2.

Let I_2 be an ideal of $2^{N \times N}$, f be an Orlicz function. Let u and v be two non-negative integers and $\mu = (\mu_{mn})$ be a sequence of non-zero reals. Then for a sequence $\eta = (\eta_{mn})$ be a double analytic sequence of strictly positive real numbers and $\left(\Delta^v_{(\mu,u)} X, \|(d_1(x_1, 0), ..., d_n(x_n, 0))\|_p\right)$ be an p-product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces. Further $\chi^2\left(p - \Delta^v_{(\mu,u)} X\right)$ denotes $\Delta^v_{(\mu,u)} X$ -valued sequence space. Now, we define the following sequence spaces:

$$\chi^{2I_2}_{\Delta^v_{(\mu,u)} f}\left[\|(d_1(x_1, 0), ..., d_n(x_n, 0))\|_p\right]^\eta$$
$$= \{x = \left(\Delta^v_{(\mu,u)} x_{mn}\right) \in \chi^2\left(p - \Delta^v_{(\mu,u)} X\right) : \forall \epsilon > 0,$$

$(r,s) \in N \times N:$

$$\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left|\Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n},\left(d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\right\|_p\right]^{\eta_{mn}}$$

$\geq r+\epsilon \in I_2,$

for every $d_1(x_1,0),\ldots,d_n(x_{n-1},0) \in \Delta^{v}_{(\mu,u)}X\}$,

$$\Lambda^{2I_2}_{\Delta^{v}_{(\mu,u)}f}\left[\left\|\left(d_1(x_1,0),\ldots,d_n(x_n,0)\right)\right\|_p\right]^{\eta}$$

$=\{x=(x_{mn}) \in \Lambda^2\left(p-\Delta^{v}_{(\mu,u)}X\right): \exists K>0,$

$(r,s) \in N \times N:$

$$\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left|\Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n},\left(d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\right\|_p\right]^{\eta_{mn}}$$

$\geq r+K \in I_2$, for every $d_1(x_1,0),\ldots,d_n(x_{n-1},0) \in \Delta^{m}X\}$,

$$\Lambda^{2}_{\Delta^{v}_{(\mu,u)}f}\left[\left\|\left(d_1(x_1,0),\ldots,d_n(x_n,0)\right)\right\|_p\right]^{\eta}$$

$=\{x=\left(\Delta^{m}x_{mn}\right) \in \Lambda^2\left(p-\Delta^{v}_{(\mu,u)}X\right): \exists K>0,$

$(r,s) \in N \times N:$

$$\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left|\Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n},\left(d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\right\|_p\right]^{\eta_{mn}}$$

$\leq r+K$, for every $d_1(x_1,0),\ldots,d_n(x_{n-1},0) \in \Delta^{v}_{(\mu,u)}X\}$.

If $\eta=\eta_{mn}=1$ for all $m,n \in \mathbb{N}$, then we obtain

$$\chi^{2I_2}_{\Delta^{v}_{(\mu,u)}f}\left[\left\|\left(d_1(x_1,0),\ldots,d_n(x_n,0)\right)\right\|_p\right]^{\eta}=\chi^{2I_2}_{\Delta^{v}_{(\mu,u)}f}\left[\left\|\left(d_1(x_1,0),\ldots,d_n(x_n,0)\right)\right\|_p\right],$$

$$\Lambda^{2I_2}_{\Delta^{v}_{(\mu,u)}f}\left[\left\|\left(d_1(x_1,0),\ldots,d_n(x_n,0)\right)\right\|_p\right]^{\eta}=\Lambda^{2I_2}_{\Delta^{v}_{(\mu,u)}f}\left[\left\|\left(d_1(x_1,0),\ldots,d_n(x_n,0)\right)\right\|_p\right],$$

$$\Lambda^{2}_{\Delta^{v}_{(\mu,u)}f}\left[\left\|\left(d_1(x_1,0),\ldots,d_n(x_n,0)\right)\right\|_p\right]^{\eta}=\Lambda^{2}_{\Delta^{v}_{(\mu,u)}f}\left[\left\|\left(d_1(x_1,0),\ldots,d_n(x_n,0)\right)\right\|_p\right].$$

The following well-known inequality will be used in this study: $0 \leq in\ f_{mn}\eta_{mn}=H_0 \leq \eta_{mn} \leq \sup_{mn}=H<\infty$, $D=\max\left(1,2^{H-1}\right)$, then

$$\left|x_{mn}+y_{mn}\right|^{\eta_{mn}} \leq D\left\{\left|x_{mn}\right|^{\eta_{mn}}+\left|y_{mn}\right|^{\eta_{mn}}\right\}$$

for all $m,n \in \mathbb{N}$ and $x_{mn},y_{mn} \in \mathbb{C}$. Also $\left|x_{mn}\right|^{\eta_{mn}/m+n} \leq \max\left(1,\left|x_{mn}\right|^{H/m+n}\right)$ for all $x_{mn} \in \mathbb{C}$.

3.1. Theorem. The classes of sequences

$$\chi^{2I_2}_{\Delta^{v}_{(\mu,u)}f}\left[\left\|\left(d_1(x_1,0),\ldots,d_n(x_n,0)\right)\right\|_p\right]^{\eta_{mn}},$$

$$\Lambda^{2I_2}_{\Delta^{v}_{(\mu,u)}f}\left[\left\|\left(d_1(x_1,0),\ldots,d_n(x_n,0)\right)\right\|_p\right]^{\eta_{mn}}$$

are linear spaces over the complex field $\mathbb{C}$.

Proof: Now we establish the result for the case $\chi^{2I_2}_{\Delta^{v}_{(\mu,u)}f}\left[\left\|\left(d_1(x_1,0),\ldots,d_n(x_n,0)\right)\right\|_p\right]^{\eta_{mn}}$, and the others can be proved similarly. Let $x,y \in \chi^{2I_2}_{\Delta^{v}_{(\mu,u)}f}\left[\left\|\left(d_1(x_1,0),\ldots,d_n(x_n,0)\right)\right\|_p\right]^{\eta_{mn}}$ and $\alpha,\beta \in \mathbb{C}$. Then

$$\left\{(r,s) \in N \times N: \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[\left\|\left(\Delta^{v}_{(\mu,u)}x_{mn}\right)^{1/m+n},\left(d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\right\|_p\right]^{\eta_{mn}} \geq r+\frac{\epsilon}{2} \in I_2\right\}\ and$$

$$\left\{(r,s) \in N \times N: \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[\|\left(\Delta^{v}_{(\mu,u)}y_{mn}\right)^{1/m+n},\left(d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\right\|_p\right]^{\eta_{mn}} \geq r+\frac{\epsilon}{2}\right\} \in I_2.$$

Since $\left\|\left(d_1(x_1,0),\ldots,d_n(x_n,0)\right)\right\|_p$ be an p-product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces and f is an Orlicz function, the following inequality holds:

$$\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(\|\frac{\left|\alpha\Delta^{v}_{(\mu,u)}x_{mn}+\beta\Delta^{v}_{(\mu,u)}y_{mn}\right|^{1/m+n}}{|\alpha|^{1/m+n}+|\beta|^{1/m+n}},\left(d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\right\|_p\right)\right]^{\eta_{mn}}$$

$$\leq \frac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[\frac{|\alpha|^{1/m+n}}{|\alpha|^{1/m+n}+|\beta|^{1/m+n}}f\left(\|\left(\left|\Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n},\left(d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\right\|_p\right)\right]^{\eta_{mn}}$$

$$+\frac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[\frac{|\beta|^{1/m+n}}{|\alpha|^{1/m+n}+|\beta|^{1/m+n}} f\left(\|\left(\left|\Delta^{v}_{(\mu,u)}y_{mn}\right|\right)^{1/m+n},\left(d_1(x_1,0),\dots,d_n(x_{n-1},0)\right)\|_p\right)\right]^{\eta_{mn}}$$

$$\leq \frac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(\|\left(\left|\Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n},\left(d_1(x_1,0),\dots,d_n(x_{n-1},0)\right)\|_p\right)\right]^{\eta_{mn}}$$

$$+\frac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(\|\left(\left|\Delta^{v}_{(\mu,u)}y_{mn}\right|\right)^{1/m+n},\left(d_1(x_1,0),\dots,d_n(x_{n-1},0)\right)\|_p\right)\right]^{\eta_{mn}}.$$

From the above inequality we get $(r,s)\in N\times N$:

$$\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(\|\frac{\left(\left|\alpha\Delta^{v}_{(\mu,u)}x_{mn}+\beta\Delta^{v}_{(\mu,u)}y_{mn}\right|\right)^{1/m+n}}{|\alpha|^{1/m+n}+|\beta|^{1/m+n}},\left(d_1(x_1,0),\dots,d_n(x_{n-1},0)\right)\|_p\right)\right]^{\eta_{mn}}$$

$\geq r+\epsilon \subset (r,s)\in N\times N$:

$$\frac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(\|\left(\left|\Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n},\left(d_1(x_1,0),\dots,d_n(x_{n-1},0)\right)\|_p\right)\right]^{\eta_{mn}}$$

$\geq r+\frac{\epsilon}{2}\subset I_2\bigcup(r,s)\in N\times N$:

$$\frac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(\|\left(\left|\Delta^{v}_{(\mu,u)}y_{mn}\right|\right)^{1/m+n},\left(d_1(x_1,0),\dots,d_n(x_{n-1},0)\right)\|_p\right)\right]^{\eta_{mn}}$$

$\geq r+\frac{\epsilon}{2}\in I_2.$

This completes the proof.

3.2. Theorem. The class of sequence $\chi^{2I_2}_{\Delta^{v}_{(\mu,u)}f}\left[\left\|\left(d_1(x_1,0),\dots,d_n(x_n,0)\right)\right\|_p\right]^{\eta}$ is a paranormed space with respect to the paranorm defined by

$$g_{rs}(x)=\inf\left\{\sup_{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(\|\left(\left|\Delta^{v}_{(\mu,u)}y_{mn}\right|\right)^{1/m+n},\left(d_1(x_1,0),\dots,d_n(x_{n-1},0)\right)\|_p\right]^{\eta_{mn}}\right)^{\frac{1}{H}}\leq r\right\},$$

for every $d_1(x_1,0),\dots,d_n(x_{n-1},0)\in X$.

Proof: $g_{rs}(\theta)=0$ and $g_{rs}(-x)=g_{rs}(x)$ are easy to prove, so we omit them. Let us take $x,y\in\chi^{2I_2}_{\Delta^{v}_{(\mu,)}f}\left[\left\|\left(d_1(x_1,0),\dots,d_n(x_n,0)\right)\right\|_p\right]^{\eta_{mn}}$. Let

$$g_{rs}(x)=\inf\left\{\sup_{rs}\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\|\left(\left(\left|\Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n},d_1(x_1,0),\dots,d_n(x_{n-1},0)\right)\|_p\right]^{\eta_{mn}}\leq r,\forall x\in X\right\} \text{ and}$$

$$g_{rs}(y)=\inf\left\{\sup_{rs}\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\|\left(\left(\left|\Delta^{v}_{(\mu,u)}y_{mn}\right|\right)^{1/m+n},d_1(x_1,0),\dots,d_n(x_{n-1},0)\right)\|_p\right]^{\eta_{mn}}\leq r,\forall x\in X\right\}.$$

Then we have

$$\sup_{rs}\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\|\left(\left(\left|\Delta^{v}_{(\mu,u)}x_{mn}+\Delta^{v}_{(\mu,u)}y_{mn}\right|\right)^{1/m+n},d_1(x_1,0),\dots,d_n(x_{n-1},0)\right)\|_p\right]^{\eta_{mn}}$$

$$\leq\sup_{rs}\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\|\left(\left(\left|\Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n},d_1(x_1,0),\dots,d_n(x_{n-1},0)\right)\|_p\right]^{\eta_{mn}}$$

$$+\sup_{rs}\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\|\left(\left(\left|\Delta^{v}_{(\mu,u)}y_{mn}\right|\right)^{1/m+n},d_1(x_1,0),\dots,d_n(x_{n-1},0)\right)\|_p\right]^{\eta_{mn}}.$$

Thus

$$\sup_{rs}\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\|\left(\left(\left|\Delta^{v}_{(\mu,u)}x_{mn}+\Delta^{v}_{(\mu,u)}y_{mn}\right|\right)^{1/m+n},d_1(x_1,0),\dots,d_n(x_{n-1},0)\right)\|_p\right]^{\eta_{mn}}\leq 1$$

and $g_{rs}(x+y)=g_{rs}(x)+g_{rs}(y)$.

Now, $\lambda^{u}_{mn}\to\lambda$, where $\lambda^{u}_{mn},\lambda\in\mathbb{C}$ and $g_{rs}\left(\Delta^{v}_{(\mu,u)}x^{u}_{mn}-\Delta^{v}_{(\mu,u)}x_{mn}\right)\to 0$ as $u\to\infty$. We have to prove that $g_{rs}\left(\lambda_{mn}\Delta^{v}_{(\mu,u)}x^{u}_{mn}-\lambda\Delta^{v}_{(\mu,u)}x_{mn}\right)\to 0$ as $u\to\infty$. Let

$$g_{rs}(x^u)=\left\{\sup_{rs}\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\|\left(\left(\left|\Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n},d_1(x_1,0),\dots,d_n(x_{n-1},0)\right)\|_p\right]^{\eta_{mn}}\leq r,\forall x\in X\right\}$$

and

$$g_{rs}(x^u-x)=\left\{\sup_{rs}\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\|\left(\left(\left|\Delta^{v}_{(\mu,u)}x^{u}_{mn}-\Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n},d_1(x_1,0),\dots,d_n(x_{n-1},0)\right)\|_p\right]^{\eta_{mn}}\leq r\right\} \text{ for all } x\in X.$$

We observe that

$$f\left(\| (\frac{\left(\left|\lambda_{mn}^{u}\Delta_{(\mu,u)}^{v}x_{mn}^{u}-\lambda\Delta_{(\mu,u)}^{v}x_{mn}\right|\right)^{1/m+n}}{\left|\lambda_{mn}^{u}-\lambda\right|^{1/m+n}+|\lambda|^{1/m+n}}\right.$$

$$\left.,d_1(x_1,0),\dots,d_n(x_{n-1},0))\|_p\right)$$

$$\leq f\left(\| (\frac{\left(\left|\lambda_{mn}^{u}x_{mn}^{u}-\lambda\Delta_{(\mu,u)}^{v}x_{mn}\right|\right)^{1/m+n}}{\left|\lambda_{mn}^{u}-\lambda\right|^{1/m+n}+|\lambda|^{1/m+n}}\right.$$

$$\left.,d_1(x_1,0),\dots,d_n(x_{n-1},0))\|_p\right)$$

$$+f\left(\| (\frac{\left(\left|\lambda\Delta_{(\mu,u)}^{v}x_{mn}^{u}-\lambda\Delta_{(\mu,u)}^{v}x_{mn}\right|\right)^{1/m+n}}{\left|\lambda_{mn}^{u}-\lambda\right|^{1/m+n}+|\lambda|^{1/m+n}}\right.$$

$$\left.,d_1(x_1,0),\dots,d_n(x_{n-1},0))\|_p\right)$$

$$\leq\frac{\left|\lambda_{mn}^{u}-\lambda\right|}{\left|\lambda_{mn}^{u}-\lambda\right|+|\lambda|}f\left(\|\left(\left(\left|\Delta_{(\mu,u)}^{v}x_{mn}^{u}\right|\right)^{1/m+n}\right.\right.$$

$$\left.,d_1(x_1,0),\dots,d_n(x_{n-1},0))\|_p\right)$$

$$+\frac{\left|\lambda_{mn}^{u}-\lambda\right|}{\left|\lambda_{mn}^{u}-\lambda\right|+|\lambda|}f\left(\|\left(\left(\left|\Delta_{(\mu,u)}^{v}x_{mn}^{u}-\Delta_{(\mu,u)}^{v}x_{mn}\right|\right)^{1/m+n}\right.\right.$$

$$\left.,d_1(x_1,0),\dots,d_n(x_{n-1},0))\|_p\right).$$

From this inequality, it follows that

$$\left[f\left(\| (\frac{\left(\left|\lambda_{mn}^{u}\Delta_{(\mu,u)}^{v}x_{mn}^{u}-\lambda\Delta^{2}x_{mn}\right|\right)^{1/m+n}}{\left|\lambda_{mn}^{u}-\lambda\right|^{1/m+n}+|\lambda|^{1/m+n}}\right.\right.$$

$$\left.\left.,d_1(x_1,0),\dots,d_n(x_{n-1},0))\|_p\right)\right]^{\eta_{mn}}\leq r$$

and consequently

$$g_{rs}\left(\left|\lambda_{mn}^{u}\Delta_{(\mu,u)}^{v}x_{mn}^{u}-\lambda\Delta_{(\mu,u)}^{v}x_{mn}\right|\right)$$

$$\leq\left(\left|\lambda_{mn}^{u}-\lambda\right|\right)^{\frac{\eta_{mn}}{H}}\inf\left\{g_{rs}\left(\Delta_{(\mu,u)}^{v}x_{mn}^{u}\right)\right\}$$

$$+\left(|\lambda|\right)^{\frac{\eta_{mn}}{H}}\inf\left\{g_{rs}\left(\Delta_{(\mu,u)}^{v}x_{mn}^{u}-x\right)\right\}$$

$$\leq\max\left\{|\lambda|,\left(|\lambda|\right)^{\frac{\eta_{mn}}{H}}\right\}g_{rs}\left(\Delta_{(\mu,u)}^{v}x_{mn}^{u}-\Delta^{m}x_{mn}\right).$$

Hence by our assumption the right hand side tends to zero as u, m and $n\to\infty$. This completes the proof.

3.3. Theorem. (i) If $0 < in\ f_{mn}\eta_{mn} = H_0 \leq \eta_{mn} < 1,$ then

$$\chi^{2I_2}_{\Delta^v_{(\mu,u)}f}\left[\left\|\left(d_1(x_1,0),\dots,d_n(x_n,0)\right)\right\|_p\right]^{\eta}$$

$$\subset\chi^{2I_2}_{\Delta^v_{(\mu,u)}f}\left[\left\|\left(d_1(x_1,0),\dots,d_n(x_n,0)\right)\right\|_p\right].$$

(ii) If $1\leq\eta_{mn}\leq\sup_{mn}\eta_{mn}=H<\infty,$ then

$$\chi^{2I_2}_{\Delta^v_{(\mu,u)}f}\left[\left\|\left(d_1(x_1,0),\dots,d_n(x_n,0)\right)\right\|_p\right]$$

$$\subset\chi^{2I_2}_{f}\left[\left\|\left(d_1(x_1,0),\dots,d_n(x_n,0)\right)\right\|_p\right]^{\eta}.$$

(iii) If $0<\eta_{mn}\leq\mu_{mn}<\infty$ and $\left\{\frac{\mu_{mn}}{\eta_{mn}}\right\}$ is double analytic, then

$$\chi^{2I_2}_{\Delta^v_{(\mu,u)}f}\left[\left\|\left(d_1(x_1,0),\dots,d_n(x_n,0)\right)\right\|_p\right]^{\eta}$$

$$\subset\chi^{2I_2}_{\Delta^v_{(\mu,u)}f}\left[\left\|\left(d_1(x_1,0),\dots,d_n(x_n,0)\right)\right\|_p\right]^{\mu}.$$

Proof: The proof can be established using standard technique.

The following result is well known.

3.4. Lemma. If a sequence space E is solid, then it is monotone.

3.5. Theorem. The class of sequence $\chi^{2I_2}_{\Delta^v_{(\mu,u)}f}\left[\left\|\left(d_1(x_1,0),\dots,d_n(x_n,0)\right)\right\|_p\right]^{\eta}$ is not solid and hence not monotone.

Proof: It is routine verification. Therefore we omit the proof.

3.6. Theorem. Let f, f_1 and f_2 be Orlicz functions. Then we have

(i)

$$\chi^{2I_2}_{\Delta^v_{(\mu,u)}f_1}\left[\left\|\left(d_1(x_1,0),\dots,d_n(x_n,0)\right)\right\|_p\right]^{\eta}$$

$$\subset\chi^{2I_2}_{\Delta^v_{(\mu,u)}f\circ f_1}\left[\left\|\left(d_1(x_1,0),\dots,d_n(x_n,0)\right)\right\|_p\right]^{\eta}$$

(ii)

$$\chi^{2I_2}_{\Delta^v_{(\mu,u)}f_1}\left[\left\|\left(d_1(x_1,0),\dots,d_n(x_n,0)\right)\right\|_p\right]^{\eta}$$

$$\cap\chi^{2I_2}_{\Delta^v_{(\mu,u)}f_2}\left[\left\|\left(d_1(x_1,0),\dots,d_n(x_n,0)\right)\right\|_p\right]^{\eta}$$

$$\subset\chi^{2I_2}_{\Delta^v_{(\mu,u)}f_1+\Delta^v_{(\mu,u)}f_2}\left[\left\|\left(d_1(x_1,0),\dots,d_n(x_n,0)\right)\right\|_p\right]^{\eta}.$$

Proof: (i) Let $in\ f_{mn}\eta_{mn} = H_0$. For given $r+\epsilon > 0$, we first choose $r+\epsilon_0 > 0$ such that $\max\left\{r+\epsilon_0^H, r+\epsilon_0^{H_0}\right\} < r+\epsilon$. Now using the continuity of *f*, choose $0<\delta<1$ such that $0<t<\delta$ implies $f(t) < r+\epsilon_0$. Let

$$\Delta^v_{(\mu,u)}x \in \chi^{2I_2}_{\Delta^v_{(\mu,u)}f_1}\left[\left\|\left(d_1(x_1,0),\ldots,d_n(x_n,0)\right)\right\|_p\right]^{\eta}..$$

We observe that

$$A(\delta) = \left\{(r,s)\in N\times N : \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f_1 \| \left(\left(\left|\Delta^v_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\|_p\right]^{\eta_{mn}} \geq \delta^H\right\} \in I_2.$$

Thus if $(r,s)\notin A(\delta)$ then

$$\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f_1 \| \left(\left(\left|\Delta^v_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\|_p\right]^{\eta_{mn}} < \delta^H$$

$$\Rightarrow \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f_1 \| \left(\left(\left|\Delta^v_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\|_p\right]^{\eta_{mn}} < rs\delta^H,$$

$$\Rightarrow \left[f_1 \| \left(\left(\left|\Delta^v_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\|_p\right]^{\eta_{mn}} < \delta^H,$$

$$for\ all\ m,n = 1,2,\ldots.$$

$$\Rightarrow f_1\left(\| \left(\left(\left|\Delta^v_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\|_p\right)^{\eta_{mn}} < \delta,$$

$$for\ all\ m,n = 1,2,\ldots.$$

Hence from above inequality and using continuity of f; we must have

$$f\left(f_1\left(\| \left(\left(\left|\Delta^v_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\|_p\right)^{\eta_{mn}} < r+\epsilon_0,$$

$$for\ all\ m,n = 1,2,\ldots.$$

$$\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(f_1 \| \left(\left(\left|\Delta^v_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\|_p\right)\right]^{\eta_{mn}} < rs\max\left\{r+\epsilon_0^H, r+\epsilon_0^{H_0}\right\} < rs(r+\epsilon)$$

$$\Rightarrow \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(f_1 \| \left(\left(\left|\Delta^v_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\|_p\right)\right]^{\eta_{mn}} < r+\epsilon.$$

Hence we have

$$\left\{(r,s)\in N\times N : \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(f_1 \| \left(\left(\left|\Delta^v_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\|_p\right)\right]^{\eta_{mn}} \geq \epsilon\right\} \subset A(\delta)\in I_2.$$

(ii) Let

$$x \in \chi^{2I_2}_{\Delta^v_{(\mu,u)}f_1}\left[\left\|\left(d_1(x_1,0),\ldots,d_n(x_n,0)\right)\right\|_p\right]^{\eta} \cap \chi^{2I_2}_{\Delta^v_{(\mu,u)}f_2}\left[\left\|\left(d_1(x_1,0),\ldots,d_n(x_n,0)\right)\right\|_p\right]^{\eta}.$$

Then the fact that

$$\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[(f_1+f_2) \| \left(\left(\left|\Delta^v_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\|_p\right)\right]^{\eta_{mn}}$$

$$\leq \frac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f_1 \| \left(\left(\left|\Delta^v_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\|_p\right)\right]^{\eta_{mn}}$$

$$+\frac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f_2 \| \left(\left(\left|\Delta^v_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\|_p\right)\right]^{\eta_{mn}}.$$

This completes the proof.

3.7. Theorem. The class of sequence $\Lambda^{2I_2}_{\Delta^v_{(\mu,u)}f}\left[\left\|d_1(x_1,0),\ldots,d_n(x_n,0)\right\|_p\right]^{\eta}$ is a sequence algebra.

Proof: Let

$$\left(\Delta^v_{(\mu,u)}x_{mn}\right),\left(\Delta^v_{(\mu,u)}y_{mn}\right) \in \Lambda^{2I_2}_{\Delta^v_{(\mu,u)}f}\left[\left\|d_1(x_1,0),\ldots,d_n(x_n,0)\right\|_p\right]^{\eta}$$

and $0<\epsilon<1$. Then the result follows from the following inclusion relation:

$$\left\{(r,s)\in N\times N : \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f \| \left(\left(\left|\Delta^v_{(\mu,u)}x_{mn}\otimes\Delta^v_{(\mu,u)}y_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\|_p\right)\right]^{\eta_{mn}} < \epsilon+K\right\} \in I_2$$

$$\supseteq \left\{ \left\{ \left\{ (r,s) \in N \times N : \frac{1}{rs} \sum_{m=1}^{r} \sum_{n=1}^{s} \left[f \parallel \left(\left(\left| \Delta_{(\mu,u)}^{v} x_{mn} \right| \right)^{1/m+n} , d_1(x_1,0),\ldots,d_n(x_{n-1},0) \right) \parallel_p \right) \right]^{\eta_{mn}} < \epsilon + K \right\} \in I_2 \right\}$$

$$\cap \left\{ \left\{ \left\{ (r,s) \in N \times N : \frac{1}{rs} \sum_{m=1}^{r} \sum_{n=1}^{s} \left[f \parallel \left(\left(\left| \Delta_{(\mu,u)}^{v} y_{mn} \right| \right)^{1/m+n} , d_1(x_1,0),\ldots,d_n(x_{n-1},0) \right) \parallel_p \right) \right]^{\eta_{mn}} < \epsilon + K \right\} \in I_2 \right\}.$$

Similarly we can prove the result for other cases.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this research paper.

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