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Fixed Point Results on Closed Ball for a New Rational Type Contractive Mappings in Complete Dislocated Metric Spaces

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Abstract In this article, we find fixed point results for a pairs of mappings satisfying the locally contractive conditions on a closed ball for a new generalized rational type contraction in complete dislocated metric space. Example has been constructed to demonstrate the novelty of our results.

Keywords: *fixed point, generalized rational type contraction, closed ball, complete dislocated metric spaces*

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1. Introduction and Preliminaries

Fixed point theory plays a fundamental role in functional analysis. Banach [4] proved significant result in the manipulation of contractive mappings in analysis. Many authors have presented a large number of theorems related to fixed point theorems. These authors have made different generalizations of the Banach's result. After that, huge number of fixed point theorems have been established by various authors and they made different generalizations of the Banach's result. Let $H:W\to W$ be a mapping. A point $w\in W$ is called a fixed point of H if $w=Hw$. In literature, there are many results about the fixed point of mappings that are contractive over all the theories. It is possible that $H:W\to W$ is not a contraction but $H:Y\to W$ is a contraction, where Y is a closed ball in W. It is possible for one to get fixed point results for mappings under different condition. It has been shown by. Hussain et al. [8] related to the results concerning the presence of fixed points of a mapping that ful.lls the conditions on closed ball (see also [1,2,3,5,15,16,17]). The idea of dislocated is proved to be useful for logic programming semantics (see [7]). Dislocated metric space (metric-like space) (see [10,13]) is an induction of partial metric space (see [11]).

In 1994, Matthews [11] proposed the idea of partial metric spaces and got numerous fixed point results. In particular, he brought about the short connection between partial metric spaces and quasi-metric spaces, and showed a partial metric generalization of Banach's contraction mapping theorem. We can observe many findings about fixed point results on cone metric spaces (see [9,12,14]). The idea of dislocated topologies presented by Hitzler and Seda [7] is called a generalized metric a dislocated metric. They have presented also fixed point results in complete dislocated metric spaces to draw conclusion from the well known Banach contraction principle. The aim of this paper is to strengthen the conclusion of fixed point results for a pair of mappings proving the contractive conditions on subspace for a new generalized rational type contraction in complete dislocated metric space.

Definition 1.1. [7] Let X be a nonempty set. A mapping $d_l:X\times X\to[0,\infty)$ is called a dislocated metric (or simply d_l-metric) if the following conditions hold, for any $x,y,z\in X$:

(1) If $d_l(x,y)=0$, then $x=y$;

(2) $d_l(x,y)=d_l(y,x)$;

(3) $d_l(x,y)\le d_l(x,z)+d_l(z,y)$.

Then d_l is called a dislocated metric on X, and the pair (X,d_l) is called dislocated metric space or d_l metric space. It is clear that if $d_l(x,y)=0$, then from (i), $x=y$. But if $x=y$, $d_l(x,y)$ may not be 0.

Example 1.2. If $X=\mathbb{R}^+\cup\{0\}$, then $d_l(x,y)=x+y$ defines a dislocated metric on X.

Definition 1.3. [7] A sequence $\{u_n\}$ in d_l-metric space is called Cauchy sequence if for given $\varepsilon>0$, there corresponds $n_0\in N$ such that for all $n,m\ge n_0$, we have $d_l(u_m,u_n)<\varepsilon$.

Definition 1.4. [7] A sequence $\{u_n\}$ in d_l-metric space converges with respect to d_l if there exists $u\in X$ such that $d_l(u_n,u)\to 0$ as $n\to\infty$. In this case, u is called limit of $\{u_n\}$ and we write $u_n\to u$.

Every metric space is a dislocated metric, but the converse may not be true.

Example 1.5. Let $X = R$ and $d_l : X \times X \to [0, \infty)$ defined by $d_l(u,v) = |u| + |v|$ for all $u, v \in X$.

Note that dl is a dislocated metric, but not a metric since $d_l(1,1) = 2 > 0$.

Definition 1.6. Let (X, d_l) be a dislocated metric space then for $u_0 \in X$, $r > 0$, the closed ball with centre u_0 and radius r is,

$$\overline{B_{d_l}(u_0, r)} = \{ y \in X : d_l(u_0, y) \le r \}.$$

Definition 1.7. [7] A d_l-metric space (X, d_l) is called complete if every Cauchy sequence in X converges to a point in X.

Example 1.8. Let $X = [0,1]$ and $d_l(u,v) = \max\{u, v\}$. Then the pair (X, d_l) is dislocated metric space, but it is not a metric space.

Definition 1.9. [7] Let (X, d_l) be a dislocated metric space. A mapping $T : X \to X$ is called contraction if there exists $0 \le \lambda < 1$ such that

$$d(T(u), T(v)) \le \lambda d(u,v), \text{ for all } u, v \in X \text{ with } u \ne v.$$

Then T has a unique fixed point in X.

The purpose of this paper is to prove common fixed point theorems for generalized rational contractions on dislocated metric spaces. We provide an example to validate our results.

2. The Results

In this section, we will prove the existance of common fixed points of two self mappings involving rational expressions in complete dislocated metric space.

Theorem 2.1. *Let (X, d_l) be a complete dislocated metric space and u_0 be any arbitrary point in X let the mappings $S, T : X \to X$ satisfy:*

$$\begin{aligned} d_l(Su, Tv) \le{} & a_1 d_l(u,v) + a_2 \frac{d_l(u,Su).d_l(v,Tv)}{d_l(u,v)} \\ & + a_3 \frac{d_l(u,Tv).d_l(v,Su)}{d_l(u,v)} \\ & + a_4 \frac{d_l(u,Su).d_l(v,Tv)}{d_l(u,Su)+d_l(u,v)+d_l(v,Tv)} \end{aligned} \quad (2.1)$$

for all $u, v \in \overline{B_{d_l}(u_0, r)}$ and $u \ne v$ with $r > 0$, and

$$d_l(u_0, Su_0) \le (1-\lambda) r \quad (2.2)$$

where $\lambda = \left(\frac{a_1 + 2a_3 + a_4}{1 - a_2 - 2a_3} \right)$ and a_1, a_2, a_3, a_4 are non negative reals with $a_1 + a_2 + 4a_3 + a_4 < 1$. Then $\{u_n\}$ is a non increasing sequence in $\overline{B_{d_l}(u_0, r)}$ for all $n \in \mathbb{N} \cup \{0\}$ and $u_n \to h \in \overline{B_{d_l}(u_0, r)}$. Then S and T have common fixed point h in $\overline{B_{d_l}(u_0, r)}$.

Proof: *Let u_0 be an arbitrary point in X and define $u_1 = Su_0$ and $u_2 = Tu_1$ such that $d_l(u_1, u_2) = d_l(Su_0, Tu_1)$.*

Then

$$\begin{aligned} d_l(u_1,u_2) \le{} & a_1 d_l(u_0,u_1) + a_2 \frac{d_l(u_0,Su_0).d_l(u_1,Tu_1)}{d_l(u_0,u_1)} \\ & + a_3 \frac{d_l(u_0,Tu_1).d_l(u_1,Su_0)}{d_l(u_0,u_1)} \\ & + a_4 \frac{d_l(u_0,Su).d_l(u_1,Tu_1)}{d_l(u_0,Su_0)+d_l(u_0,u_1)+d_l(u_1,Tu_1)}, \\ \le{} & a_1 d_l(u_0,u_1) + a_2 \frac{d_l(u_0,u_1).d_l(u_1,u_2)}{d_l(u_0,u_1)} \\ & + a_3 \frac{d_l(u_0,u_2).d_l(u_1,u_1)}{d_l(u_0,u_1)} \\ & + a_4 \frac{d_l(u_0,u_1).d_l(u_1,u_2)}{d_l(u_0,u_1)+d_l(u_0,u_1)+d_l(u_1,u_2)} \\ \le{} & a_1 d_l(u_0,u_1) + a_2 \frac{d_l(u_0,u_1).d_l(u_1,u_2)}{d_l(u_0,u_1)} \\ & + a_3 \frac{d_l(u_0,u_2).\{d_l(u_1,u_0)+d_l(u_0,u_1)\}}{d_l(u_0,u_1)} \\ & + a_4 \frac{d_l(u_0,u_1).d_l(u_1,u_2)}{d_l(u_0,u_1)+d_l(u_0,u_1)+d_l(u_1,u_2)}, \\ \le{} & a_1 d_l(u_0,u_1) + a_2 d_l(u_1,u_2) + 2a_3 d_l(u_0,u_2) \\ & + a_4 \frac{d_l(u_0,u_1).d_l(u_1,u_2)}{d_l(u_0,u_1)+d_l(u_0,u_2)}. \end{aligned}$$

As (owing to triangular inequality),

$$\begin{aligned} d_l(u_1,u_2) \le{} & a_1 d_l(u_0,u_1) + a_2 d_l(u_1,u_2) + 2a_3 d_l(u_0,u_1) \\ & + +2a_3 d_l(u_1,u_2) + a_4 \frac{d_l(u_0,u_1).d_l(u_1,u_2)}{d_l(u_0,u_1)+d_l(u_0,u_2)}. \end{aligned}$$

Where

$$d_l(u_1,u_2) \le d_l(u_1,u_0) + d_l(u_0,u_2).$$

Hence

$$\begin{aligned} d_l(u_1,u_2) & \le \left(\frac{a_1 + 2a_3 + a_4}{1 - a_2 - 2a_3} \right) d_l(u_0,u_1) \\ & \le \lambda d_l(u_0,u_1) \le \lambda(1-\lambda) r, \text{ by using (2.2)} \\ d_l(u_1,u_2) & \le \lambda(1-\lambda) r. \end{aligned}$$

Where $\lambda = \frac{a_1 + 2a_3 + a_4}{1 - a_2 - 2a_3}$. Now,

$$\begin{aligned} d_l(u_0,u_2) & \le d_l(u_0,u_1) + d_l(u_1,u_2) \\ & \le (1-\lambda) r + \lambda(1-\lambda) r \\ & \le (1-\lambda)^2 r \le r \end{aligned}$$

$$d_l(u_0,u_2) \le r.$$

This implies that $u_2 \in \overline{B_{d_l}(u_0,r)}$. Similarly, by repeating the same process for

$$d_l(u_2,u_3) = d_l(Tu_1,Su_2) = d_l(Su_2,Tu_1),$$

we get

$$d_l(u_2,u_3) \le \lambda^2 d_l(u_0,u_1).$$

Consequently, $u_3,u_4,...,u_j \in \overline{B_{d_l}(u_0,r)}$, for some $j \in N$. If $j = 2i+1$, where $i = 0, 1, 2, ..., \frac{j-1}{2}$ we get

$$d_l(u_{2i+1},u_{2i+2}) \le \lambda d_l(u_{2i},u_{2i+1}). \tag{2.3}$$

Similarly, if $j = 2i+2$, where $i = 0, 1, 2, ..., \frac{j-2}{2}$, we have

$$d_l(u_{2i+2},u_{2i+3}) \le \lambda d_l(u_{2i+1},u_{2i+2}). \tag{2.4}$$

Now, (2.3) implies that

$$d_l(u_{2i+1},u_{2i+2}) \le \lambda^{2i+1} d_l(u_0,u_1). \tag{2.5}$$

Also, (2.4) implies that

$$d_l(u_{2i+2},u_{2i+3}) \le \lambda^{2i+2} d_l(u_0,u_1). \tag{2.6}$$

Now, by combining (2.5) and (2.6), we have

$$d_l(u_j,u_{j+1}) \le \lambda^j d_l(u_0,u_1) \text{ for all } j \in N. \tag{2.7}$$

Now,

$$\begin{aligned} d_l(u_0,u_{j+1}) &\le d_l(u_0,u_1) + d_l(u_1,u_2) + ... + d_l(u_j,u_{j+1}) \\ &\le d_l(u_0,u_1) + \lambda d_l(u_0,u_1) + ... + \lambda^j d_l(u_0,u_1) \\ &\qquad \text{by (2.7)} \\ &\le \left(1+\lambda+\lambda^2+...+\lambda^j\right) d_l(u_0,u_1) \\ &\le \frac{1\left(1-\lambda^j\right)}{1-\lambda}(1-\lambda)r \quad as\ j \to \infty \\ &\le r. \end{aligned}$$

Thus, $u_{j+1} \in \overline{B_{d_l}(u_0,r)}$. Hence $u_n \in \overline{B_{d_l}(u_0,r)}$ for all $n \in N$, therefore $\{u_n\}$ is a sequence in $\overline{B_{d_l}(u_0,r)}$. Now, the inequality (2.7) can be written as

$$d_l(u_n,u_{n+1}) \le \lambda^n d_l(u_0,u_1) \text{ for all } n \in N. \tag{2.8}$$

Hence for any $m > n$,

$$\begin{aligned} d_l(u_n,u_m) &< d_l(u_n,u_{n+1}) + d_l(u_{n+1},u_{n+2}) \\ &\quad + ... + d_l(u_{m-1},u_m), \\ &< \left(\lambda^n + \lambda^{n+1} + ... + \lambda^{m-1}\right) d_l(u_0,u_1), \\ &\quad \text{by using (2.8)} \\ &< \frac{\lambda^n}{1-\lambda} d_l(u_0,u_1). \end{aligned}$$

And

$$d_l(u_n,u_m) < \frac{\lambda^n}{1-\lambda} d_l(u_0,u_1).$$
$$\to 0, \text{ as } m,n \to \infty.$$

This implies that $\{u_n\}$ is a Cauchy sequence in $\overline{B_{d_l}(u_0,r)}$. Since $\overline{B_{d_l}(u_0,r)}$ is closed and complete, there exists a point $h \in \overline{B_{d_l}(u_0,r)}$ such that $u_n \to h$. It follows that $h = Sh$, otherwise $d(h,Sh) = z > 0$ and we would then have

$$\begin{aligned} d_l(h,Sh) &\le d_l(h,u_{2n+2}) + d_l(u_{2n+2},Sh) \\ d_l(h,Sh) &\le d_l(h,u_{2n+2}) + d_l(Tu_{2n+1},Sh) \\ d_l(h,Sh) &\le d_l(h,u_{2n+2}) + d_l(Sh,Tu_{2n+1}) \end{aligned}$$

$$\begin{aligned} d_l(h,Sh) &\le d_l(h,u_{2n+2}) + a_1 d_l(h,u_{2n+1}) \\ &\quad + a_2 \frac{d_l(h,Sh).d_l(u_{2n+1},Tu_{2n+1})}{d_l(h,u_{2n+1})} \\ &\quad + a_3 \frac{d_l(h,Tu_{2n+1}).d_l(u_{2n+1},Sh)}{d_l(h,u_{2n+1})} \\ &\quad + a_4 \frac{d_l(h,Sh)d_l(u_{2n+1},Tu_{2n+1})}{d_l(h,Sh)+d_l(h,u_{2n+1})+d_l(u_{2n+1},Tu_{2n+1})} \\ &\le d_l(h,u_{2n+2}) + a_1 d_l(h,u_{2n+1}) \\ &\quad + a_2 \frac{d_l(h,Sh).d_l(u_{2n+1},u_{2n+2})}{d_l(h,u_{2n+1})} \\ &\quad + a_3 \frac{d_l(h,u_{2n+2}).d_l(u_{2n+1},Sh)}{d_l(h,u_{2n+1})} \\ &\quad + a_4 \frac{d_l(h,Sh).d_l(u_{2n+1},u_{2n+2})}{d_l(h,Sh)+d_l(h,u_{2n+1})+d_l(u_{2n+1},u_{2n+2})}. \end{aligned}$$

This implies that

$$\begin{aligned} z &\le d_l(h,u_{2n+2}) + a_1 d_l(h,u_{2n+1}) + a_2 \frac{z.d_l(u_{2n+1},u_{2n+2})}{d_l(u,u_{2n+1})} \\ &\quad + a_3 \frac{d_l(h,u_{2n+2}).d_l(u_{2n+1},Sh)}{d_l(h,u_{2n+1})} \\ &\quad + a_4 \frac{z.d_l(u_{2n+1},u_{2n+2})}{d_l(h,Sh)+d_l(h,u_{2n+1})+d_l(u_{2n+1},u_{2n+2})}, \end{aligned}$$

which on making $n \to \infty$, gives rise $d_l(h,Sh) = 0$ a contradiction so that $h = Sh$. Similarly, one can show that $h = Th$.

Example 2.2. Let $X = \mathbb{R}^+ \cup \{0\}$ be a dislocated metric space $d_l : X \times X \to X$ defined by

$$d_l(u,v) = \max\{u,v\}.$$

Let $S : X \to X$ defined by

$$Su = \begin{Bmatrix} \frac{u}{3} \text{ if } u \in [0,1] \\ 2u \text{ if } u \in (1,\infty) \end{Bmatrix}.$$

And $T: X \to X$ defined by

$$Tv = \begin{Bmatrix} \frac{2v}{3} & \text{if } u \in [0,1] \\ 3v & \text{if } u \in (1,\infty) \end{Bmatrix}.$$

Take $a_1 = \frac{1}{3}, a_2 = \frac{1}{4}, a_3 = \frac{1}{16}, a_4 = \frac{1}{7}, u_0 = \frac{1}{6}, r = 5,$ then $\overline{B_{d_l}(u_0,r)} = [0,1] \cap X$. We have $\lambda = \left(\frac{a_1 + 2a_3 + a_4}{1 - a_2 - 2a_3}\right) = \frac{101}{105}$ with

$$(1-\lambda)r = \left(1 - \frac{101}{105}\right)5 = \frac{4}{21}$$

and

$$d_l(u_0, Su_0) = \max\left\{\frac{1}{6}, \frac{1}{18}\right\} = \frac{1}{6} < (1-\lambda)r.$$

Also if $u,v \in (1,\infty) \cap X$, then

$$\begin{aligned} d_l(Su,Tv) &= \max\{2u,3v\} \\ &\geq \frac{1}{3}\max\{u,v\} + \frac{1}{4}\frac{\max\{u,2u\}.\max\{v,3v\}}{\max\{u,v\}} \\ &\quad + \frac{1}{16}\frac{\max\{u,3v\}.\max\{v,2u\}}{\max\{u,v\}} \\ &\quad + \frac{1}{7}\frac{\max\{u,2u\}.\max\{v,3v\}}{\max\{u,3v\} + \max\{u,v\} + \max\{v,2u\}} \\ &\geq a_1 d_l(u,v) + a_2\frac{d_l(u,Su).d_l(v,Tv)}{d_l(u,v)} \\ &\quad + a_3\frac{d_l(u,Tv).d_l(v,Su)}{d_l(u,v)} \\ &\quad + a_4\frac{d_l(u,Su)d_l(v,Tv)}{d_l(u,Tv) + d_l(u,v) + d_l(v,Su)}. \end{aligned}$$

Taking $u = 2$ and $v = 3$, then clearly $u,v \in (1,\infty) \cap X$, and $u,v > 1$, we have

$$d_l(Su,Tv) = \max\{2u,3v\}$$
$$d_l(S2,T3) = \max\{4,9\} = 9.$$

Now,

$$\begin{aligned} \max\{4,9\} &\geq \frac{1}{3}\max\{2,3\} + \frac{1}{4}\frac{\max\{2,4\}.\max\{3,9\}}{\max\{2,3\}} \\ &\quad + \frac{1}{16}\frac{\max\{2,9\}.\max\{3,4\}}{\max\{2,3\}} \\ &\quad + \frac{1}{7}\frac{\max\{2,4\}.\max\{3,9\}}{\max\{2,9\} + \max\{2,3\} + \max\{3,4\}} \\ &\geq \frac{1}{3}\times 3 + \frac{1}{4}\frac{4.9}{3} + \frac{1}{16}\frac{9.4}{3} + \frac{1}{7}\frac{4.9}{9+3+4} \\ &\geq 1 + 3 + \frac{3}{4} + \frac{36}{112} \end{aligned}$$
$$9 \geq 5.0714$$

Hence, clearly whole space does not satisfy the contractive condition. Also if $u,v \in \overline{B_{d_l}(u_0,r)}$, then

$$\begin{aligned} d_l(Su,Tv) &= \max\left\{\frac{u}{3}, \frac{2v}{3}\right\} \\ &\leq \frac{1}{3}\max\{u,v\} + \frac{1}{4}\frac{\max\left\{u,\frac{u}{3}\right\}.\max\left\{v,\frac{2v}{3}\right\}}{\max\{u,v\}} \\ &\quad + \frac{1}{16}\frac{\max\left\{v,\frac{2v}{3}\right\}.\max\left\{v,\frac{u}{3}\right\}}{\max\{u,v\}} \\ &\quad + \frac{1}{7}\frac{\max\left\{u,\frac{u}{3}\right\}.\max\left\{v,\frac{2v}{3}\right\}}{\max\left\{u,\frac{2v}{3}\right\} + \max\{u,v\} + \max\left\{v,\frac{u}{3}\right\}} \\ &\leq a_1 d_l(u,v) + a_2\frac{d_l(u,Su).d_l(v,Tv)}{d_l(u,v)} \\ &\quad + a_3\frac{d_l(u,Tv).d_l(v,Su)}{d_l(u,v)} \\ &\quad + a_4\frac{d_l(u,Su)d_l(v,Tv)}{d_l(u,Tv) + d_l(u,v) + d_l(v,Su)}. \end{aligned}$$

Hence, all contractive conditions of theorem 2.1 are satisfied.

Corollary 2.3. *Let (X,d_l) be a complete dislocated metric space and u_0 be any arbitrary point in X let the mappings $S,T: X \to X$ satisfy:*

$$\begin{aligned} d_l(Su,Tv) &\leq a_1 d_l(u,v) + a_2\frac{d_l(u,Su).d_l(v,Tv)}{d_l(u,v)} \\ &\quad + a_3\frac{d_l(u,Su).d_l(v,Tv)}{d_l(u,Tv) + d_l(u,v) + d_l(v,Su)} \end{aligned}$$

for all $u,v \in \overline{B_{d_l}(u_0,r)}$ with $r > 0$, and

$$d_l(u_0,Su_0) \leq (1-\lambda)r$$

where $\lambda = \left(\frac{a_1 + a_3}{1 - a_2}\right)$ and, a_1, a_2, a_3, are non negative reals with $a_1 + a_2 + a_3 < 1$. If, $\{u_n\}$ is a non increasing sequence in $\overline{B_{d_l}(u_0,r)}$ for all $n \in \mathbb{N} \cup \{0\}$ and $u_n \to h \in \overline{B_{d_l}(u_0,r)}$. Then S and T have common fixed point h in $\overline{B_{d_l}(u_0,r)}$.

Proof: By putting $a_3 = 0$ in Theorem 2.1, we get the required result:

Theorem 2.4. *Let (X,d_l) be a complete dislocated metric space and v_0 be any arbitrary point in X let the mappings $S,T: X \to X$ satisfy:*

$$d_l(S(v),T(f)) \leq a d_l(v,f) + b\frac{d_l(v,S(v))d_l(f,T(f))}{1 + d_l(v,f)} \quad (2.9)$$

for all $v, f \in \overline{B_{d_l}(u_0, r)}$ *and* $v \neq f$ *with* $r > 0$,

$$d_l(v_0, Sv_0) \leq (1-\lambda)r \qquad (2.10)$$

where $\lambda = \left(\frac{a}{1-b}\right)$ *and* a, b *are nonnegative reals with* $a+b<1$. *If,* $\{v_n\}$ *is a non increasing sequence in* $\overline{B_{d_l}(v_0, r)}$ *for all* $n \in \mathbb{N}\cup\{0\}$ *and* $v_n \to u \in \overline{B_{d_l}(v_0, r)}$. *Then S and T have common fixed point u in* $\overline{B_{d_l}(v_0, r)}$.

Proof: *Let* v_0 *be an arbitrary point in X and define* $v_1 = S(v_0)$ *and* $v_2 = T(v_1)$ *such that*

$$d_l(v_1, v_2) = d_l(S(v_0), T(v_1)).$$

Then

$$d_l(v_1, v_2) \leq a d_l(v_0, v_1) + b\frac{d_l(v_0, S(v_0))\, d_l(v_1, T(v_1))}{1 + d_l(v_0, v_1)},$$
$$\leq a d_l(v_0, v_1) + b\frac{d_l(v_0, v_1)\, d_l(v_1, v_2)}{1 + d_l(v_0, v_1)},$$
$$\leq a d_l(v_0, v_1) + b d_l(v_1, v_2)\left(\frac{d_l(v_0, v_1)}{1 + d_l(v_0, v_1)}\right),$$
$$\leq a d_l(v_0, v_1) + b d_l(v_1, v_2).$$

This implies that

$$d_l(v_1, v_2) \leq \left(\frac{a}{1-b}\right) d_l(v_0, v_1),$$
$$\leq \lambda d_l(v_0, v_1),$$
$$\leq \lambda(1-\lambda)r \text{ by using (2.10)}$$

Where $\left(\frac{a}{1-b}\right) = \lambda$. Now

$$d_l(v_0, v_2) \leq d_l(v_0, v_1) + d_l(v_1, v_2),$$
$$\leq (1-\lambda)r + \lambda(1-\lambda)r,$$
$$\leq \left(1-\lambda^2\right)r \leq r$$

this implies that $v_2 \in \overline{B_{d_l}(v_0, r)}$ similarly,

$$d_l(v_2, v_3) = d_l(v_3, v_2) = d_l(S(v_2), T(v_1))$$

$$d_l(S(v_2), T(v_1))$$
$$\leq a d_l(v_2, v_1) + b\frac{d_l(v_2, S(v_2)) d_l(v_1, T(v_1))}{1 + d_l(v_2, v_1)},$$
$$\leq a d_l(v_2, v_1) + b\frac{d_l(v_2, v_3) d_l(v_1, v_2)}{1 + d_l(v_2, v_1)},$$
$$\leq a d_l(v_2, v_1) + b d_l(v_2, v_3)\left(\frac{d_l(v_1, v_2)}{1 + d_l(v_2, v_1)}\right),$$
$$d_l(v_2, v_3) \leq a d_l(v_1, v_2) + b d_l(v_2, v_3).$$

This implies that

$$d_l(v_2, v_3) \leq \left(\frac{a}{1-b}\right) d_l(v_1, v_2),$$
$$\leq \lambda.\lambda d_l(v_0, v_1),$$
$$\leq \lambda^2 d_l(v_0, v_1)$$
$$\leq \lambda^2(1-\lambda)r \leq r.$$

Consequently, $v_3, v_4, ..., v_j \in \overline{B_{d_l}(v_0, r)}$, for some $j \in N$. If $j = 2i+1$, where $i = 0, 1, 2, ..., \frac{j-1}{2}$ we get

$$d_l(v_{2i+1}, v_{2i+2}) \leq \lambda d_l(v_{2i}, v_{2i+1}). \qquad (2.11)$$

Similarly, if $j = 2i+2$, where $i = 0, 1, 2, ..., \frac{j-2}{2}$, we have

$$d_l(v_{2i+2}, v_{2i+3}) \leq \lambda d_l(v_{2i+1}, v_{2i+2}). \qquad (2.12)$$

Now, (2.11) implies that

$$d_l(v_{2i+1}, v_{2i+2}) \leq \lambda^{2i+1} d_l(v_0, v_1). \qquad (2.13)$$

Also, (2.12) implies that

$$d_l(v_{2i+2}, v_{2i+3}) \leq \lambda^{2i+2} d_l(v_0, v_1). \qquad (2.14)$$

Now, by combining (2.13) and (2.14), we have

$$d_l(v_j, v_{j+1}) \leq \lambda^j d_l(v_0, v_1) \text{ for all } j \in N. \qquad (2.15)$$

Now,

$$d_l(v_0, v_{j+1}) \leq d_l(v_0, v_1) + d_l(v_1, v_2) + ... + d_l(v_j, v_{j+1})$$
$$\leq d_l(v_0, v_1) + \lambda d_l(v_0, v_1) + ... + \lambda^j(v_0, v_1) \text{ by (2.15)}$$
$$\leq \left(1 + \lambda + \lambda^2 + ... + \lambda^j\right) d_l(v_0, v_1)$$
$$\leq \frac{1\left(1-\lambda^j\right)}{1-\lambda}(1-\lambda)r \text{ as } j \to \infty$$
$$\leq r.$$

Thus $v_{j+1} \in \overline{B_{d_l}(v_0, r)}$. Hence $v_n \in \overline{B_{d_l}(v_0, r)}$ for all $n \in N$, therefore $\{v_n\}$ is a sequence in $\overline{B_{d_l}(v_0, r)}$. Now, the inequality (2.15) can be written as

$$d_l(v_n, v_{n+1}) \leq \lambda^n d_l(v_0, v_1) \text{ for all } n \in N. \qquad (2.16)$$

To prove that $\{v_n\}$ is a Cauchy sequence, we have for any $m > n$,

$$d_l(v_n, v_m)$$
$$\leq d_l(v_n, v_{n+1}) + d_l(v_{n+1}, v_{n+2}) + ... + d_l(v_{m-1}, v_m),$$
$$\leq \lambda^n d_l(v_0, v_1) + \lambda^{n+1} d_l(v_0, v_1) + ... + \lambda^{m-1} d_l(v_0, v_1)$$
$$\leq \left(\lambda^n + \lambda^{n+1} + ... + \lambda^{m-1}\right) d_l(v_0, v_1),$$
$$\leq \left(\frac{\lambda^n}{1-\lambda}\right) d_l(v_0, v_1),$$
$$\to 0 \text{ as } m, n \to \infty.$$

This implies that $\{v_n\}$ is a Cauchy sequence in $\overline{B_{d_l}(v_0,r)}$. Since $\overline{B_{d_l}(v_0,r)}$ is closed and complete, there exists a point $u \in \overline{B_{d_l}(v_0,r)}$ such that $v_n \to u$, and suppose $\theta = d_l(u,Su)$. Therefore we have

$$\begin{aligned} d_l(u,Su) &\le d_l(u,v_{2n+2}) + d_l(v_{2n+2},Su) \\ &= d_l(u,v_{2n+2}) + d_l(T(v_{2n+1}),Su) \\ &= d_l(u,v_{2n+2}) + d_l(Su,T(v_{2n+1})) \\ &\le d_l(v_{2n+2},u) + ad_l(u,v_{2n+1}) \\ &\quad + b\frac{d_l(u,Su)d_l(v_{2n+1},T(v_{2n+1}))}{1+d_l(u,v_{2n+1})} \\ &\le d_l(v_{2n+2},u) + ad_l(u,v_{2n+1}) \\ &\quad + b\frac{d_l(u,Su)d_l(v_{2n+1},v_{2n+2})}{1+d_l(u,v_{2n+1})} \end{aligned}$$

$$\theta \le d_l(u,v_{2n+2}) + ad_l(u,v_{2n+1}) + b\frac{\theta + d_l(v_{2n+1},v_{2n+2})}{1+d_l(u,v_{2n+1})}$$

letting $n \to \infty$, and $v_n \to u$ we get,

$$(1-b)\theta \le 0$$

$$(1-b) \ne 0$$

$$\theta = d_l(u,Su) \le 0.$$

which implies that $u = Su$. It follows similarly that $u = Tu$. Now, we show that S and T have a unique common fixed point. For this, assume that h in X is a second common fixed point of S and T. Then

$$\begin{aligned} d_l(u,h) &= d_l(Su,Th) \\ &\le ad_l(u,h) + b\frac{d_l(u,Su)d_l(h,Th)}{1+d_l(u,h)} \\ &\le ad_l(u,h). \end{aligned}$$

This implies that,

$$(1-a)d_l(u,h) \le 0$$

$$1-a \ne 0$$

$$d_l(u,h) = 0.$$

This implies that $u = h$, completing the proof of the theorem.

Example 2.5. Let $X = \{0\} \cup \mathbb{Q}^+$ be a dislocated metric space $d_l : X \times X \to X$ defined by

$$d_l(v,f) = v + f.$$

Let $S : X \to X$ defined by

$$S(v) = \begin{Bmatrix} \frac{v}{2} & \text{if } v \in [0,1] \cap X \\ 2v & \text{if } v \in (1,\infty) \cap X \end{Bmatrix}.$$

And $T : X \to X$ defined by

$$T(f) = \begin{Bmatrix} \frac{3f}{4} & \text{if } f \in [0,1] \cap X \\ 5f & \text{if } f \in (1,\infty) \cap X \end{Bmatrix}.$$

Take $a = \frac{1}{3}, b = \frac{1}{4}, v_0 = \frac{1}{6}, r = 5,$ then $\overline{B_{d_l}(v_0,r)} = [0,1] \cap X$. We have $\lambda = \left(\frac{a}{1-b}\right) = \frac{4}{9}$ with

$$(1-\lambda)r = \left(1-\frac{4}{9}\right)5 = \frac{25}{9}$$

and

$$d_l(v_0,Sv_0) = \frac{1}{6} + \frac{1}{12} = \frac{1}{4} < (1-\lambda)r.$$

Also if $v, f \in (1,\infty) \cap X$, then

$$\begin{aligned} d_l(S(v),T(f)) &= 2v + 5f \\ &\ge \frac{1}{3}(v+f) + \frac{1}{4}\frac{(v+2v).(f+5f)}{1+(v+f)} \\ &\ge \frac{1}{3}(v+f) + \frac{9vf}{2\{1+(v+f)\}} \end{aligned}$$

$$d_l(S(v),T(f)) \ge ad_l(v,f) + b\frac{d_l(v,S(v))d_l(f,T(f))}{1+d_l(v,f)}.$$

Then $v = 2$ and $f = 3$, then clearly $v, f \in (1,\infty) \cap X$, and $v, f \ge 1$, we have

$$19 \ge \frac{5}{3} + \frac{1}{4} \times \frac{54}{12}$$
$$19 \ge 2.785.$$

Hence, clearly whole space does not satisfy the contractive condition. Also if $v, f \in \overline{B_{d_l}(v_0,r)}$, then

$$\begin{aligned} &d_l(S(v),T(f)) \\ &= \frac{v}{2} + \frac{3f}{4} \le \frac{1}{3}(v+f) + \frac{1}{4}\frac{\left(v+\frac{v}{2}\right).\left(f+\frac{3f}{4}\right)}{1+(v+f)} \\ &d_l(S(v),T(f)) \\ &\le ad_l(v,f) + b\frac{d_l(v,S(v))d_l(f,T(f))}{1+d_l(v,f)}. \end{aligned}$$

Hence, the contractive conditions of Theorem (2.4) are satisfied.

As an application of Theorems (2.1) and (2.4) we prove the following theorem for two finite families of mappings.

Theorem 2.6. *If $\{T_p\}_1^m$ and $\{S_p\}_1^n$ are two finite pair wise commuting finite families of self mappings de.ned on complete dislocated metric space (X,d_l) such that the mappings T and S satisfy the conditions of theorems* (2.1) *and* (2.4), *then the component maps of the two families $\{T_p\}_1^m$ and $\{S_p\}_1^n$ have a unique common fixed point.*

Proof. In view of theorems (2.1) and (2.4), one can infer that T and S have a unique common fixed point q i.e. $Tq = Sq = q$. Now we are required to show that q is common fixed point of all the components maps of both the families. In view of pairwise commutativity of families of $\{T_p\}_1^m$ and $\{S_p\}_1^n$, (for every $1 \le i \le m$) we can write

$$T_i q = T_i Sq = ST_i q \text{ and } T_i q = T_i Tq = TT_i q$$

which shows that $T_i q$ (for every i) is also a common fixed point of T and S. By using the uniqueness of common fixed point, we can write $T_i q = q$ (for every i) which shows that q is the common fixed point of the family $\{T_p\}_1^m$. Using the foregoing arguments, one can also shows that (for every $1 \le i \le n$) $S_i q = q$. This completes the proof of the theorem.

Corollary 2.7. *If $S : X \to X$ be a self mapping de.ned on a complete dislocated metric space (X, d_l) satisfying the condition*

$$d_l(Su,Sv) \le a d_l(u,v) + b\frac{d_l(u,Su).d_l(v,Sv)}{d_l(u,v)} + c\frac{d_l(u,Sv).d_l(v,Su)}{d_l(u,v)} + d\frac{d_l(u,Su).d_l(v,Sv)}{d_l(u,Sv)+d_l(u,v)+d_l(v,Su)}$$

for all $u, v \in \overline{B_{d_l}(u_0,r)}$ with $r > 0$,

$$d_l(u_0,Su_0) \le (1-\lambda)r$$

where $\lambda = \left(\frac{a+2c+d}{1-b-2a_3}\right)$ and a, b, c, d are non negative reals with $a+b+4c+d<1$. Then $\{u_n\}$ is a non increasing sequence in $\overline{B_{d_l}(u_0,r)}$ for all $n \in \mathbb{N} \cup \{0\}$ and $u_n \to h \in \overline{B_{d_l}(u_0,r)}$. Then S has common fixed point h in $\overline{B_{d_l}(u_0,r)}$.

Corollary 2.8. *If $S : X \to X$ be a self mapping defined on a complete on dislocated metric space (X, d_l) satisfying the condition:*

$$d_l(Su,Sv) \le a_1 d_l(u,v) + a_2\frac{d_l(u,Su).d_l(v,Sv)}{d_l(u,v)}$$

for all $u, v \in \overline{B_{d_l}(u_0,r)}$ with $r > 0$,

$$d_l(u_0,Su_0) \le (1-\lambda)r$$

where $\lambda = \left(\frac{a_1}{1-a_2}\right)a_1$ and a_2 are non negative reals with $a_1 + a_2 < 1$. If, $\{u_n\}$ is a non increasing sequence in $\overline{B_{d_l}(u_0,r)}$ for all $n \in \mathbb{N} \cup \{0\}$ and $u_n \to h \in \overline{B_{d_l}(u_0,r)}$ Then S has common fixed point h in $\overline{B_{d_l}(u_0,r)}$.

Conflict of Interests

The authors declare that they have no competing interests.

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