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Hyers-Ulam Stability of Generalized Tribonacci Functional Equation

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Abstract In this paper we study Hyers-Ulam stability of the generalized Tribonacci functional equation, $f(x)=af(x-1)+bf(x-2)+cf(x-3)$, where a, b and c are non- zero constants. The functional equation is solved and its stability is established in the class of functions $f:\mathbb{R}\to X$, where X is a Banach space.

Keywords: *Fibonacci functional equation, Hyers-Ulam stability, Tribonacci functional equation*

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1. Introduction

Stability problems of functional equations have been extensively studied. (see [1,5,6] and references therein). The importance of the topic lies in the fact that stability of functional equation is associated with notions of Controlled Chaos [11] and Shadowing [14]. In [7] and [8] Jung discusses the stability problem for Fibonacci functional equation and its extension in Banach Space respectively whereas in [3] the problem is discussed in Modular Functional space. In [10] *k*-Fibonacci functional equation is discussed. In [2] and [4] solution and stability of Tribonacci functional equation in non-Archimedean Banach spaces and 2-normed spaces have been discussed respectively. Stability of Tribonacci and k-Tribonacci functional equations in Modular spaces is discussed in [13]. In [9], authors investigate the solution of generalized linear Tribonacci functional equation in terms of Fibonacci numbers. In this paper we show that the generalized linear Tribonacci functional equation is associated with generalized Tribonacci sequence and also obtain its stability in the class of functions $f:\mathbb{R}\to X$, where X is a real (or complex) Banach space. Like Fibonacci numbers, Tribonacci numbers also play an important role in problems of combinatorics [12] and also in evaluation of determinants of circulant matrices [15].
We now define the generalized Tribonacci sequence.

Definition 1. *The generalized Tribonacci sequence is defined by*

$$T_{n+2}=aT_{n+1}+bT_n+cT_{n-1}, \tag{1.1}$$

$$T_0=0,\ T_1=0 \text{ and } T_2=1, \text{ for all } n\in\mathbb{Z},$$

where a, b and c are non-zero fixed real numbers and T_n is the n^{th} term of (1.1) given by Binet type formula,

$$T_n=\frac{(\alpha-\beta)\gamma^n-(\alpha-\gamma)\beta^n+(\beta-\gamma)\alpha^n}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}. \tag{1.2}$$

where α, β and γ are distinct roots of the characteristics equation corresponding to (1.1).

Definition 2. *Let X be a real (or complex) Banach space. A function $f:\mathbb{R}\to X$ is called a generalized Tribonacci function if it satisfies generalized Tribonacci functional equation*

$$f(x)=af(x-1)+bf(x-2)+cf(x-3),\ \forall x\in\mathbb{R}. \tag{1.3}$$

2. Existence of General Solution to the Equation (1.3)

In this section we solve equation (1.3) and prove that its solution is in terms of (1.2). To prove this result, we require the following lemma.

Lemma 3. *If α, β and γ are distinct roots of the characteristics equation corresponding to (1.1), then the generalized Tribonacci function $f:\mathbb{R}\to X$ satisfies*

$$\begin{aligned}f(x)&=T_{n+2}f(x-n)\\&+(bT_{n+1}+cT_n)f(x-n-1)+cT_{n+1}f(x-n-2),\end{aligned} \tag{2.1}$$

$\forall x\in\mathbb{R}$ *and* $\forall n\in\mathbb{Z}$.

Proof. Characteristics equation corresponding to (1.1) is

$$\lambda^3-a\lambda^2-b\lambda-c=0. \tag{2.2}$$

Since α, β and γ are distinct roots of (2.2), we get $a=\alpha+\beta+\gamma$, $b=-(\alpha\gamma+\beta\gamma+\alpha\beta)$ and $c=\alpha\beta\gamma$.

Substituting *a, b* and c in (1.3), we have

$$\begin{aligned}f(x)&=(\alpha+\beta+\gamma)f(x-1)\\&-(\alpha\gamma+\beta\gamma+\alpha\beta)f(x-2)+(\alpha\beta\gamma)f(x-3),\end{aligned}$$

which implies

$$\begin{aligned} &f(x)-(\alpha+\beta)f(x-1)+(\alpha\beta)f(x-2) \\ &=\gamma\big(f(x-1)-(\alpha+\beta)f(x-2)+(\alpha\beta)f(x-3)\big). \end{aligned} \tag{2.3}$$

Replacing x by $x-1$ in (2.3), we get

$$\begin{aligned} &f(x-1)-(\alpha+\beta)f(x-2)+(\alpha\beta)f(x-3) \\ &=\gamma\big(f(x-2)-(\alpha+\beta)f(x-3)+(\alpha\beta)f(x-4)\big) \end{aligned}$$

and hence (2.3) yields

$$\begin{aligned} &f(x)-(\alpha+\beta)f(x-1)+(\alpha\beta)f(x-2) \\ &=\gamma^2\big(f(x-2)-(\alpha+\beta)f(x-3)+(\alpha\beta)f(x-4)\big). \end{aligned}$$

By induction on n, we get

$$\begin{aligned} &f(x)-(\alpha+\beta)f(x-1)+(\alpha\beta)f(x-2) \\ &=\gamma^n\big(f(x-n)-(\alpha+\beta)f(x-n-1)+(\alpha\beta)f(x-n-2)\big). \end{aligned} \tag{2.4}$$

Similarly, we have

$$\begin{aligned} &f(x)-(\alpha+\gamma)f(x-1)+(\alpha\gamma)f(x-2) \\ &=\beta^n\big(f(x-n)-(\alpha+\gamma)f(x-n-1)+(\alpha\gamma)f(x-n-2)\big) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &f(x)-(\beta+\gamma)f(x-1)+(\beta\gamma)f(x-2) \\ &=\alpha^n\big(f(x-n)-(\beta+\gamma)f(x-n-1)+(\beta\gamma)f(x-n-2)\big), \end{aligned} \tag{2.6}$$

$\forall x\in\mathbb{R}$ *and* $\forall n\in\mathbb{N}\cup\{0\}$.

Now replacing *x* by *x* + 1 in (2.3), we get

$$\begin{aligned} &f(x+1)-(\alpha+\beta)f(x)+(\alpha\beta)f(x-1) \\ &=\gamma\big(f(x)-(\alpha+\beta)f(x-1)++(\alpha\beta)f(x-2)\big). \end{aligned}$$

Therefore,

$$\begin{aligned} &f(x)-(\alpha+\beta)f(x-1)+(\alpha\beta)f(x-2) \\ &=\gamma^{-1}\big(f(x+1)-(\alpha+\beta)f(x)+(\alpha\beta)f(x-1)\big) \end{aligned}$$

Thus by induction on n, we get

$$\begin{aligned} &f(x)-(\alpha+\beta)f(x-1)+(\alpha\beta)f(x-2) \\ &=\gamma^{-n}\big(f(x+n)-(\alpha+\beta)f(x+n-1)+(\alpha\beta)f(x+n-2)\big). \end{aligned}$$

Similarly, we have

$$\begin{aligned} &f(x)-(\alpha+\gamma)f(x-1)+(\alpha\gamma)f(x-2) \\ &=\beta^{-n}\big(f(x+n)-(\alpha+\gamma)f(x+n-1)+(\alpha\gamma)f(x+n-2)\big) \end{aligned}$$

and

$$\begin{aligned} &f(x)-(\beta+\gamma)f(x-1)+(\beta\gamma)f(x-2) \\ &=\alpha^{-n}\big(f(x+n)-(\beta+\gamma)f(x+n-1)+(\beta\gamma)f(x+n-2)\big), \end{aligned}$$

$\forall x\in\mathbb{R}$ *and* $\forall n\in\mathbb{N}\cup\{0\}$.

Therefore equations (2.4), (2.5) and (2.6) are true $\forall x\in\mathbb{R}$ and $\forall n\in\mathbb{Z}$.

Now multiplying the equations (2.4), (2.5) and (2.6) by $\gamma^2(\alpha-\beta)$, $-\beta^2(\alpha-\gamma)$, $\alpha^2(\beta-\gamma)$ respectively, and adding we get

$$\begin{aligned} &f(x) \\ &=\left(\frac{\gamma^{n+2}(\alpha-\beta)-\beta^{n+2}(\alpha-\gamma)+\alpha^{n+2}(\beta-\gamma)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\right)f(x-n) \\ &+\left(\begin{array}{l} b\dfrac{\gamma^{n+1}(\alpha-\beta)-\beta^{n+1}(\alpha-\gamma)+\alpha^{n+1}(\beta-\gamma)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)} \\ +c\dfrac{\gamma^{n}(\alpha-\beta)-\beta^{n}(\alpha-\gamma)+\alpha^{n}(\beta-\gamma)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)} \end{array}\right)f(x-n-1) \\ &+c\left(\frac{\gamma^{n+1}(\alpha-\beta)-\beta^{n+1}(\alpha-\gamma)+\alpha^{n+1}(\beta-\gamma)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\right)f(x-n-2). \end{aligned}$$

Using (1.2), this gives

$$\begin{aligned} f(x)&=T_{n+2}f(x-n)+(bT_{n+1}+cT_n)f(x-n-1) \\ &\quad+cT_{n+1}f(x-n-2), \end{aligned}$$

$\forall x\in\mathbb{R}$ *and* $\forall n\in\mathbb{Z}$.

We use Lemma 3 to prove the following result.

Theorem 4. *A function* $f:\mathbb{R}\to X$ *is a solution of functional equation* (1.3) *if and only if there exists a function* $h:[-2,1)\to X$ *such that*

$$\begin{aligned} f(x)&=T_{\lfloor x\rfloor+2}h(x-\lfloor x\rfloor)+b\big(T_{\lfloor x\rfloor+1}+cT_{\lfloor x\rfloor}\big)h(x-\lfloor x\rfloor-1) \\ &\quad+cT_{\lfloor x\rfloor+1}h(x-\lfloor x\rfloor-2), \end{aligned} \tag{2.7}$$

where T_n is given by (1.2).

Proof. If $f(x)$ is a solution of (1.3), then by lemma 3 $f(x)$ satisfies (2.1). Putting $n=\lfloor x\rfloor$ in (2.1), we get

$$\begin{aligned} f(x)&=T_{\lfloor x\rfloor+2}f(x-\lfloor x\rfloor)+\big(bT_{\lfloor x\rfloor+1}+cT_{\lfloor x\rfloor}\big)f(x-\lfloor x\rfloor-1) \\ &\quad+cT_{\lfloor x\rfloor+1}f(x-\lfloor x\rfloor-2). \end{aligned}$$

Since $0\le x-\lfloor x\rfloor<1$, $-1\le x-\lfloor x\rfloor<0$ and $-2\le x-\lfloor x\rfloor-2<-1$, we define a function $h:[-2,1)\to X$ by $h:=f|_{[-2,1)}$, then *f* is of the form (2.7).

Now we assume that *f* is a function of the form (2.7) and prove that it is a solution of (1.3).

Consider

$$\begin{aligned} &f(x)-af(x-1)-bf(x-2)-cf(x-3) \\ &=\big(T_{\lfloor x\rfloor+2}-aT_{\lfloor x\rfloor+1}-bT_{\lfloor x\rfloor}-cT_{\lfloor x\rfloor-1}\big)h(x-\lfloor x\rfloor) \\ &+b\big(T_{\lfloor x\rfloor+1}-aT_{\lfloor x\rfloor}-bT_{\lfloor x\rfloor-1}-cT_{\lfloor x\rfloor-2}\big)h(x-\lfloor x\rfloor-1) \\ &+c\big(T_{\lfloor x\rfloor}-aT_{\lfloor x\rfloor-1}-bT_{\lfloor x\rfloor-2}-cT_{\lfloor x\rfloor-3}\big)h(x-\lfloor x\rfloor-1) \\ &+c\big(T_{\lfloor x\rfloor+1}-aT_{\lfloor x\rfloor}-bT_{\lfloor x\rfloor-1}-cT_{\lfloor x\rfloor-2}\big)h(x-\lfloor x\rfloor-2) \end{aligned}$$

$=0$, for any $x\in\mathbb{R}$ and arbitrary function $h:[-2,1)\to X$.

Therefore (2.7) is a solution of (1.3). Hence the theorem is proved.

We next illustrate this result.

Example 5. *Consider the following functional equation*

$$f(x)=\frac{23}{4}f(x-1)-\frac{31}{8}f(x-2)+\frac{5}{8}f(x-3). \quad (2.8)$$

Define the function $h:[-2,1)\to X$ *by*

$$h(x)=x^3-\frac{23}{4}x^2+\frac{31}{8}x-\frac{5}{8}. \quad (2.9)$$

Note tha $\frac{1}{2}$, $\frac{1}{4}$, and 5 are distinct roots of the characteristic equation corresponding to Tribonacci sequence defined by

$$T_{n+2}=\frac{23}{4}T_{n+1}-\frac{31}{8}T_n+\frac{5}{8}T_{n-1}, \quad (2.10)$$

$$T_0=0, T_1=0, T_2=1,$$

$\forall n\in Z$. *Therefore* (2.7) *implies*

$$\begin{aligned}f(x)&=T_{\lfloor x\rfloor+2}h(x-\lfloor x\rfloor)\\&+\left(-\frac{31}{8}T_{\lfloor x\rfloor+1}+\frac{5}{8}T_{\lfloor x\rfloor}\right)h(x-\lfloor x\rfloor-1) \quad (2.11)\\&+\frac{5}{8}T_{\lfloor x\rfloor+1}h(x-\lfloor x\rfloor-2),\end{aligned}$$

where h(*x*) *is given by* (2.9).

3. Hyers-Ulam Stability

In this section we prove the Hyers-Ulam stability of functional equation (1.3) by assuming that roots α,β and γ are distinct and $0<|\alpha|$, $|\gamma|<1$, $|\beta|>1$. We first prove the lemma required to prove Hyers-Ulam stability of functional equation (1.3).

Lemma 6. *If a function* $f:\mathbb{R}\to X$ *satisfies,*

$$\left\|f(x)-af(x-1)-bf(x-2)-cf(x-3)\right\|\le\epsilon,\forall x\in\mathbb{R},$$

for some $\epsilon\ge 0$ *and* α,β *and* γ *are distinct roots of (2.2) such that* $0<|\alpha|$, $|\gamma|<1$, $|\beta|>1$, *then there exists Tribonacci functions* $F_1,F_2,F_3:\mathbb{R}\to X$ *defined by*

$$\begin{aligned}F_1(x)=\lim_{n\to\infty}\gamma^n[&f(x-n)\\&-(\alpha+\beta)f(x-n-1)+(\alpha\beta)f(x-n-2)],\end{aligned}$$

$$\begin{aligned}F_2(x)=\lim_{n\to\infty}\alpha^n[&f(x-n)\\&-(\beta+\gamma)f(x-n-1)+(\beta\gamma)f(x-n-2)]\end{aligned}$$

and

$$\begin{aligned}F_3(x)=\lim_{n\to\infty}\beta^{-n}[&f(x+n)\\&-(\alpha+\gamma)f(x+n-1)+(\alpha\gamma)f(x+n-2),\end{aligned}$$

such that

$$\begin{aligned}\|f(x)-(\alpha+\beta)f(x-1)+(\alpha\beta)f(x-2)&\\-F_1(x)\|\le\frac{\epsilon}{1-|\gamma|},&\end{aligned} \quad (3.1)$$

$$\begin{aligned}\|f(x)-(\beta+\gamma)f(x-1)+(\beta\gamma)f(x-2)&\\-F_2(x)\|\le\frac{\epsilon}{1-|\alpha|}&\end{aligned} \quad (3.2)$$

and

$$\begin{aligned}\|F_3(x)-\big(f(x)-(\alpha+\gamma)f(x-1)+(\alpha\gamma)f(x-2)\big)\|&\\\le\frac{\epsilon}{|\beta|-1},&\end{aligned} \quad (3.3)$$

$\forall x\in\mathbb{R}$.

Proof. Using (2.4) with n = 1 in given condition on *f*, we have

$$\begin{aligned}&\|f(x)-(\alpha+\beta)f(x-1)+(\alpha\beta)f(x-2)\\&-\gamma[f(x-1)-(\alpha+\beta)f(x-2)++(\alpha\beta)f(x-3)]\|\le\epsilon.\end{aligned}$$

Replacing x by $x-k$, we have

$$\begin{aligned}&\|f(x-k)-(\alpha+\beta)f(x-k-1)+(\alpha\beta)f(x-k-2)\\&-\gamma[f(x-k-1)-(\alpha+\beta)f(x-k-2)+(\alpha\beta)f(x-k-3)]\|\le\epsilon.\end{aligned}$$

Multiplying both side by $|\gamma|^k$,

$$\begin{aligned}&\|\gamma^k[f(x-k)-(\alpha+\beta)f(x-k-1)+(\alpha\beta)f(x-k-2)]\\&-\gamma^{k+1}[f(x-k-1)-(\alpha+\beta)f(x-k-2)+(\alpha\beta)f(x-k-3)]\|\\&\le|\gamma|^k\epsilon,\forall x\in\mathbb{R}\text{ and }\forall k\in\mathbb{Z}.\end{aligned} \quad (3.4)$$

Further,

$$\begin{aligned}&\|f(x)-(\alpha+\beta)f(x-1)+(\alpha\beta)f(x-2)\\&-\gamma^n[f(x-n)-(\alpha+\beta)f(x-n-1)+(\alpha\beta)f(x-n-2)]\|\\&\le\sum_{k=0}^{n-1}\|\gamma^k[f(x-k)-(\alpha+\beta)f(x-k-1)+(\alpha\beta)f(x-k-2)]\\&\qquad-\gamma^{k+1}[f(x-k-1)-(\alpha+\beta)f(x-k-2)+(\alpha\beta)f(x-k-3)]\|\\&\qquad\le\sum_{k=0}^{n-1}|\gamma|^k\epsilon,\forall x\in\mathbb{R}\text{ and }\forall k\in\mathbb{Z}.\end{aligned}$$

Therefore

$$\begin{aligned}&\|f(x)-(\alpha+\beta)f(x-1)+(\alpha\beta)f(x-2)\\&-\gamma^n[f(x-n)-(\alpha+\beta)f(x-n-1)+(\alpha\beta)f(x-n-2)]\|\\&\le\sum_{k=0}^{n-1}|\gamma|^k\epsilon,\forall x\in\mathbb{R},\forall k\in\mathbb{Z}\text{ and }\forall n\in\mathbb{N}.\end{aligned} \quad (3.5)$$

Thus for $0<|\gamma|<1$ and for any $x\in\mathbb{R}$, (3.5) implies that the sequence $\{\gamma^n[f(x-n)-(\alpha+\beta)f(x-n-1)+(\alpha\beta)f(x-n-2)]\}$ is a cauchy sequence.

Therefore, since X is Banach space, we can define a function $F_1:\mathbb{R}\to X$ by

$$\begin{aligned}F_1(x)=\lim_{n\to\infty}\gamma^n[&f(x-n)\\&-(\alpha+\beta)f(x-n-1)+(\alpha\beta)f(x-n-2)].\end{aligned}$$

We now prove that $F_1(x)$ satisfies (1.3).

Consider,

$$aF_1(x-1)+bF_1(x-2)+cF_1(x-3)$$
$$=a\gamma^{-1}\lim_{n\to\infty}\gamma^{n+1}[f(x-(n+1))$$
$$-(\alpha+\beta)f(x-(n+2))+(\alpha\beta)f(x-(n+3))]$$
$$+b\gamma^{-2}\lim_{n\to\infty}\gamma^{n+2}[f(x-(n+2))$$
$$-(\alpha+\beta)f(x-(n+3))+(\alpha\beta)f(x-(n+4))]$$
$$+c\gamma^{-3}\lim_{n\to\infty}\gamma^{n+3}[f(x-(n+3))$$
$$-(\alpha+\beta)f(x-(n+4))+(\alpha\beta)f(x-(n+5))]$$
$$+F_1(x)\left(a\gamma^{-1}+b\gamma^{-2}+c\gamma^{-3}\right),$$
$$=F_1(x),\ \forall x\in\mathbb{R}.$$

Hence F_1 is a Tribonacci function.

Now taking $n\to\infty$, (3.5) implies

$$\left\|f(x)-(\alpha+\beta)(x-1)+(\alpha\beta)f(x-2)-F_1(x)\right\|$$
$$\le\lim_{n\to\infty}\sum_{k=0}^{n-1}\left|\gamma^k\right|\epsilon=\frac{\epsilon}{1-|\gamma|}.$$

Similarly since $0<|\alpha|<1$, using equation (2:6) with $n=1$ we can prove that there exists a Tribonacci function $F_2:\mathbb{R}\to X$ given by

$$F_2(x)=\lim_{n\to\infty}\alpha^n[f(x-n)$$
$$-(\beta+\gamma)f(x-n-1)+(\beta\gamma)f(x-n-2)]$$

such that

$$\left\|f(x)-(\beta+\gamma)f(x-1)+(\beta\gamma)f(x-2)-F_2(x)\right\|$$
$$\le\lim_{n\to\infty}\sum_{k=0}^{n-1}\left|\alpha^k\right|\epsilon=\frac{\epsilon}{1-|\alpha|}.$$

From equation (2.5) with n = 1, it follows that

$$\|f(x)-(\alpha+\gamma)f(x-1)+(\alpha\gamma)f(x-2)$$
$$-\beta\left[f(x-1)-(\alpha+\gamma)f(x-2)+(\alpha\gamma)f(x-3)\right]\|\le\epsilon.$$

Replacing x by $x+k$ and multiplying both sides by $|\beta|^{-k}$, we get

$$\|\beta^{-k}\left[f(x+k)-(\alpha+\gamma)f(x+k-1)+(\alpha\gamma)f(x+k-2)\right]$$
$$-\beta^{-k+1}\left[f(x+k-1)-(\alpha+\gamma)f(x+k-2)+(\alpha\gamma)f(x+k-3)\right]\|$$
$$\le|\beta|^{-k}\epsilon,\forall x\in IR\text{ and }\forall k\in\mathbb{Z}. \tag{3.6}$$

For $n\in\mathbb{N}$,

$$\|\beta^{-n}\left[f(x+n)-(\alpha+\gamma)f(x+n-1)+(\alpha\gamma)f(x+n-2)\right]$$
$$-\left[f(x)-(\alpha+\gamma)f(x-1)+(\alpha\gamma)f(x-2)\right]\|$$
$$\le\sum_{k=1}^{n}\|\beta^{-k}\left[f(x+k)-(\alpha+\gamma)f(x+k-1)+(\alpha\gamma)f(x+k-2)\right]$$
$$-\beta^{-k-1}\left[f(x+k-1)-(\alpha+\gamma)f(x+k-2)+(\alpha\gamma)f(x+k-3)\right]\|$$
$$\le\sum_{k=1}^{n}|\beta|^{-k}\epsilon.$$

Thus, for all $x\in\mathbb{R}$ and $\forall_n\in\mathbb{N}$,

$$\|\beta^{-n}\left[f(x+n)-(\alpha+\gamma)f(x+n-1)+(\alpha\gamma)f(x+n-2)\right]$$
$$-\left[f(x)-(\alpha+\gamma)f(x-1)+(\alpha\gamma)f(x-2)\right]\| \tag{3.7}$$
$$\le\sum_{k=1}^{n}|\beta|^{-k}\epsilon.$$

Equation (3.7) implies that

$$\left\{\beta^{-n}\left[f(x+n)-(\alpha+\gamma)f(x+n-1)+(\alpha\gamma)f(x+n-2)\right]\right\}$$

is a cauchy sequence for any fixed $x\in\mathbb{R}$. Since X is a Banach space, we can define a function $F_3:\mathbb{R}\to X$ by

$$F_3(x)=\lim_{n\to\infty}\beta^{-n}[f(x+n)$$
$$-(\alpha+\gamma)f(x+n-1)+(\alpha\gamma)f(x+n-2).$$

Now Consider,

$$aF_3(x-1)+bF_3(x-2)+cF_3(x-3)$$
$$=a\beta^{-1}\lim_{n\to\infty}\beta^{-n+1}[f(x+(n-1))$$
$$-(\alpha+\gamma)f(x+(n-2))+(\alpha\gamma)f(x+(n-3))]$$
$$+b\beta^{-2}\lim_{n\to\infty}\beta^{n+2}[f(x+(n-2))$$
$$-(\alpha+\gamma)f(x+(n-3))+(\alpha\gamma)f(x+(n-4))]$$
$$+c\beta^{-3}\lim_{n\to\infty}\beta^{n+3}[f(x+(n-3))$$
$$-(\alpha+\gamma)f(x+(n-4))+(\alpha\gamma)f(x+(n-5))$$
$$=F_3(x)\left(a\beta^{-1}+b\beta^{-2}+c\beta^{-3}\right)$$
$$=F_3(x),\forall x\in\mathbb{R}.$$

Therefore $F_3(x)$ is a Tribonacci function. So as $n\to\infty$, we have

$$\left\|F_3(x)-\left(f(x)-(\alpha+\gamma)f(x-1)+(\alpha\gamma)f(x-2)\right)\right\|$$
$$\le\lim_{n\to\infty}\sum_{k=1}^{n-1}|\beta|^{-k}\epsilon=\frac{\epsilon}{|\beta|-1},\forall x\in IR.$$

We prove the following theorem using Lemma 6.

Theorem 7. *If a function* $f:\mathbb{R}\to X$ *satisfies the inequality,*

$$\left\|f(x)-af(x-1)-bf(x-2)-cf(x-3)\right\|\le\epsilon,\forall x\in\mathbb{R} \tag{3.8}$$

and for some $\epsilon\ge 0$, *then there exists a unique solution function* $F:\mathbb{R}\to X$ *of the functional equation (1.3) such that*

$$\left\|f(x)-F(x)\right\|$$
$$\le\frac{\epsilon}{|\alpha-\beta||\beta-\gamma||\alpha-\gamma|}\left(\frac{(|\alpha|-|\beta|)|\gamma|^2}{1-|\gamma|}+\frac{(|\alpha|-|\gamma|)|\beta|^2}{|\beta|-1}\right.$$
$$\left.+\frac{(|\beta|-|\gamma|)|\alpha|^2}{1-|\alpha|}\right),\forall x\in IR.$$

Proof. From (3.1), (3.2) and (3.3), we see that

$$\left\| f(x) - \frac{\left[(\alpha-\beta)\gamma^2 F_1(x) - (\alpha-\gamma)\beta^2 F_3(x) + (\beta-\gamma)\alpha^2 F_2(x)\right]}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)} \right\|$$

$$= \frac{1}{|\alpha-\beta||\beta-\gamma||\alpha-\gamma|}\Big[\Big\|(\alpha-\beta)\gamma^2$$
$$\big(f(x) - (\alpha+\beta) f(x-1) + (\alpha\beta) f(x-2) - F_1(x)\big)\|\Big]$$
$$+ \frac{1}{|\alpha-\beta||\beta-\gamma||\alpha-\gamma|}\Big[\Big\|(\beta-\gamma)\alpha^2$$
$$\big(f(x) - (\beta+\gamma) f(x-1) + (\beta\gamma) f(x-2) - F_2(x)\big)\Big\|\Big]$$
$$+ \frac{1}{|\alpha-\beta||\beta-\gamma||\alpha-\gamma|}\Big[\Big\|(\alpha-\gamma)\beta^2$$
$$\big[F_3(x) - \big(f(x) - (\alpha+\gamma) f(x-1) + (\alpha\gamma) f(x-2)\big)\big]\Big\|\Big]$$
$$\le \frac{\epsilon}{|\alpha-\beta||\beta-\gamma||\alpha-\gamma|}\left(\frac{(\alpha-\beta)|\gamma|^2}{1-|\gamma|} + \frac{(\beta-\gamma)|\alpha|^2}{1-|\alpha|} + \frac{(\alpha-\gamma)|\beta|^2}{|\beta|-1}\right), \forall x \in IR.$$

We now define a function $F : \mathbb{R} \to X$ by

$$F(x) = \frac{(\alpha-\beta)\gamma^2 F_1(x) - (\alpha-\gamma)\beta^2 F_3(x) + (\beta-\gamma)\alpha^2 F_2(x)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)},$$
$\forall x \in IR.$

Consider,

$$aF(x-1) - bF(x-2) - cF(x-3)$$
$$= a\frac{(\alpha-\beta)\gamma^2 F_1(x-1) - (\alpha-\gamma)\beta^2 F_3(x-1) + (\beta-\gamma)\alpha^2 F_2(x-1)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}$$
$$+ b\frac{(\alpha-\beta)\gamma^2 F_1(x-2) - (\alpha-\gamma)\beta^2 F_3(x-2) + (\beta-\gamma)\alpha^2 F_2(x-2)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}$$
$$+ c\frac{(\alpha-\beta)\gamma^2 F_1(x-3) - (\alpha-\gamma)\beta^2 F_3(x-3) + (\beta-\gamma)\alpha^2 F_2(x-3)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}$$
$$= \frac{(\alpha-\beta)\gamma^2 F_1(x) - (\alpha-\gamma)\beta^2 F_3(x) + (\beta-\gamma)\alpha^2 F_2(x)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}$$
$$= F(x), \forall x \in IR.$$

Therefore F is a solution of (1.3). Now we prove the uniqueness of $F(x)$.

Assume that $F, \hat{F} : \mathbb{R} \to X$ are solutions of (1.3) and that there exist positive constants C1 and C2 such that $\|f(x) - F(x)\| \le C_1$ and $\|f(x) - \hat{F}(x)\| \le C_2$ for all $x \in \mathbb{R}$.

Therefore by Theorem 4, there exist $h, g : [-2,1) \to X$ such that

$$F(x) = T_{\lfloor x \rfloor + 2} h(x - \lfloor x \rfloor) + \left(bT_{\lfloor x \rfloor + 1} + cT_{\lfloor x \rfloor}\right) h(x - \lfloor x \rfloor - 1) + \left(cT_{\lfloor x \rfloor + 1}\right) h(x - \lfloor x \rfloor - 2) \quad (3.9)$$

and

$$\hat{F}(x) = T_{\lfloor x \rfloor + 2} g(x - \lfloor x \rfloor) + \left(bT_{\lfloor x \rfloor + 1} + cT_{\lfloor x \rfloor}\right) g(x - \lfloor x \rfloor - 1) + \left(cT_{\lfloor x \rfloor + 1}\right) g(x - \lfloor x \rfloor - 2), \quad (3.10)$$

for any $x \in \mathbb{R}$.

Fix t with $0 \le t < 1$. It then follows from (3.9) and (3.10) that

$$\|T_{n+2}\big(h(t) - g(t)\big) + \big(bT_{n+1} + cT_n\big)\big(h(t-1) - g(t-1)\big)$$
$$+ \big(cT_{n+1}\big)\big(h(t-2) - g(t-2)\big)\|$$
$$= \left\|F(n) - \hat{F}(n)\right\|$$
$$\le \left\|F(n+t) - f(n+t)\right\| + \left\|f(n+t) - \hat{F}(n+t)\right\|$$
$$\le C_1 + C_2, \text{ for each } n \in \mathbb{Z}.$$

That is,

$$\|T_{n+2}\big(h(t) - g(t)\big) + \big(bT_{n+1} + cT_n\big)\big(h(t-1) - g(t-1)\big)$$
$$+ \big(cT_{n+1}\big)\big(h(t-2) - g(t-2)\big)\|$$
$$\le C_1 + C_2.$$

This implies

$$\Bigg\|\frac{(\alpha-\beta)\gamma^{n+2} - (\alpha-\gamma)\beta^{n+2} + (\beta-\gamma)\alpha^{n+2}}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\big(h(t) - g(t)\big)$$
$$+ \left(b\frac{(\alpha-\beta)\gamma^{n+1} - (\alpha-\gamma)\beta^{n+1} + (\beta-\gamma)\alpha^{n+1}}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)} + c\frac{(\alpha-\beta)\gamma^{n} - (\alpha-\gamma)\beta^{n} + (\beta-\gamma)\alpha^{n}}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\right)\big(h(t-1) - g(t-1)\big)$$
$$+ \left(c\frac{(\alpha-\beta)\gamma^{n+1} - (\alpha-\gamma)\beta^{n+1} + (\beta-\gamma)\alpha^{n+1}}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\right)\big(h(t-2) - g(t-2)\big)\|$$
$$\le C_1 + C_2. \quad (3.11)$$

Dividing both sides by $|\beta|^n$ and by letting $n \to \infty$, we obtain

$$\|-(\alpha-\gamma)\beta^2\big(h(t) - g(t)\big)$$
$$+ \big(-b(\alpha-\gamma)\beta - c(\alpha-\gamma)\big)\big(h(t-1) - g(t-1)\big)$$
$$-c(\alpha-\gamma)\beta\big(h(t-2) - g(t-2)\big) = 0$$

$$\therefore \|\beta^2\big(h(t) - g(t)\big) + (b\beta + c)\big(h(t-1) - g(t-1)\big) + c\beta\big(h(t-2) - g(t-2)\big)\| = 0. \quad (3.12)$$

Analogously, if we divide both sides of by $|\alpha|^n$ and $|\gamma|^n$ and let $n \to -\infty$, then we get

$$\|\alpha^2\big(h(t) - g(t)\big) + (b\alpha + c)\big(h(t-1) - g(t-1)\big) + c\alpha\big(h(t-2) - g(t-2)\big)\| = 0 \quad (3.13)$$

and

$$\|\gamma^2\left(h(t)-g(t)\right)+(b\gamma+c)\left(h(t-1)-g(t-1)\right)$$
$$+c\gamma\left(h(t-2)-g(t-2)\right)\|=0. \quad (3.14)$$

Rewritting equations (3.12), (3.13) and (3.14) in matrix form, we get

$$\begin{bmatrix}\gamma^2 & b\gamma+c & c\gamma\\ \alpha^2 & b\alpha+c & c\alpha\\ \beta^2 & b\beta+c & c\beta\end{bmatrix}\begin{bmatrix}\left(h(t)-g(t)\right)\\ \left(h(t-1)-g(t-1)\right)\\ \left(h(t-2)-g(t-2)\right)\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}. \quad (3.15)$$

Note that since $c\neq 0$ and α,β,γ are distinct roots,

$$\begin{vmatrix}\gamma^2 & b\gamma+c & c\gamma\\ \alpha^2 & b\alpha+c & c\alpha\\ \beta^2 & b\beta+c & c\beta\end{vmatrix}=c^2\begin{vmatrix}\gamma^2 & 1 & \gamma\\ \alpha^2 & 1 & \alpha\\ \beta^2 & 1 & \beta\end{vmatrix}=c^2(\alpha-\gamma)(\beta-\gamma)(\alpha-\beta)\neq 0.$$

Therefore (3.15) has only trivial solution and we have, $h(t)=g(t),\ h(t-1)=g(t-1),\ h(t-2)=g(t-2), \forall t\in[0,1)$.

That is, $h(x)=g(x)$, for all $x\in[-2,1)$. Therefore, we conclude that $F(x)=\hat{F}(x)$ for all $x\in\mathbb{R}$.

We illustrate this result.

Example 8. *Consider the functional equation*

$$f(x)=\frac{23}{4}f(x-1)-\frac{31}{8}f(x-2)+\frac{5}{8}f(x-3)$$

and Tribonacci recurrence relation associated to it.

$$T_{n+2}=\frac{23}{4}T_{n+1}-\frac{31}{8}T_n+\frac{5}{8}T_{n-1},$$
$$T_0=0, T_1=0, T_2=1, \forall n\in Z.$$

Roots of the equation $\lambda^3-\frac{23}{4}\lambda^2+\frac{31}{8}\lambda^1+\frac{5}{8}=0$ *are* $\frac{1}{2},\ \frac{1}{4}$ *and* 5. *Let* $\alpha=\frac{1}{2},\ \beta=5$ *and* $\gamma=\frac{1}{4}$. *Note that roots* α, β, γ *are distinct and* $|\alpha|<1,\ |\gamma|<1$ *and* $|\beta|>1$.

Hence the solution is given by

$$F(x)=\frac{(\alpha-\beta)\gamma^2F_1(x)-(\alpha-\gamma)\beta^2F_3(x)+(\beta-\gamma)\gamma^2F_2(x)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}$$

where

$$F_1(x)=\lim_{n\to\infty}\left(\frac{1}{4}\right)^n[f(x-n)$$
$$-(\alpha+\beta)f(x-n-1)+(\alpha\beta)(x-n-2)]$$

$$F_2(x)=\lim_{n\to\infty}\left(\frac{1}{2}\right)^n[f(x-n)$$
$$-(\beta+\gamma)f(x-n-1)+(\beta\gamma)(x-n-2)$$

$$F_2(x)=\lim_{n\to\infty}(5)^{-n}[f(x+n)$$
$$-(\alpha+\gamma)f(x+n-1)+(\alpha\gamma)(x+n-2).$$

Therefore

$$F(x)=\frac{\left(\frac{-9}{2}\times\frac{1}{16}\right)F_1(x)-\left(\frac{1}{4}\times 25\right)F_3(x)+\left(\frac{19}{4}\times\frac{1}{4}\right)F_2(x)}{\left(\frac{-9}{2}\right)\left(\frac{19}{4}\right)\left(\frac{1}{4}\right)}$$
$$=\frac{9F_1(x)+200F_3(x)-38F_2(x)}{171}$$

and $\|f(x)-F(x)\|\leq\frac{114}{171}\epsilon.$

4. Conclusion

Hyers-Ulam stability of generalized Tribonacci functional equation is discussed after obtaining the solution of generalized Tribonacci functional equation in terms of generalized Tribonacci numbers.

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