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Some Fixed Point Theorems for Multivalued Mappings in Banach Algebras and Application to Integral Inclusions

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Abstract In this paper, we present new multivalued analogues of the krasnoselskii fixed point theorems, for the sum $AB+C$, where the operators *A;B* and *C* are *D*-set Lipcshitzian with respect to a measure of non-compactness which satisfies condition (*m*). Our results generalize, prove and extend well-known results in the literature. An application to solving non linear integral inclusion is given.

Keywords: *measure of noncompatness, Banach algebras, condensing multimap, integral equations*

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1. Introduction

The study of the linear integral equation in Banach algebra was initiated by Dhage via fixed point theorems [14,16,17]. Recently, many authors are interested in the study of non linear integral equations and integral inclusions, in Banach algebras, by using the argument of measure of noncompactness [1,2,5,6,7,13,15,18,19]. Given a Banach algebra *X* and a closed bounded interval *J* = [0, *a*] in *IR*, we consider the following functional integral inclusion

$$x(t) \in f(t,x(t))\left(q(t)+\int_0^t F(s,x(s))ds\right) + \int_0^t G(s,x(s))ds,\ t \in J, \tag{1}$$

where $f: J\times X \to X$, $q: J \to X$, and $F,G: J\times X \to \mathcal{P}(X)$. The functional integral inclusion (1) can be transformed to the following fixed point problem

$$x \in Ax.Bx + Cx,$$

where *A*, *B* and *C* are multivalued operators acting on the Banach algebra *C*(*J*, *X*). Note that a particular case of (1) has been studied by B.C Dhage [11] where *X* is the Banach algebra *IR*.

In order to study problem (1), we look at the nonlinear equation

$$x \in Ax.Bx + Cx,\ x \in S,$$

where *S* is a non empty closed convex subset (not necessarily bounded) of a Banach algebra *X* and *A*, *B*, *C* are upper semi-continuous or closed multivalued mappings acting on *X*.

In [11], by using a propriety of the Hausdorff measure of noncompatness, Dhage has examined the nonlinear equation $x \in Ax.Bx$, where *A* and *B* are multivalued operators acting on a Banach algebra. In [8], authors introduced a new class of multivalued mappings of the form $\left(\frac{I-C}{A}\right)$ where *A* and *C* are multivalued operators acting a on Banach algebra. Using this concept, they presented some fixed point theorems for the operator *AB* + *C* in weak topology setting.

The present paper is organized as follows. In section 2, we give some useful preliminaries and definitions which will be needed in the sequel. In section 3, by using the propriety (*m*) of a measure of non compactness ν on *X*, we present some hybrid fixed point theorems for the operator *AB* + *C*. For this, we use the propriety of *D*-set lipschitzian with respect to ν for the multivalued operators *A*, *B* and *C*. On another direction, in section 4, we use the class of multivalued operator $\left(\frac{I-C}{A}\right)$ to prove some fixed point theorems for the sum *AB* +*C*. This is based on the propriety of μ-condensing of the operato $\left(\frac{I-C}{A}\right)^{-1}$ for any measure of non compactness μ on *X*. Our results generalize, prove and extend well known results, for single valued mappings [13,14,15,16,19] and for multivalued mappings [11].

Section 5 is devoted to establishing an existence theorem for the nonlinear integral inclusions (1), where *X* is a separable Banach algebra, as a consequence of our Theorem 3.2.

2. Preliminaries

Throughout this paper, Let X be a Banach space endowed with the norm $\|.\|$. For convenience let

$$\mathcal{P}(X)=\{D\subset X: D \text{ is non-empty}\},$$
$$\mathcal{P}_{\text{bd}}(X)=\{D\subset X: D \text{ is non-empty and bounded}\},$$
$$\mathcal{P}_{\text{cv}}(X)=\{D\subset X: D \text{ is non-empty and convex}\},$$
$$\mathcal{P}_{\text{cl,bd}}(X)=\{D\subset X: D \text{ is non-empty closed and bounded}\},$$
$$\mathcal{P}_{\text{cp}}(X)=\{D\subset X: D \text{ is compact}\}.$$

We denote by $B(x, r)$ the closed ball centered at x with radius r. For a subset S of X, we write $\overline{S}$, $convS$, and $\overline{convS}$, to denote the closure, the convex hull and the closed convex hull of the subset S, respectively. We write $x_n \to x$ to denote the convergence with respect to the norm of X.

Let X and Y be two Banach spaces. A correspondence $T: X\to \mathcal{P}(Y)$ is called a multivalued operator or a multivalued mapping on X into Y. For any subset A of X, define

$$T^{+}(A)=\{x\in X, T(x)\subset A\},$$
$$T^{-}(A)=\{x\in X, T(x)\cap A\neq \emptyset\}$$
$$\text{and } T(A)=\bigcup_{x\in A} T(x).$$

The range of T is the image $T(X)$ of X. We can identify T with its graph GrT the subset of $X \times Y$ given by

$$GrT=\{(x,y)\in X\times Y, y\in T(x)\}.$$

A point $x\in X$ is called a fixed point of T if $x\in T(x)$.

Definition 2.1 *Let $T: X\to \mathcal{P}(Y)$ be a multivalued mapping.*

- We say that T is upper semi-continuous if $T^{+}(U)$ is an open set in X, for all open subset U of Y.

- We say that T is closed if its graph is closed in $X\times Y$. In other words, if $\{(x_n, y_n)\}$ be a sequence in GrphT such that $(x_n, y_n)\to (x,y)$, then we have that $(x,y)\in GrphT$

- We say that un element x in X is a fixed point of T if $x\in T(x)$.

- We say that T is compact if its range T(X) is relatively compact in Y.

- We say that T is completely continuous if it is upper semi-continuous and, for all bounded subset Ω of X, $T(\Omega)$ is relatively compact.

In the following proposition we recall some essential properties of upper semi-continuous and closed multivalued mappings in Banach spaces.

Proposition 2.1 [26]. *Let X and Y be Banach spaces, and let $T: X\to 2^X$ be a multivalued mapping.*

1) If T is upper semi-continuous and compact-valued, then for each compact subset K of X, T(K) is compact.

2) If T is upper semi-continuous and compact-valued, then T is closed.

3) If Y is compact, then T is closed if and only if it is upper semi-continuous and closed-valued.

4) If T is compact-valued, then T is upper semi-continuous if and only if for all sequence $\{(x_n, y_n)\}$ in GrphT such that $x_n\to x$, the sequence $\{y_n\}$ has a limit point in T(x).

We recall the KaKutani-Fan fixed point principle for upper semi-continuous multivalued mappings.

Theorem 2.1 [20]. *Let K be a compact convex subset of a Banach space E and let $T: K\to \mathcal{P}_{cp,cv}(K)$ be an upper semi-continuous multivalued mapping. Then T has a fixed point.*

We cite the following generalizations of the KaKutani-Fan fixed point principle.

Theorem 2.2 *(Bohmenblust-Karlin, see [21])) Let S be a closed, bounded and convex subset of a Banach space E and let $T: K\to \mathcal{P}_{cp,cv}(K)$ be an upper semi-continuous multivalued mapping with compact range. Then T has a fixed point.*

Theorem 2.3 *(O'Regan [24]). Let S be a closed, bounded and convex subset of a Banach space E and let $T: K\to \mathcal{P}_{cp,cv}(K)$ be closed compact multivalued mapping. Then T has a fixed point.*

The compactness of the operator T is weakened to condensing operator by using the notion of measure of noncompactness in Banach spaces.

Definition 2.2 *A function $\mu: \mathcal{P}_{bd}(K)\to IR_{+}$ is said to be a measure of noncompactness on X (MNC, for short), if it satisfies the following properties*:

1. $\mu\left(\overline{conv}(\Omega)\right)=\mu(\Omega)$, *for all bounded subsets* $\Omega\subseteq X$,

2. *Monotonicity: For any bounded subsets* Ω_1, Ω_2 *of X we have*

$$\Omega_1\subseteq\Omega_2 \Rightarrow \mu(\Omega_1)\le \mu(\Omega_2).$$

3. *Nonsingularity:* $\mu(\Omega\cup\{a\})=\mu(\Omega)$ *for all* $a\in X, \Omega$ *bounded set of* X.

4. $\mu(\Omega)=0$ *if and only if* Ω *is relatively compact in X.*

5. *If* Ω_n *is a decreasing sequence of sets in* $\mathcal{P}_{bd}(X)$ *satisfying* $\lim_{n\to\infty}\mu(\Omega_n)=0$, *then the limiting set* $\Omega_\infty=\lim_{n\to\infty}\overline{\Omega_n}$ *is non empty.*

The MNC μ is said to be positive homogeneous provided

$$\mu(\lambda\Omega)=\lambda\mu(\Omega), \text{ for all } \lambda>0 \text{ and } \Omega\in\mathcal{P}_{bd}(X).$$

The MNC μ is said to be subadditive if

$$\mu(\Omega_1+\Omega_2)\le \mu(\Omega_1)+\mu(\Omega_2),$$
$$\text{for all } \Omega_1,\Omega_2\in \mathcal{P}_{bd}(X).$$

The above notion is a generalization of the Hausdorff measure of noncompactness χ defined on each bounded set Ω of X by

$$\chi(\Omega)=\inf\left\{r>o:\Omega=\bigcup_{i=1}^{n}B(x_i,r),\ x_i\in\Omega\right\}.$$

It is well known that χ is homogenous, subadditive and satisfies the set additivity property. The details of measure of noncompactness appear in Deimling [10] and Zeidler [25].

Definition 2.3 *Let μ be a MNC on X, we say that the multivalued mapping $T:S\to\mathcal{P}(X)$ is μ-condensing, if for any $\Omega\in\mathcal{P}_{\text{bd}}(S)$, $T(\Omega)$ is bounded and $\mu(T(\Omega))<\mu(\Omega)$ for $\mu(\Omega)\neq 0$.*

We say that the multivalued mapping $T:S\to\mathcal{P}(X)$ is a $\mathcal{D}$-set-Lipschitzian (with respect to μ) if there exists a continuous nondecreasing function $\phi:IR_+\to IR_+$ with $\phi(0)=0$ such that

$$\mu(T(\Omega))\le\phi(\mu(D)),$$

for all bounded subset $\Omega\subset X$ with $T(\Omega)\in\mathcal{P}_{\text{bd}}(X)$. If $\phi(r)=kr$, we say that T is a k-set-Lipschitzian and if k < 1, then T is called a k-set-contraction. If $\phi(r)<r$, for r > 0, then T is called a nonlinear $\mathcal{D}$-set-contraction.

For any $\Omega\in\mathcal{P}_{\text{bd}}(X)$, let

$$\|\Omega\|=\sup\{\|a\|,\ a\in\Omega\}.$$

For $\Omega_1,\Omega_2\in\mathcal{P}_{\text{cl,bd}}(X)$ and $a\in\Omega_1$, let

$$D(a,\Omega_2)=\inf\{\|a-b\|,\ b\in\Omega_2\}$$
$$\text{and } \rho(\Omega_1,\Omega_2)=\sup\{D(a,\Omega_2),\ a\in\Omega_1\}.$$

The function $d_H:\mathcal{P}_{\text{cl,bd}}(X)\times\mathcal{P}_{\text{cl,bd}}(X)\to IR_+$ defined by

$$d_H(\Omega_1,\Omega_2)=\max(\rho(\Omega_1,\Omega_2),\rho(\Omega_2,\Omega_1))$$

is a metric on $\mathcal{P}_{\text{cl,bd}}(X)$ and is called the Hausdorff metric on X (see [26]). It is clear that $H(0,\Omega)=\|\Omega\|$, for any $\Omega\in\mathcal{P}_{\text{cl,bd}}(X)$.

Proposition 2.2 [26] *If $\Omega_1,\Omega_2\in\mathcal{P}_{\text{cl,bd}}(X)$, then $d_H(\Omega_1,\Omega_2)\le r$ is equivalent to the following assertion: $\Omega_1\subset\Omega_2+B_r(0)$ and $\Omega_2\subset\Omega_1+B_r(0)$.*

Definition 2.4 *A multivalued mapping $T:S\to\mathcal{P}_{\text{cl,bd}}(X)$ is called $\mathcal{D}$-Lipschitzian if there exists a continuous nondecreasing function $\phi:IR_+\to IR_+$ with $\phi(0)=0$ such that*

$$d_H(T(x),T(y))\le\phi\|x-y\|,$$

for all $x,y\in S$. The function ϕ is called a $\mathcal{D}$-function of T on X. If $\phi(r)=Kr$ for r > 0, then T is called a Lipschitzian multivalued mapping. In particular, if K < 1, then T is called a multivalued contraction. If ϕ satisfies $\phi(r)<r$, then T is called a nonlinear contraction multivalued mapping with contraction function ϕ.

3. Fixed Point Theory in Banach Algebras

Definition 3.1 *An algebra X is a vector space endowed with an inner operation noted by (.) which is associative and bilinear.*

A normed algebra is an algebra endowed with a norm such that

$$\forall x,y\in X,\ \|x.y\|\le\|x\|\|y\|.$$

A complete normed algebra is called a Banach algebra.

Definition 3.2 [5]. *We state that a measure of noncompactness μ on a Banach algebra X satisfies the condition (m) if for arbitrary bounded sets Ω_1,Ω_2 of X, the following inequality is satisfied*

$$\mu(\Omega_1\Omega_2)\le\|\Omega_1\|\mu(\Omega_2)+\|\Omega_2\|\mu(\Omega_1).$$

Lemma 3.1 [3]. *For any bounded subsets Ω_1 and Ω_2 of X, we have*

$$\chi(\Omega_1\Omega_2)\le\|\Omega_1\|\chi(\Omega_2)+\|\Omega_2\|\chi(\Omega_1).$$

According to Lemma 3.1, the Hausdorff measure of noncompactness ϕ satisfies condition (*m*).

We note that condition (*m*) was used for the first time in [3] for measures of noncompactness defined on the Banach algebra *C*[*a*, *b*].

Example 3.1 *Let X = $BC(IR_+)$ the Banach space of continuous and bounded functions on IR_+ equipped with the standard norm $\|x\|=\sup\{|x(t)|,\ t\in IR_+\}$. Obviously $BC(IR_+)$ has also the structure of Banach algebra with the standard multiplication of functions. For all $\Omega\in\mathcal{P}_{\text{bd}}(X)$, $\epsilon>0$, L > 0 and $x\in X$, we pose*

$$\omega^L(x,\epsilon)=\sup\{|x(t)-x(s)|,t,s\in[0,L],|t-s|\le\epsilon\},$$
$$\omega^L(\Omega,\epsilon)=\sup\{\omega(x,\epsilon),\ x\in\Omega\},$$
$$\omega_0^L(\Omega)=\lim_{\epsilon\to 0}\omega^L(\Omega,\epsilon),$$
$$\omega_0^\infty(\Omega)=\lim_{L\to\infty}\omega_0^L(\Omega),$$
$$\mu_c(\Omega)=\omega_0^\infty(\Omega)+\limsup_{t\to\infty} diam(\Omega(t)).$$

According to [4], μ_c is a measure of noncompactness on $BC(IR_+)$which satisfies condition (m) on the family of nonnegative functions in $BC(IR_+)$ (see [1]).

We begin this section by proving the following fixed point theorem which extends Theorem 2.2 in [12].

Theorem 3.1 *Let X be a Banach space, μ a MNC on X and S a closed convex subset of X. Let $T:S\to\mathcal{P}_{\text{cl,cv}}(S)$ be a closed an μ-condensing mapping such that T(S) is bounded. Then T has a fixed point.*

Proof. Let x_0 be an arbitrary element in *S*. We pose

$$\mathcal{A}=\left\{A\subseteq S,\ \overline{conv}(A)=A,\ x_0\in A \text{ and } T(A)\subseteq A\right\}.$$

The set $\mathcal{A} \neq \emptyset$ since $S \in \mathcal{A}$. Let $L = \bigcap_{A \in \mathcal{A}} A$. We Show that $L = \overline{conv}\left(T(L) \cup \{x_0\}\right)$. Clearly L is a closed convex subset of S and $T(L) \subseteq L$. Thus, $L \in \mathcal{A}$. This implies $\overline{conv}\left(T(L) \cup \{x_0\}\right) \subseteq L$. Hence,

$$T\left(\overline{conv}\left(T(L) \cup \{x_0\}\right)\right) \subseteq T(L) \subseteq \overline{conv}\left(T(L) \cup \{x_0\}\right).$$

Consequently, $\overline{conv}\left(T(L) \cup \{x_0\}\right) \in \mathcal{A}$. Hence, $L \subseteq \overline{conv}\left(T(L) \cup \{x_0\}\right)$. As a result

$$L = \overline{conv}\left(T(L) \cup \{x_0\}\right).$$

Suppose that $\mu(L) > 0$, we have

$$\mu(L) = \mu\left(\overline{conv}\left(T(L) \cup \{x_0\}\right)\right) = \mu(T(L)) < \mu(L).$$

Then $\mu(L) = 0$ and consequently L is compact and convex. By Theorem 2.3, the multivalued mapping $T : L \to \mathcal{P}_{\text{cl,cv}}(L)$ has a fixed point.

Theorem 3.2 *Let S be a closed convex subset of a Banach algebra X and μ a subadditive MNC on X satisfying condition (m). Let $A, B : S \to \mathcal{P}_{\text{cp}}(X)$ and $C : S \to \mathcal{P}_{\text{cl}}(X)$ be multivalued mappings such that:*

1. A, B are upper semi-continuous and C is closed,

2. A, B and C are D-set Lipschitzian (with respect to μ) with D-function ϕ_A, ϕ_B and ϕ_C, respectively,

3. for all $x \in S, A(x).B(x)+C(x)$ is a convex subset of S,

4. A(S), B(S) and C(S) are bounded.

Then the equation $x \in A(x).B(x)+C(x)$ has at least one solution provided

$$\|A(S)\|\phi_B(r) + \|B(S)\|\phi_A(r) + \phi_C(r) < r, \text{ for all } r > 0.$$

Proof. Let

$$T : S \to \mathcal{P}_{\text{cl,cv}}(S),\ x \mapsto (x).B(x)+C(x).$$

Since A, B have compact values and C has closed values, assumption 3) guarantees that T is well defined. We show that T has closed graph. Let $\{x_n\} \subset S$, $x_n \to x \in S$ and

$$y_n \in A(x_n)B(x_n) + C(x_n) \to y \in S.$$

Let $y_n = a_n b_n + c_n$ such that $a_n \in A(x_n)$, $b_n \in B(x_n)$, and $c_n \in C(x_n)$. Since A and B are upper semicontinous, by Proposition 2.1 we can suppose that $a_n \to a \in A(x)$, $b_n \to b \in B(x)$. It yields that $c_n = y_n - a_n b_n \to y - ab$. Since C has closed graph, we get $y - ab \in C(x)$, then $y \in A(x)B(x)+C(x)$. Hence $AB + C$ has closed graph. Let Ω be a bounded subset of S such that $\mu(\Omega) \neq 0$. It is clear that $A(\Omega)B(\Omega)+C(\Omega)$ is bounded and we have

$$\begin{aligned}
&\mu\left(A(\Omega).B(\Omega)+C(\Omega)\right) \\
&\leq \|A(\Omega)\|\mu(B(\Omega)) + \mu(A(\Omega))\|B(\Omega)\| + \mu(C(\Omega)) \\
&\leq \|A(\Omega)\|\phi_B(\mu(\Omega)) + \|B(\Omega)\|\phi_A(\mu(\Omega)) + \phi_C(\mu(\Omega)) \\
&< \mu(\Omega).
\end{aligned}$$

Thus, the multivalued mapping $AB + C$ is μ-condensing. From Theorem 3.1, $AB + C$ has a fixed point.

If we suppose that the multivalued mapping B is completely continuous, we obtain the following result.

Theorem 3.3 *Let S be a closed convex subset of a Banach algebra X and μ a subadditive MNC on X satisfying condition (m). Let $A, B : S \to \mathcal{P}_{\text{cp}}(X)$ and $C : S \to \mathcal{P}_{\text{cl}}(X)$ be multivalued mappings such that:*

1. A is upper semi-continuous and C is closed,

2. A and C are D-set Lipschitzian (with respect to μ) with D-functions ϕ_A and ϕ_C, respectively,

3. B is completely continuous,

4. for all $x \in S, A(x).B(x)+C(x)$ is a convex subset of S,

5. A(S), B(S) and C(S) are bounded.

Then the equation $x \in A(x).B(x)+C(x)$ has at least one solution provided

$$\|B(S)\|\phi_A(r) + \phi_C(r) < r, \text{ for all } r > 0.$$

Proof. As in the proof of Theorem 3.2, the operator

$$T : S \to \mathcal{P}_{\text{cl,cv}}(S),\ x \mapsto A(x).B(x)+C(x).$$

is well defined and has a closed graph. Let Ω be a bounded subset of S such that $\mu(\Omega) \neq 0$. Since B is completely continuous $\mu(B(\Omega)) = 0$ and we have

$$\begin{aligned}
&\mu\left(A(\Omega).B(\Omega)+C(\Omega)\right) \\
&\leq \|A(\Omega)\|\mu(B(\Omega)) + \mu(A(\Omega))\|B(\Omega)\| + \mu(C(\Omega)) \\
&\leq \|B(\Omega)\|\phi_A(\mu(\Omega)) + \phi_C(\mu(\Omega)) \\
&< \mu(\Omega).
\end{aligned}$$

Then T is μ-condensing and the proof is concluded by Theorem 3.1.

In the following result, we interested in the case where A, B and C are D-Lipschitzian. We need the following lemmas which are essential for the proof.

Lemma 3.2 *Let X be a Banach space, S be a non-empty subset of X and $T : S \to \mathcal{P}_{\text{cl,bd}}(S)$ be a $\mathcal{D}$-Lipschitzian multivalued mapping with $\mathcal{D}$-function ϕ_T. Then for any bounded subset Ω of S, $T(\Omega)$ is bounded.*

Proof. Let Ω be a bounded subset of S. Then there is constant $r > 0$ such that $\|x\| \leq r$ for all $x \in \Omega$. Fix $y \in \Omega$. Since T is $\mathcal{D}$-Lipschitzian, for all $x \in \Omega$ we have

$$\begin{aligned}
\|T(x)\| &\leq d_H(T(x), 0) \\
&\leq d_H(T(x), T(y)) + d_H(T(y), 0) \\
&\leq \phi_T(\|x - y\|) + \|T(y)\| \\
&\leq \phi_T(2\|\Omega\|) + \|T(y)\|.
\end{aligned}$$

Hence $T(\Omega)$ is bounded.

Lemma 3.3 *Let X be a Banach space and let $T: X \to \mathcal{P}_{cp}(X)$ be a D-Lipschitzian multivalued mapping with D-function ϕ. Then T is upper semi-continuous.*

Proof. Let $\{x_n\}$ be a sequence in X converging to a point $x \in X$ and let $\{y_n\}$ such that $y_n \in T(x_n)$, for all integer n. We have

$$D(y_n, T(x)) \le H(T(x_n), T(x)) \le \phi(\|x_n - x\|).$$

Then $\lim_{n\to\infty} D(y_n, T(x)) = 0$. Let $\epsilon > 0$, there exists an integer N such that, for all $n \ge N$, $D(y_n, T(x)) \le \epsilon$ This implies that

$$\{y_n\} \subset \{y_n, n \le N\} \cup (T(x) + \epsilon),$$

and consequently $\{y_n\}$ is relatively compact. Then there exists a subsequence $\{y_{n_k}\}$ which converges to $y \in T(x)$. According to Proposition 2.1, T is upper semi-continuous.

Lemma 3.4 *Let X be a Banach space and $T: X \to \mathcal{P}_{cp}(X)$ be a $\mathcal{D}$-Lipschitzian multivalued mapping with a $\mathcal{D}$-function ϕ. Then T is $\mathcal{D}$-set-Lipschitzian with respect to the Hausdorff measure of noncompactness.*

Proof. Let Ω be a bounded subset of X. From Lemma 3.2, $T(\Omega)$ is bounded. Let $r > 0$ such that $\chi(\Omega) < r$, there exits a finite subset $\{x_1,\ldots,x_n\}$ of Ω such that $\Omega \subset \bigcup_{i=1}^{n} B(x_i, r)$. Let $x \in \Omega$, there exists x_i such that $\|x - x_i\| \le r$. Since T is D-Lipschitzian, we have

$$H(T(x), T(x_i)) \le \phi\|x - x_i\| \le \phi(r).$$

According to Proposition 2.2, $T(x) \subset T(x_i) + \phi(r)$. Consequently

$$T(\Omega) \subset \bigcup_{i=1}^{n} T(x_i) + \phi(r) \subset T(\{x_1,\ldots,x_n\}) + \phi(r).$$

On the other hand, from Lemma 3.3 T is upper semi-continous and, by Proposition 2.1, $T(\{x_1,\ldots,x_n\})$ is compact. For each $\epsilon > 0$, there exists $\{y_1,\ldots,y_p\}$ in $T(\{x_1,\ldots,x_n\})$ such that

$$T(\{x_1,\ldots,x_n\}) \subset \bigcup_{i=1}^{p} B(y_i, \epsilon).$$

Consequently,

$$T(\Omega) \subset \bigcup_{i=1}^{p} B(y_i, \epsilon) + \phi(r).$$

That is $\chi(T(\Omega)) \le \phi(r) + \epsilon$, for each $\epsilon > 0$. Letting $r \to \chi(\Omega)$ and by the continuity of ϕ, we deduce that $\chi(\Omega) \le \phi(\chi(\Omega))$. Hence T is D-set Lipchitzian with D-function ϕ.

Now we are ready to prove the following result.

Theorem 3.4 *Let S be a closed convex subset of a Banach algebra X. Let $A, B, C: S \to \mathcal{P}_{cp}(X)$ be multivalued mappings such that:*

1. A, B and C are D-Lipschitzian with D-functions ϕ_A, ϕ_B and ϕ_C, respectively,

2. for all $x \in S$, $A(x).B(x) + C(x)$ is a convex subset of S,

3. A(S), B(S) and C(S) are bounded.

Then the equation $x \in A(x).B(x) + C(x)$ has at least one solution provided

$$\|A(S)\|\phi_B(r) + \|B(S)\|\phi_A(r) + \phi_C(r) < r, \text{ for all } r > 0.$$

Proof. From Lemma 3.3, the mappings A, B and C are upper semi-continuous, in particular, by Proposition 2.1, the operator C is closed. Further, by Lemma 3.4, the mappings A, B and C are D-set Lipschitzian with respect to χ. Since the measure χ satisfies condition (m), all assumptions of Theorem 3.2 are satisfies and the proof is concluded.

The following result is a direct consequence of Theorem 3.3 and Lemma 3.4.

Theorem 3.5 *Let S be a closed convex subset of a Banach algebra X. Let $A, C: S \to \mathcal{P}_{cp}(X)$ and $B: S \to \mathcal{P}_{cl}(X)$ three multivalued mappings such that:*

1. A and C are D-Lipschitzian with D-functions ϕ_A and ϕ_B, respectively,

2. B is completely continuous,

3. for all $x \in S$, $A(x).B(x) + C(x)$ is a convex subset of S,

4. A(S), B(S) and C(S) are bounded.

Then the equation $x \in A(x).B(x) + C(x)$ has at least one solution provided

$$\|B(S)\|\phi_A(r) + \phi_C(r) < r, \text{ for all } r > 0.$$

In the particular case where A and C are Lipschitzian, we obtain the following corollary which extends Theorem 3.5 in [11].

Corollary 3.1 *Let S be a closed convex subset of a Banach algebra X. Let $A, C: S \to \mathcal{P}_{cp}(X)$ be multivalued mappings and let $B: S \to \mathcal{P}_{cl}(X)$ such that:*

1. A and C are Lipschitzian with Lipschitz constant k_1 and k_2, respectively,

2. B is completely continuous,

3. for all $x \in S$, $A(x).B(x) + C(x)$ is a convex subset of S,

4. A(S), B(S) and C(S) are bounded.

Then the equation $x \in A(x).B(x) + C(x)$ has at least one solution provided

$$k_1\|B\|+k_2<1.$$

In the particular case where A, B and C are single valued mappings, we obtain the following result which generalizes Theorem 1.4 in [19].

Corollary 3.2 *Let S be a closed convex subset of a Banach algebra X and let* $A,B,C:S\to X$ *such that:*

1. A and C are D-Lipschitzian with D-functions ϕ_A *and* ϕ_C *respectively,*

2. B is completely continuous,

3. for all $x\in S$, $A(x).B(x)+C(x)$ *is a an element of S,*

4. A(S), B(S) and C(S) are bounded.

Then the equation $x\in A(x).B(x)+C(x)$ *has at least one solution provided*

$$\|A\|\phi_B(r)+\|B\|\phi_A(r)+\phi_C(r)<r,\ \textit{for all } r>0.$$

4. Another Direction

In the following, we introduce the operator $\frac{I-C}{A}$ for multivalued mappings and we well use it to prove existence theorems of the equation $x\in A(x)B(x)+C(x)$.

Definition 4.1 [8] *Let X be a Banach algebra and* $A,C:X\to\mathcal{P}(X)$ *be multivalued mappings. We say that the mapping* $\frac{I-C}{A}$ *is well defined on* $x\in X$ *and we write*

$$y\in\left(\frac{I-C}{A}\right)(x)\Leftrightarrow x\in A(x)y+C(x).$$

Theorem 4.1 *Let X be a Banach algebra,* μ *a MNC on X and* ν *a subadditive MNC on X which satisfies condition (m). Let S be a non-empty closed convex subset of X and let* $A:X\to\mathcal{P}_{\mathrm{cp,cv}}(X)$, $B:S\to\mathcal{P}_{\mathrm{cp}}(X)$, $C:X\to\mathcal{P}_{\mathrm{cl,cv}}(X)$ *be three multivalued mappings satisfying the following properties*:

1. A, B are upper semi-continuous and C is closed,

2. A and C are $\mathcal{D}$*-set-Lipschitzian (with respect to* ν*) with* $\mathcal{D}$*-functions* ϕ_A *and* ϕ_C*, respectively,*

3. A(X), C(X) and B(S) are bounded,

4. for all y $y\in S$, $\left(\frac{I-C}{A}\right)^{-1}B(y)$ *is convex and* $\left(x\in A(x)B(y)+C(x),\ y\in S\right)\Rightarrow x\in S$,

5. $\left(\frac{I-C}{A}\right)^{-1}B$ *is* μ*-condensing.*

Then the equation $x\in A(x)B(x)+C(x)$ *has at least one solution in S if*

$$\|B(S)\|\phi_A(r)+\phi_C(r)<r,\ \textit{for all } r>0.$$

Proof. Fix $y\in S$ and let $z\in B(y)$. Consider

$$\varphi_z:X\to\mathcal{P}_{\mathrm{cv,cl}}(X),\ x\mapsto Ax.z+Cx.$$

Since $A(x)$ is compact and $C(x)$ is closed, it is clear that φ_z is well defined. We claim that φ_z has closed graph. Let $\{x_n\}$ be a sequence converging to $x\in X$ and $\{y_n\}$ such that $x_n\in\varphi_z(x_n)$ with $y_n\to y$. There exist $\alpha_n\in A(x_n)$ and $\beta_n\in B(x_n)$ such that $y_n=\alpha_n+\beta_n$. Since A is upper semi-continuous and has compact values, according to Proposition 2.1, there exists a subsequence, we note also $\{\alpha_n\}$, such that $\alpha_n\to\alpha\in A(x_n)$. Consequently $\beta_n=y_n-\alpha_n z\to y-\alpha z$. Since C has closed graph, we deduce that $y-\alpha z\in C(x)$ which implies that $y\in\varphi_z(x)$. Hence φ_z has closed graph.

We show that φ_z is ν-condensing. Fixing a bounded subset Ω of X such that $\nu(\Omega)>0$. It is clear that $\varphi_z(\Omega)$ is bounded. Since ν is subadditive and satisfies condition (m), we have

$$\begin{aligned}\nu(\varphi_z(\Omega))&\le\nu(A(\Omega),z)+\nu(C(\Omega))\\&\le\|z\|\nu(A(\Omega))+\nu(C(\Omega))\\&\le\|z\|\phi_A(\nu(\Omega))+\phi_C(\nu(\Omega))\\&\le M\phi_A(\nu(\Omega))+\phi_C(\nu(\Omega))\\&<\nu(\Omega).\end{aligned}$$

So, φ_z is ν-condensing. Now all assumptions of Theorem 3.1 are satisfied for the operator φ_z, then there exists $x\in X$ such that $x\in A(x)z+C(x)$. Thus $z\in\left(\frac{I-C}{A}\right)(x)$ and, so, $\left(\frac{I-C}{A}\right)(x)\cap(By)\neq\emptyset$. Consequently the multivalued mapping $\left(\frac{I-C}{A}\right)^{-1}$ is well defined on $B(S)$. Note that, for all $x\in\left(\frac{I-C}{A}\right)^{-1}(y)$ is equivalent to $x\in A(x)B(y)+C(x)$. By assumption 4), we deduce that

$$T=\left(\frac{I-C}{A}\right)^{-1}B:S\to\mathcal{P}_{cv}(S)$$

is well defined. We show that T has closed graph. Let $\{x_n\}$ be a sequence converging to $x\in X$ and $\{y_n\}$ such that $x_n\in T(x_n)$ with $y_n\to y$. We have $y_n\in A(x_n)B(y_n)+C(x_n)$, then $y_n=\alpha_n\beta_n+\gamma_n$, where $\alpha_n\in A(x_n)$, $\beta_n\in B(y_n)$ and $\gamma_n\in C(x_n)$. Since A and B are upper semi-continuous with compact values, by Proposition 2.1, we can suppose that $\alpha_n\to\alpha\in A(x)$ and $\beta_n\to\beta\in B(y)$. Then

$$\gamma_n=y_n-\alpha_n\beta_n\to y-\alpha\beta\in C(x)$$

and so

$$y\in A(x)B(y)+C(x)\Rightarrow y\in\left(\frac{I-C}{A}\right)^{-1}B(x)=T(x).$$

According to Theorem 3.1, it suffices to verify that T is bounded. In fact

$$T(S) \subset A(T(S))B(S) + C(T(S)).$$

then assumption 3) guarantees that $T(S)$ is bounded and the proof is concluded.

As consequence of Theorem 4.1, we derive the following corollary.

Corollary 4.1 *Let X be a Banach algebra, S be a nonempty closed convex subset of X and μ a MNC on X. Let $A, C : X \to \mathcal{P}_{cp,cv}(X)$ and $B : S \to \mathcal{P}_{cp}(X)$ be three multivalued mappings satisfying the following properties*:

1. B is upper semi-continuous,

2. A and C are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions ϕ_A and ϕ_C, respectively,

3. $A(X)$, $C(X)$ and $B(S)$ are bounded,

4. for all $y \in S$, $\left(\frac{I-C}{A}\right)^{-1} B(y)$ is convex and $(x \in A(x)B(y) + C(x),\ y \in S) \Rightarrow x \in S$,

5. $\left(\frac{I-C}{A}\right)^{-1} B$ is μ-condensing.

Then the equation $x \in A(x)B(y) + C(x)$ has at least one solution in S if

$$\|B(S)\|\phi_A(r) + \phi_C(r) < r, \text{ for all } r > 0.$$

Proof. By Lemma 3.3 and Lemma 3.4, the multivalued mappings A and C are upper semi-continuous and D-set Lipschitzian with respect to χ. Further, by Proposition 2.1, C has closed graph. All assumption of Theorem 4.1 are satisfied and the proof is concluded.

In the case where B is completely continuous, we can omit assumption 5) in Theorem 4.1 and we get the following result.

Theorem 4.2 *Let X be a Banach algebra, S be a non-empty closed convex subset of X and ν a subadditive MNC on X which satisfies condition (m). Let $A : X \to \mathcal{P}_{cp,cv}(X)$, $B : S \to \mathcal{P}(X)$ and $C : X \to \mathcal{P}_{cl,cv}(X)$ be three multivalued mappings satisfying the following properties*:

1. A is upper semi continuous, B is completely continuous and C have closed graph,

2. A and C are $\mathcal{D}$-set-Lipschitzian (with respect to ν) with D-functions ϕ_A and ϕ_C, respectively,

3. $A(X)$, $C(X)$ and $B(S)$ are bounded,

4. for all $y \in S$, $\left(\frac{I-C}{A}\right)^{-1} B(y)$ is convex and $(x \in A(x)B(y) + C(x),\ y \in S) \Rightarrow x \in S$. Then the equation $x \in A(x)B(y) + C(x)$ has at least one solution in S if

$$\|B(S)\|\phi_A(r) + \phi_C(r) < r, \text{ for all } r > 0.$$

Proof. As in the proof of Theorem 4.1, the operator

$$T = \left(\frac{I-C}{A}\right)^{-1} B : S \to \mathcal{P}_{cv}(S)$$

is well defined and has closed graph. We show that T is ν-condensing. Let N be a bounded subset of S, it is clear that $T(N)$ is bounded. Further, we have

$$T(N) \subset A(T(N)).B(N) + C(T(N)).$$

Then

$$\begin{aligned}\nu(T(N)) &\le \|B(S)\|\nu(A(T(N))) + \nu(C(T(N))) \\ &\le \|B(S)\|\phi_A(\nu(T(N))) + \phi_C(\nu(T(N))). \\ &< \nu(T(N)).\end{aligned}$$

Hence T is ν-condensing.

The following result is a direct consequence of Theorem 4.2, Lemma 3.3 and Lemma 3.4.

Corollary 4.2 *Let X be a Banach algebra and S be a non-empty closed convex subset of X. Let $A, C : X \to \mathcal{P}_{cp,cv}(X)$ and $B : S \to \mathcal{P}_{cp}(X)$ be three multivalued mappings satisfying the following properties*:

1. B is completely continuous,

2. A and C are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions ϕ_A and ϕ_C, respectively,

3. $A(X)$, $C(X)$ and $B(S)$ are bounded,

4. for all $y \in S$, $\left(\frac{I-C}{A}\right)^{-1} B(y)$ is convex and $(x \in A(x)B(y) + C(x),\ y \in S) \Rightarrow x \in S$. Then the equation $x \in A(x)B(y) + C(x)$ has at least one solution in S if

$$\|B(S)\|\phi_A(r) + \phi_C(r) < r, \text{ for all } r > 0.$$

In the particular case where A, B and C are single valued mappings, we obtain the following result which extends Theorem 1.5 in [19], Theorem 2.1 in [15] and Theorem 2.3 in [16].

Theorem 4.3 *Let X be a Banach algebra and S be a non-empty closed convex subset of X. Let $A, C : X \to X$ and $B : S \to X$ be three mappings satisfying the following properties*:

1. B is completely continuous,

2. A and C are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions ϕ_A and ϕ_C, respectively,

3. $A(X)$, $C(X)$ and $B(S)$ are bounded with $\|B(S)\| = M$,

4. $(x = A(x)B(y) + C(x), y \in S) \Rightarrow x \in S$.

Then the equation $x = A(x)B(y) + C(x)$ has at least one solution in S if

Proof. From Theorem 4.2 it suffices to verify that $\left(\frac{I-C}{A}\right)^{-1} B$ is a single-valued mapping from S into itself.

Let $y \in S$ be fixed and consider

$$\varphi_y : X \to X, x \mapsto Ax.y + Cx.$$

Let $x_1, x_2 \in X$, we have

$$\begin{aligned}&\left\|\varphi_y(x_1)-\varphi_y(x_2)\right\|\\ &\le \|Ax_1By-Ax_2By\|+\|Cx_1Cx_2\|\\ &\le \|Ax_1-Ax_2\|\|By\|+\|Cx_1-Cx_2\|\\ &\le \|B(S)\|\phi_A(\|x_1-x_2\|)+\phi_C(\|x_1-x_2\|)\\ &< \|x_1-x_2\|.\end{aligned}$$

From a fixed point theorem of Boyd and Wong [9], there is a unique element $x \in X$ such that $x = AxBy+Cx$ which is equivalent to $x \in \left(\frac{I-C}{A}\right)^{-1} B(y)$ (here $\left(\frac{I-C}{A}\right)^{-1} B$ is seen as a multivalued mapping). Moreover, bearing in mind 4) we have that there exists a unique $x \in S$ such that $x \in \left(\frac{I-C}{A}\right)^{-1} B(y)$. Hence $\left(\frac{I-C}{A}\right)^{-1} B : S \to S$ is well defined as a single-valued mapping.

Remark 1 *Assumption 4) in Theorem 4.3 is satisfied if we suppose that $A(x)B(y)+C(x) \in S$ for all $x, y \in S$. Then, we obtain the following corollary which extends and proves a result due to Dhage cited in [13] (Theorem 2.3) and proved in* [14].

Corollary 4.3 *Let S be a non-empty closed convex subset of a Banach algebra X and let $A, C : X \to X$ and $B : S \to X$ be three mappings satisfying: the following properties*:

1. *B is completely continuous,*
2. *A and C are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions ϕ_A and ϕ_C, respectively,*
3. *A(X), C(X) and B(S) are bounded,*
4. *$A(x)B(y)+C(x) \in S$, for all $x, y \in S$.*

Then the equation x = A(x)B(x) + C(x) has at least one solution in S if

$$\|B(S)\|\phi_A(r)+\phi_C(r) < r \text{ for all } r > 0.$$

5. Functional Integral Inclusion

In the following, we suppose that $(X, |.|)$ is a separable Banach algebra and $F, G : J \times X \to \mathcal{P}_{cp,cv}(X)$. By a solution of (1) we mean a function $x \in \mathcal{C}(J, X)$ that satisfies

$$x(t) = f(t, x(t))\left(q(t)+\int_0^t v(s)ds\right)+\int_0^t w(s)ds, t \in J,$$

for some $v, w \in L^1(J, X)$ satisfying $v(s) \in F(s, x(s))$ and $w(s) \in G(s, x(s))$ a.e. for $s \in J$.

A multivalued mapping $T : J \to P(X)$ is said to be measurable if for any $y \in X$, the function $t \to d(y, T(t))$ is measurable. Further T is said to be integrably bounded if there exists $h \in L^1(J, X)$ such that $\|v\| \le h(t)$ a.e $t \in J$ for all $v \in F(t)$.

For $T : J \to P(X)$, we pose

$$S_T^1 = \left\{v \in L^1(J, X), v(s) \in T(s)\ a.e.\ s \in J\right\}.$$

It is known that this set is nonempty if and only if $s \to \inf\{\|x\|, x \in T(s)\} \in L^1$ (see [22]). This is the case if T is integrably bounded.

A multifunction $\beta : J \times X \to P_{cl,bd}(X)$ is called Carathéodory if

(i) $t \to \beta(t, x)$ is measurable for each $x \in X$,

(ii) $x \to \beta(t, x)$ is upper semi-continuous a.e. for $t \in J$.

A Carathéodory multivalued mapping $\beta : J \times X \to P_{cl,bd}(X)$ is called L^1-Carathéodory if for every real number $r > 0$ there exists a function $h_r \in L^1(J, IR)$ such that

$$\|\beta(t, x)\| = \sup\{\|u\|, u \in \beta(t, x)\} \le h_r(t), a.e. t \in J$$

for all $x \in X$ with $\|x\| \le r$. Denote

$$S_\beta^1(x) = \left\{v \in L^1(J, X), v(s) \in \beta(s, x(s))\ a.e.\ s \in J\right\}.$$

To discuss equation (1), we list the following hypotheses.

(H_1) The mapping f is bounded and there exists a bounded function $l : J \to IR$ such that

$$\|f(t, x)-f(t, y)\| \le l(t)\|x-y\|, a.e. t \in J.$$

(H_2) The multivalued mapping F is L^1-Carathéodory with growth functions h_r, for all $r > 0$.

(H_3) There exists $\phi \in L^1(J, IR)$ such that, for all bounded $\Omega \subset X$,

$$\chi(F(t, \Omega)) \le \phi(t)\chi(\Omega), a.e. t \in J.$$

(H_4) For all $x \in C(J, X)$, the mapping $t \to G(t, x(t))$ is integrably bounded.

(H_5) There exists $\psi \in L^1(J, IR)$ such that

$$H(G(t, x), G(t, y)) \le \psi(t)\|x-y\|\ a.e.\ t \in J.$$

(H_6) The function $q : J \to X$ is continuous with bounded $\|q\|$.

Theorem 5.1 *Assume that the hypotheses (H_1)-(H_6) hold. Suppose that there exists r > 0 such that*

$$\|f\|\left(\|q\|+\|h_r\|_{L^1}\right)+\|\psi\|_{L^1} r + L < r$$

and

$$\|f\|\|\varphi\|+\left(\|q\|+\|h_r\|\right)\|l\|+\|\psi\|_{L^1}<1,$$

where $L=\int_0^t\|G(s,0)\|ds.$ *Then (1) has a solution in* $C(J,X)$.

Proof. *For all* $x\in C(J,X)$, *we pose*

$$Ax(t)=f(t,x(t)),$$

$$Bx(t)=\left\{q(t)+\int_0^t v(s)ds,\ v(s)\in F(s,x(s)),\ a.e\ s\in J\right\},$$

$$Cx(t)=\left\{\int_0^t v(s)ds,\ v(s)\in G(s,x(s)),\ a.e\ s\in J\right\}.$$

We show that operators A, B and C satisfy all assumptions of Theorem 3.2. We pose $E = C(J, X)$ and

$$S=\left\{x\in C(J,X),\ \|x\|\le r\right\}.$$

Step 1. Let $x\in E$ and $t,t'\in J$. We have

$$\|Ax(t)-Ax(t')\|=\|f(t,x(t))-f(t',x(t'))\|$$
$$\le\|l\|\|x(t)-x(t')\|.$$

Since x is continuous, then Ax is also continuous. Hence, the operator $A:S\to E$ is well defined as a single valued mapping. For all $x,y\in E$, we have

$$\|Ax-Ay\|=\sup_{t\in J}\|f(t,x(t))-f(t,y(t))\|\le\|l\|\|x-y\|.$$

Then A is Lipschitzian with constant $\|l\|$. It is clear that $\|A(S)\|\le\|f\|$.

Step 2. We prove that $C:S\to P_{cp}(E)$ is well defined and D-Lipschitzian. From assumption (H_4), $S_G^1(x)$ is non empty and, so, Bx is non empty. Let $x\in E$, $v\in S_G^1(x)$ and $u(t)=\int_0^t v(s)ds$, for all $t\in J$. Since $v\in L^1(J,IR)$, it is clear that $u\in C(J,X)$.

We show that Cx is compact, for all $x\in E$. Let (y_n) be a sequence in Cx such that

$$y_n(t)=\int_0^t v_n(s)ds,\ \textit{for all}\ t\in J.$$

Since $v_n(s)\in G(s,x(s))$ and $G(s,x(s))$ is compact, then for all $s\in J$ the subset $\{v_n(s),n\in IN\}$ is relatively compact in X. The pointwise topology coincides with the product topology on X^J, then

$$\{v_n,n\in IN\}=\prod_{s\in J}\{v_n(s),n\in IN\}$$

is relatively compact in X^J, with respect to the pointwise topology. Hence, there exists a subsequence, for simplicity we note also (v_n), such that $v_n(s)\to v(s)$, for all $s\in J$.

For all n, $\|v_n\|\le\|h\|$ (h the growth function of $G(x)$). By the convergent dominate theorem, we get $v\in L^1(J,X)$ and

$$y_n(t)=\int_0^t v_n(s)ds\to\int_0^t v(s)ds.$$

We deduce that $\{y_n(t),n\in IN\}$ is relatively compact, for all $t\in J$. For all $t,t'\in J$, we have

$$\|y_n(t)-y_n(t')\|\le\int_{t'}^t\|v_n(s)\|ds\le\|h\|_{L^1}|t-t'|.$$

It follows that the family $\{y_n,n\in IN\}$ is equicontinuous. By Ascoli theorem's, we deduce that $\{y_n,n\in IN\}$ is relatively compact in $C(J,X)$. Then, there exists a subsequence (y_{n_k}) which converge uniformly to $y(t)=\int_0^t v(s)ds$. On the other hand, for all $s\in J$, $G(s,x(s))$ is closed, so $v_n(s)\to v(s)\in G(s,x(s))$. Then, $v\in S_G^1(x)$ and $y\in C(x)$. Hence Cx is compact.

We show that C is D-Lipshitzian. Let $x_1,x_2\in E$ and $y_1\in Cx_1$ such that

$$y_1(t)=\int_0^t v(s),\ v(s)\in G(s,x_1(t)),\ a.e\ s\in J.$$

Since

$$H(G(s,x_1(s)),G(s,x_2(s)))\le\psi(s)\|x_1(s)-x_2(s)\|,$$

by Proposition 2,2 there exists $w:J\to X$, such that $w(s)\in G(s,x_2(s))$, for all $s\in J$, and

$$\|v(s)-w(s)\|\le\psi(s)\|x_1(s)-x_2(s)\|.$$

Further the mapping w is measurable (see [18]). We pose $y_2(t)=\int_0^t w(s)ds$, we have

$$\|y_1(t)-y_2(t)\|\le\int_0^t\|v(s)-w(s)\|ds\le\|\psi\|_{L^1}\|x_1-x_2\|.$$

It follows that $\|y_1-y_2\|\le\|\psi\|_{L^1}\|x_1-x_2\|$. By Proposition 2.2, we deduce that

$$H(Cx_1,Cx_2)\le\|\psi\|_{L^1}\|x_1-x_2\|.$$

Hence the multivalued mapping C is Lipschitzian with constant $\|\psi\|_{L^1}$. By Proposition 2.1 and Lemma 3.3, we deduce that C has closed graph. On the other hand, for all $z\in X$, we have

$$\|G(s,z)\|=H(G(s,z),0)$$
$$\le H(G(s,z),G(s,0))+H(G(s,0),0)$$
$$\le\psi(s)\|z\|+\|G(s,0)\|.$$

For all $u\in Cx$ with $u(t)=\int_0^t w(s)ds$, and $v\in S_G^1(x)$, we get

$$\|u\| \le \|\psi\|_{L^1} r + L.$$

Step 3. Since the multivalued map F satisfies (H_1) and (H_2), then the multivalued operator B is upper semicontinuous with compact values (see [23], Theorem 5.1.2 and corollary 5.1.2). We show that the multivalued mapping B is Lipschitzian, with respect to the Hausdorff measure of nocompactness on $C(J, X)$, also noted χ. Let Ω be a subset of S and $u \in B(\Omega)$, where

$$u(t) = q(t) + \int_0^t v(s)ds,\ v \in S_F^1(x),$$

for some $x \in \Omega$. For all $t, t' \in J$, we have

$$\|u(t) - u(t')\| \le \|q(t) - q(t')\| + \int_{t'}^t \|v(s)\|ds$$
$$\le |q(t) - q(t')| + \int_{t'}^t h_r(s)ds.$$

Then, the subset $B(\Omega)$ is equicontinuous in $C([0,a], X)$. By the properties of χ, we have

$$\chi(B(\Omega)) = \sup_{t\in J} \chi(B(\Omega)(t))$$
$$= \sup_{t\in J} \chi\left(\left\{q(t) + \int_0^t v(s),\ v \in S_F^1(x),\ x \in \Omega\right\}\right)$$
$$= \sup_{t\in J} \chi\left(\left\{\int_0^t v(s),\ v \in S_F^1(x),\ x \in \Omega\right\}\right)$$
$$\le \sup_{t\in J} \chi\left(\int_0^t F(s, s(\Omega))ds\right).$$

The multivalued mapping $s \to F(s, s(\Omega)) = \bigcup_{x\in\Omega} F(s, x(s))$ is integrably bounded with growth function h_r. Further, we have

$$\chi(B(\Omega)) \le \sup_{t\in J} \chi\left(\int_0^t F(s, s(\Omega))ds\right)$$
$$\le \sup_{t\in J} \chi(\Omega)\left(\int_0^t \phi(t)ds\right) \le \|\varphi\| \chi(\Omega).$$

Hence, the multivalued operator B is lipschitzian with respect to χ.

For all $u \in Bx$, we have

$$|u(t)| = \left|q(t) + \int_0^t v(s)ds\right| \le \|q\| + \|h_r\|_{L^1}.$$

Then $\|B(S)\| \le \|q\| + \|h_r\|_{L^1}$.

Step 4. *We show that $Ax.Bx + Cx$ is a convex subset of S, for each $x \in S$. Let*

$$u_1(t) = f(t, x(t))\left(q(t) + \int_0^t v_1(s)ds\right) + \int_0^t w_1(s)ds$$

and

$$u_2(t) = f(t, x(t))\left(q(t) + \int_0^t v_2(s)ds\right) + \int_0^t w_2(s)ds$$

where $v_1(s), v_2(s) \in F(s, x(s))$ and

$$w_1(s), w_2(s) \in G(s, x(s)),\ a.e\ s \in J.$$

Then, for all $\lambda \in [0,1]$, we have

$$\lambda u_1(t) + (1-\lambda)u_2(t)$$
$$= f(t, x(t)).\int_0^t (\lambda v_1(s) + (1-\lambda)v_2(s))ds$$
$$+ \int_0^t (\lambda w_1(s) + (1-\lambda)w_2(s))ds.$$

Since $F(s, x(s))$ and $G(s, x(s))$ are convex subsets of X, we get

$$\lambda u_1(t) + (1-\lambda)u_2(t) = f(t, x(t)).\int_0^t v(s)ds + \int_0^t w(s)ds,$$

where $v \in S_F^1(x)$ and $w \in S_G^1(x)$.

Let $u \in Ax.Bx + Cx$ with

$$u(t) = f(t, x(t)).\left(q(t) + \int_0^t v(s)ds\right) + \int_0^t w(s)ds.$$

Then

$$|u(t)| \le \|f\|(\|q\| + \|h_r\|_{L^1}) + \|\psi\|_{L^1} r + L < r.$$

Since

$$\|f\|\|\varphi\| + (\|q\| + \|h_r\|)\|l\| + \|\psi\|_{L^1} < 1.$$

By Lemma 3.3 and Lemma 3.4, the multivalued mappings $A;B$ and C satisfy all the conditions of Theorem 3.2 and equation $x \in Ax.Bx + Cx$ has a solution in $C(J, X)$.

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