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Fixed Point Theorems for Expansive Mappings in G-metric Spaces

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Abstract In this paper we prove some fixed point theorems for contractive as well as for expansive mappings in G-metric space by using integral type contraction. Finally, we present an example.

Keywords: *G-metric space, fixed point, integral type contractive mapping, expansive mapping*

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1. Introduction

In 2006, Mustafa and Sims [10], introduced the concept of G-metric spaces and presented some fixed point theorems in G-metric spaces. Mehdi et al. [9] gave new approach to G-metric spaces and proved some fixed point theorems in G-metric spaces. Further we note that many researchers have studied G-metric spaces see, [1-21]. Moreover, In 2002, A. Branciari [8] introduced the concept of integral type contractive mappings in complete metric spaces and study the existence of fixed points for mappings which is defined on complete metric space satisfies integral type contraction. Recently A. Zada et al. [22], presented the concept of integral type contraction with respect to G-metric spaces and proved some coupled coincidence fixed point results for two pairs in such spaces, by using the notion of integral type contractive mappings given by A. Branciari [8]. In section 3, we presented some fixed point theorems of integral type contractive mapping in setting of G-metric spaces. Moreover in section 4, we proved some fixed point theorems of integral type contraction for expansive mapping. Also we give suitable example that support our main results.

2. Preliminaries

Consistent with Mustafa & Sims [10] and Branciari [8]. The following definitions and results will be needed in this paper.

Definition 2.1 [10] *Let Y be a non-empty set and $G:Y\times Y\times Y\to R^+$ is a function that satisfies the following conditions:*

(1) $G(a,b,c)=0$ if $a=b=c$,

(2) $G(a,b,c)>0$ for all $a,b\in Y$ with $a\neq b$,

(3) $G(a,a,b)\leq G(a,b,c)$, for all $a,b,c\in Y$ with $b\neq c$,

(4) $G(a,b,c)=G(a,c,b)=G(b,c,a)=\ldots$ symmetry in all three variables,

(5) $G(a,a,b)\leq G(a,s,s)+G(s,b,c)$ (rectangle inequality) for all $a,b,c,s\in Y$.

Then the function G is called a generalized metric and the pair (Y,G) is called a G-metric space.

Example 2.2 [10] *Let $Y=\{x,y\}$. Define G on $Y\times Y\times Y$ by*

$$G(x,x,x)=G(y,y,y)=0,$$
$$G(x,x,y)=1, G(x,y,y)=2$$

and extend G to $Y\times Y\times Y$ by using the symmetry in the variables. Then it is clear that (Y,G) is a G-metric space.

Definition 2.3 [10] *Let (Y,G) be a G-metric space and (a_n) a sequence of points of Y. A point $a\in Y$ is said to be the limit of the sequence (a_n), if $\lim_{n,m\to+\infty} G(a,a_n,a_m)=0$ and we say that the sequence (a_n) is G-convergent to a.*

Proposition 2.4 [10] *Let (Y,G) be a G-metric space. Then the following are equivalent:*

(1) (a_n) *is G-convergent to a.*

(2) $G(a_n,a_n,a)\to 0$ *as* $n\to+\infty$.

(3) $G(a_n,a,a)\to 0$ *as* $n\to+\infty$.

(2) $G(a_n,a_m,a)\to 0$ *as* $n,m\to+\infty$.

Definition 2.5 [10] *Let (Y,G) be a G-metric space. A sequence (a_n) is called G-Cauchy if for every $\epsilon>0$, there is $k\in\mathbf{N}$ such that $G(a_n,a_m,a_l)<\epsilon$, for all $n,m,l\geq k$, that is $G(a_n,a_m,a_l)\to 0$ as $n,m,l\to+\infty$.*

Proposition 2.6 [10] *Let (Y,G) be a G-metric space. Then the following are equivalent:*

(1) The sequence (a_n) is *G*-Cauchy.

(2) For every $\epsilon > 0$, there is $k \in \mathbf{N}$ such that $G(a_n, a_m, a_m) < \epsilon$, for all $n, m \geq k$.

Definition 2.7 [10] *A G-metric space (Y,G) is called G-complete if every G-Cauchy sequence in (Y,G) is G-convergent in (Y,G).*

Lemma 2.8 [11] *By the rectangle inequality* (5) *together with the symmetry* (4), *we have*

$$\begin{aligned} &G(a,b,b) = G(b,b,a) \\ &\leq G(b,a,a) + G(a,b,a) = 2G(b,a,a) \end{aligned}$$

In 2002, Branciari in [8] introduced a general contractive condition of integral type as follows.

Theorem 2.9 [8] *Let (Y,d) be a complete metric space, $\alpha \in (0,1)$, and $f : Y \times Y$ is a mapping such that for all $x, y \in Y$,*

$$\int_0^{d(f(x),f(y))} \phi(t)dt \leq \alpha \int_0^{d(x,y)} \phi(t)dt$$

where $\phi : [0,+\infty) \to [0,+\infty)$ is nonnegative and Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $[0,+\infty)$ such that for each $\epsilon > 0$, $\int_0^{\epsilon} \phi(t)dt > 0$, then f has a unique fixed point $a \in Y$, such that for each $x \in Y$, $\lim_{n\to\infty} f^n(x) = a$.

Lemma 2.10 *By rectangle inequality* (5) *together with the symmetry* (4) *of definition 2.1, we have*

$$\begin{aligned} &\int_0^{G(a,b,b)} \varphi(t)dt = \int_0^{G(b,b,a)} \varphi(t)dt \\ &\leq \int_0^{G(b,a,a)} \varphi(t)dt + \int_0^{G(a,b,a)} \varphi(t)dt \\ &= 2\int_0^{G(b,a,a)} \varphi(t)dt. \end{aligned}$$

In this paper by using the notion of integral type given by Branciari in [8], we presented some fixed point theorems in G-metric space.

3. Main Results

In this section, we prove some fixed point theorems in generalize metric space by using integral type contractive mappings. Our first main result is follow as,

Theorem 3.1 *Let (Y,G) be a G-metric space. Suppose $H : Y \to Y$ be a mapping satisfy the following condition for all $a, b \in Y$:*

$$\int_0^{G(Ha,Hb,Hb)} \varphi(t)dt \leq k\int_0^{G(a,Ha,b)} \varphi(t)dt, \qquad (3.1)$$

where $k \in [0,1)$ and $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$. Then H has a unique fixed point in Y.

Proof. Choose $a_0 \in Y$ and define a_n by $a_n := H^n(a_0)$. Notice that if $a_{n'} = a_{n'+1}$ for some $n' \in N$, then obviously H has a fixed point. Thus, we suppose that

$$a_n \neq a_{n+1} \text{ for all } n \in N.$$

that is, we have

$$\int_0^{G(a_n,a_{n+1},a_{n+1})} \varphi(t)dt \neq k\int_0^{G(a_{n-1},a_n,a_n)} \varphi(t)dt,$$

continuing this process, we get

$$\int_0^{G(a_n,a_{n+1},a_{n+1})} \varphi(t)dt \leq k^n \int_0^{G(a_0,a_1,a_1)} \varphi(t)dt.$$

Moreover, for all $n, m \in N$; $n < m$, we have

$$\begin{aligned} &\int_0^{G(a_n,a_m,a_m)} \varphi(t)dt \\ &\leq \int_0^{G(a_n,a_{n+1},a_{n+1})} \varphi(t)dt + \int_0^{G(a_{n+1},a_{n+2},a_{n+2})} \varphi(t)dt \\ &+\int_0^{G(a_{n+2},a_{n+3},a_{n+3})} \varphi(t)dt + \ldots + \int_0^{G(a_{m-1},a_m,a_m)} \varphi(t)dt \\ &\leq \left(k^n + k^{n+1} + k^{n+2} + \ldots + k^{m-1}\right)\int_0^{G(a_0,a_1,a_1)} \varphi(t)dt \\ &\leq \frac{k^n}{1-k}\int_0^{G(a_0,a_1,a_1)} \varphi(t)dt \to 0. \end{aligned}$$

So,

$$\lim_{n,m\to\infty} \int_0^{G(a_n,a_m,a_m)} \varphi(t)dt = 0.$$

Thus

$$\lim_{n,m\to\infty} G(a_n,a_m,a_m) = 0.$$

This means that (a_n) is G-Cauchy sequence. Due to completeness of (Y,G), there exists $l \in Y$ such that an is convergent to l.

Suppose that $Hl \neq l$, then

$$\int_0^{G(a_n,Hl,Hl)} \varphi(t)dt \leq k\int_0^{G(a_{n-1},a_n,l)} \varphi(t)dt.$$

Taking limit as $n \to \infty$, and using the fact that function G is continuous, then

$$\int_0^{G(l,Hl,Hl)} \varphi(t)dt \leq k\int_0^{G(l,l,l)} \varphi(t)dt.$$

This contradiction implies that $Hl = l$.

For uniqueness, let $l \neq p$ such that $Hp = p$ and use Lemma 2.10, then

$$\begin{aligned} &\int_0^{G(l,l,p)} \varphi(t)dt = \int_0^{G(Hl,Hl,Hp)} \varphi(t)dt \\ &\leq k\int_0^{G(l,Hl,p)} \varphi(t)dt = k\int_0^{G(l,l,p)} \varphi(t)dt, \end{aligned}$$

which implies that $l = p$. The proof is completed.

Corollary 3.2 Let (Y,G) be a G-metric space. Suppose $H : Y \to Y$ be a mapping satisfy the following condition for all $a, b, c \in Y$:

$$\int_0^{G(Ha,Hb,Hc)}\varphi(t)dt \le \alpha\int_0^{G(a,Ha,c)}\varphi(t)dt+\beta\int_0^{G(a,Ha,b)}\varphi(t)dt$$

where $0\le\alpha+\beta<1$ and $\varphi:[0,\infty)\to[0,\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\epsilon>0$, $\int_0^{\epsilon}\varphi(t)dt>0$. Then H has a unique fixed point in Y.

Theorem 3.3 *Let (Y,G) be a complete G-metric space. Suppose $H:Y\to Y$ be a mapping satisfy the following condition for all $a,b\in Y$, where $x+y+z+w<1$*

$$\int_0^{G\left(Ha,Hb,H^2c\right)}\varphi(t)dt \le x\int_0^{G\left(a,Ha,H^2a\right)}\varphi(t)dt+y\int_0^{G\left(b,Hb,H^2b\right)}\varphi(t)dt +z\int_0^{G(a,Ha,Hb)}\varphi(t)dt+w\int_0^{G\left(b,Hb,H^3a\right)}\varphi(t)dt.$$

And $\varphi:[0,\infty)\to[0,\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\epsilon>0$, $\int_0^{\epsilon}\varphi(t)dt>0$. Then H has a unique fixed point in Y.

Proof. Choose $a_0\in Y$. We construct sequence in the following way:

$$a_{n+1}=Ha_n \text{ for all } n=0,1,2,...$$

Notice that if $a_{n'}=a_{n'+1}$ for some $n'\in N$, then obviously H has a fixed point. Thus, we suppose that

$$a_n\ne a_{n+1} \text{ for all } n\in N.$$

That is, we have

$$\int_0^{G(a_n,a_{n+1},a_{n+2})}\varphi(t)dt>0.$$

From above condition, with $a=a_{n-1}$ and $b=a_n$, we have

$$\int_0^{G\left(Ha_{n-1},Ha_n,H^2a_n\right)}\varphi(t)dt \le x\int_0^{G\left(a_{n-1},Ha_{n-1},H^2a_{n-1}\right)}\varphi(t)dt+y\int_0^{G\left(a_n,Ha_n,H^2a_n\right)}\varphi(t)dt +z\int_0^{G(a_{n-1},Ha_{n-1},Ha_n)}\varphi(t)dt+w\int_0^{G\left(a_n,Ha_n,H^3a_{n-1}\right)}\varphi(t)dt,$$

implies that

$$\int_0^{G(a_n,a_{n+1},a_{n+2})}\varphi(t)dt \le x\int_0^{G(a_{n-1},a_n,a_{n+1})}\varphi(t)dt+y\int_0^{G(a_n,a_{n+1},a_{n+2})}\varphi(t)dt +z\int_0^{G(a_{n-1},a_n,a_{n+1})}\varphi(t)dt+w\int_0^{G(a_n,a_{n+1},a_{n+2})}\varphi(t)dt.$$

So,

$$\int_0^{G(a_n,a_{n+1},a_{n+2})}\varphi(t)dt\le k\int_0^{G(a_{n-1},a_n,a_{n+1})}\varphi(t)dt,$$

where $k=\dfrac{(x+z)}{(1-y-w)}<1$. Then

$$\int_0^{G(a_n,a_{n+1},a_{n+2})}\varphi(t)dt\le k^n\int_0^{G(a_0,a_1,a_2)}\varphi(t)dt,$$

for all $n\in N$. From definition of G-metric space, we know that

$$\int_0^{G(a_n,a_n,a_{n+1})}\varphi(t)dt\le\int_0^{G(a_n,a_{n+1},a_{n+2})}\varphi(t)dt,$$

with $a_n\ne a_{n+1}$, also by Lemma 2.10, we know that

$$\int_0^{G(a_{n+1},a_{n+1},a_n)}\varphi(t)dt\le 2\int_0^{G(a_n,a_n,a_{n+1})}\varphi(t)dt.$$

Then by above inequality, we have

$$\int_0^{G(a_{n+1},a_{n+1},a_n)}\varphi(t)dt\le 2k^n\int_0^{G(a_0,a_1,a_2)}\varphi(t)dt.$$

Moreover, for all $n,m\in N$; $n<m$, we have

$$\begin{aligned}&\int_0^{G(a_m,a_m,a_n)}\varphi(t)dt\\ &\le\int_0^{G(a_n,a_{n+1},a_{n+1})}\varphi(t)dt+\int_0^{G(a_{n+1},a_{n+2},a_{n+2})}\varphi(t)dt\\ &+\int_0^{G(a_{n+2},a_{n+3},a_{n+3})}\varphi(t)dt+...+\int_0^{G(a_{m-1},a_m,a_m)}\varphi(t)dt\\ &\le 2\left(k^n+k^{n+1}+k^{n+3}+...+k^{m-1}\right)\int_0^{G(a_0,a_1,a_1)}\varphi(t)dt\\ &\le\frac{2k^n}{1-k}\int_0^{G(a_0,a_1,a_1)}\varphi(t)dt\to 0.\end{aligned}$$

So,

$$\lim_{n,m\to\infty}\int_0^{G(a_n,a_m,a_m)}\varphi(t)dt=0.$$

Thus

$$\lim_{n,m\to\infty}G(a_n,a_m,a_m)=0.$$

This means that (a_n) is G-Cauchy sequence. Due to completeness of (Y,G), there exists $p\in Y$ such that a_n is convergent to p. From the above condition of theorem, with $a=a_n$ and $b=p$, we have

$$\int_0^{G\left(Ha_n,Hp,H^2p\right)}\varphi(t)dt \le x\int_0^{G\left(a_n,Ha_n,H^2a_n\right)}\varphi(t)dt+y\int_0^{G\left(p,Hp,H^2p\right)}\varphi(t)dt +z\int_0^{G(a_n,Ha_n,Hp)}\varphi(t)dt+w\int_0^{G\left(p,Hp,H^3a_n\right)}\varphi(t)dt.$$

Then

$$\int_0^{G\left(a_{n+1},Hp,H^2p\right)}\varphi(t)dt \le x\int_0^{G(a_n,a_{n+1},a_{n+2})}\varphi(t)dt+y\int_0^{G\left(p,Hp,H^2p\right)}\varphi(t)dt +z\int_0^{G(a_n,a_{n+1},Hp)}\varphi(t)dt+w\int_0^{G(p,Hp,a_{n+3})}\varphi(t)dt.$$

Taking limit as $n \to \infty$ of above inequality, we have

$$\int_0^{G\left(p,Hp,H^2p\right)} \varphi(t)dt \le \frac{(z+w)}{(1-y)} \int_0^{G(p,p,Hp)} \varphi(t)dt.$$

Now, if $Hp = H^2p$, then H has a fixed point. Hence, we suppose that $Hp \ne H^2p$. Therefore, by definition of G-metric space, we get

$$\int_0^{G\left(p,Hp,H^2p\right)} \varphi(t)dt \le \frac{(z+w)}{(1-y)} \int_0^{G(p,p,Hp)} \varphi(t)dt$$

$$\le \frac{(z+w)}{(1-y)} \int_0^{G\left(p,Hp,H^2p\right)} \varphi(t)dt,$$

which implies that

$$\int_0^{G\left(p,Hp,H^2p\right)} \varphi(t)dt = 0,$$

i.e., $p = Hp = H^2p$. The proof is completed.

4. Integral Type Contraction for Expansive Mappings

In this section of our paper, we prove some fixed point theorems for expansive mapping of integral type contraction in G-metric spaces.

Theorem 4.1 *Let (Y,G) be a complete G-metric space. Suppose $H: Y \to Y$ be an onto mapping satisfy the following condition for all $a,b \in Y$, where $\alpha > 1$ holds*

$$\int_0^{G\left(Ha,H^2a,Hb\right)} \varphi(t)dt \ge \alpha \int_0^{G(a,Ha,b)} \varphi(t)dt.$$

And $\varphi: [0,\infty) \to [0,\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$. Then H has a unique fixed point in Y.

Proof. Choose $a_0 \in Y$, as H is onto map, then there exists $a_1 \in Y$ such that $a_0 = Ha_1$. If we continue this process, we can get $a_n = Ha_{n+1}$ for all $n \in N \cup 0$. In case $a_{n_0} = a_{n_0+1}$, for some $n_0 \in N \cup 0$, then clearly a_{n_0} is a fixed point of H. Next, we suppose that $a_n \ne a_{n+1}$ for all n. From above condition of this result, with $a = a_{n+1}$ and $b = a_n$ we have

$$\int_0^{G(a_n,a_{n-1},a_{n-1})} \varphi(t)dt$$

$$= \int_0^{G(Ha_{n+1},a_{n-1},a_{n-1})} \varphi(t)dt$$

$$\ge \alpha \int_0^{G(a_{n+1},Ha_{n+1},a_n)} \varphi(t)dt$$

$$= \alpha \int_0^{G(a_{n+1},a_n,a_n)} \varphi(t)dt,$$

which implies that

$$\int_0^{G(a_{n+1},a_n,a_n)} \varphi(t)dt \le h \int_0^{G(a_n,a_{n-1},a_{n-1})} \varphi(t)dt,$$

where $h = \frac{1}{\alpha} < 1$. Then we have

$$\int_0^{G(a_{n+1},a_n,a_n)} \varphi(t)dt \le h^n \int_0^{G(a_0,a_1,a_1)} \varphi(t)dt.$$

By Lemma (2.10), we get

$$\int_0^{G(a_n,a_{n+1},a_{n+1})} \varphi(t)dt$$

$$\le 2\int_0^{G(a_{n+1},a_n,a_n)} \varphi(t)dt$$

$$\le 2h^n \int_0^{G(a_0,a_1,a_1)} \varphi(t)dt.$$

If we follow the lines of the proof of result 3.1, we derive that (a_n) is a G-Cauchy sequence. Since (Y,G) is complete, there exists $p \in Y$ such that $a_n \to p$ as $n \to \infty$. Consequently, since H is onto, so there exists $l \in Y$ such that $p = Hl$. From above condition of this theorem, with $a = a_{n+1}$ and $b = l$, we have

$$\int_0^{G(a_n,a_{n-1},p)} \varphi(t)dt = \int_0^{G\left(Ha_{n+1},H^2a_n,Hl\right)} \varphi(t)dt$$

$$\ge \alpha \int_0^{G(a_{n+1},Ha_{n+1},p)} \varphi(t)dt$$

$$= \alpha \int_0^{G(a_{n+1},a_n,p)} \varphi(t)dt.$$

Taking limit as $n \to \infty$ in above inequality, we get

$$\int_0^{G(u,u,l)} \varphi(t)dt = \lim_{n\to\infty} \int_0^{G(a_n,a_{n-1},u)} \varphi(t)dt = 0.$$

That is, $u = l$. Then $u = Hl = Hu$.

For uniqueness, let $r \ne s$ such that $Hs = s$ and $Hr = r$. By above condition of result, we get

$$\int_0^{G(r,r,s)} \varphi(t)dt = \int_0^{G\left(Hr,H^2r,Hs\right)} \varphi(t)dt$$

$$\ge \alpha \int_0^{G(r,Hr,s)} \varphi(t)dt$$

$$\ge \alpha \int_0^{G(r,r,s)} \varphi(t)dt$$

$$> \int_0^{G(r,r,s)} \varphi(t)dt,$$

which arise contradiction, Hence $r = s$. The proof is completed.

Theorem 4.2 *Let (Y,G) be a complete G-metric space. Suppose $H: Y \to Y$ be an onto mapping satisfy the following condition for all $a,b \in Y$, where $\alpha > 1$*

$$\int_0^{G\left(Ha,Hb,H^2b\right)} \varphi(t)dt \ge \alpha \int_0^{G\left(a,Ha,H^2a\right)} \varphi(t)dt. \quad (4.1)$$

And $\varphi: [0,\infty) \to [0,\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$. Then H has a unique fixed point in Y.

Proof. Choose $a_0 \in Y$, as H is onto map, then there exists $a_1 \in Y$ such that $a_0 = Ha_1$. If we continue this process, we can get $a_n = Ha_{n+1}$ for all $n \in N \bigcup 0$. In case $a_{n_0} = a_{n_0+1}$, for some $n_0 \in N \bigcup 0$, then clearly a_{n_0} is a fixed point of H. Next, we suppose that $a_n \neq a_{n+1}$ for all n. From (4.1), with $a = a_{n+1}$ and $b = a_n$ we have

$$\int_0^{G\left(Ha_{n+1},Ha_n,H^2a_n\right)} \varphi(t)dt \geq \alpha \int_0^{G\left(a_{n+1},Ha_{n+1},H^2a_{n+1}\right)} \varphi(t)dt,$$

implies that

$$\int_0^{G(a_n,a_{n-1},a_{n-2})} \varphi(t)dt \geq \alpha \int_0^{G(a_{n+1},a_n,a_{n-1})} \varphi(t)dt,$$

and so,

$$\int_0^{G(a_{n+1},a_n,a_{n-1})} \varphi(t)dt \leq h \int_0^{G(a_n,a_{n-1},a_{n-2})} \varphi(t)dt,$$

where $h = \frac{1}{\alpha} < 1$. By the proof of Theorem 3.1, we can show that (a_n) is a G-Cauchy sequence. Since (Y,G) is complete, it exists $p \in Y$ such that $a_n \to p$ as $n \to \infty$. Consequently, since H is onto, so there exists $l \in Y$ such that $p = Hl$. From 4.1, with $a = l$ and $b = a_{n+1}$, we have

$$\int_0^{G(p,a_n,a_{n-1})} \varphi(t)dt = \int_0^{G\left(Hl,Ha_{n+1},H^2a_{n+1}\right)} \varphi(t)dt \geq \alpha \int_0^{G\left(l,Hl,H^2l\right)} \varphi(t)dt.$$

On taking limit $n \to \infty$ in above inequality, we have $G\left(l,Hl,H^2l\right) = 0$. That is, $l = Hl = H^2l$. For uniqueness, let $r \neq s$ such that $Hs = s$ and $Hr = r$.

Now, by using condition 4.1, we have

$$\int_0^{G(r,r,s)} \varphi(t)dt > \int_0^{G(r,r,s)} \varphi(t)dt,$$

which is contradiction. Hence $r = s$. The proof is completed.

5. Example

In this section, we present an example, which indicates that how our results can be applied to different problems.

Example 5.1 *Let* $Y = [0,\infty)$ *and*

$$G(a,b,c) = \begin{cases} 0, & if\ a = b = c \\ \max\{a,b,c\}, & otherwise \end{cases}$$

be a G-metric space on Y. Define $H: Y \to Y$ *by* $Ha = \frac{1}{5}a.$

Then the condition of Theorem 3.1 holds. In fact,

$$\int_0^{G(Ha,Hb,Hb)} \varphi(t)dt = \int_0^{G\left(\frac{1}{5}a,\frac{1}{5}b,\frac{1}{5}b\right)} \varphi(t)dt,$$

and

$$\int_0^{G(a,Ha,b)} \varphi(t)dt = \int_0^{\max\{a,b\}} \varphi(t)dt,$$

and so,

$$\int_0^{G(Ha,Hb,Hb)} \varphi(t)dt \leq \frac{1}{4} \int_0^{G(a,Ha,b)} \varphi(t)dt.$$

That is, condition of Theorem 3.1 holds with $k = \frac{1}{4} \in [0,1)$.

References

[1] R. Agarwal, E, Karapinar, *Remarks on some coupled fixed point theorems in G-metric spaces*, Fixed Point Theory Appl., 2013, 2 (2013).

[2] H. Aydi, M. Postolache, W. Shatanawi, *Coupled fixed point results for (ϕ-ψ)-weakly contractive mappings in ordered G-metric spaces*, Comput. Math. Appl., vol. 63(1), 2012, pp. 298-309.

[3] H. Aydi, W. Shatanawi, C. Vetro, *On generalized weakly G-contraction mappings in G-metric spaces*, Comput. Math. Appl., vol. 62, 2011, pp. 4222-4229.

[4] H. Aydi, B. Damjanovic, B. Samet, W. Shatanawi, *Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces*, Math. Comput. Model., vol. 54, 2011, pp. 2443-2450.

[5] H. Aydi, E. Karapinar, W. Shatanawi, *Tripled fixed point results in generalized metric spaces*, J. Appl. Math., 2012, Article ID 314279.

[6] H. Aydi, E. Karapinar, Z. Mustafa. *On common fixed points in G-metric spaces using (E.A) property*, Comput. Math. Appl., vol. 64(6), 2012, pp. 1944-1956.

[7] H. Aydi, E. Karapinar, W. Shatanawi, *Tripled common fixed point results for generalized contractions in ordered generalized metric spaces*, Fixed Point Theory Appl., 2012, 101 (2012).

[8] A. Branciari. *A fixed point theorem for mappings satisfying a general contractive condition of integral type*. International Journal of Mathematics and Mathematical Sciences, vol. 29, no. 9, 2002, pp. 531-536.

[9] A. Mehdi, K. Erdal, S. Peyman, *A new approach to G-metric and related fixed point theorems*, J. Ineq. Apps., 2013, 2013:454.

[10] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., vol. 7 (2), 2006, pp. 289-297.

[11] Z. Mustafa, *A new structure for generalized metric spaces with applications to fixed point theory*, PhD thesis, The University of Newcastle, Australia, 2005.

[12] Z. Mustafa, M. Khandaqji, W. Shatanawi, *Fixed point results on complete G-metric spaces*, Studia Sci. Math. Hung., vol. 48, 2011, pp. 304-319.

[13] Z. Mustafa, H. Aydi, E. Karapinar, *Mixed g-monotone property and quadruple fixed point theorems in partially ordered metric space*, Fixed Point Theory Appl., 2012, 71.

[14] Z. Mustafa, *Common fixed points of weakly compatible mappings in G-metric space*s, Appl. Math. Sci., vol. 6(92), 2012, pp. 4589-4600.

[15] Z. Mustafa, *Some new common fixed point theorems under strict contractive conditions in G-metric spaces*, J. Appl. Math., 2012, Article ID 248937.

[16] Z. Mustafa, *Mixed g-monotone property and quadruple fixed point theorems in partially ordered G-metric spaces using (ϕ-ψ) contractions*, Fixed Point Theory Appl., 2012, 199.

[17] Z. Mustafa, W. Shatanawi, M. Bataineh, *Existence of fixed point results in G-metric spaces*, Int. J. Math. Math. Sci., 2009, Article ID 283028 (2009).

[18] R. Shah, A. Zada and I. Khan, *Some fixed point theorems of integral type contraction in cone b-metric spaces*, Turkish. J. Ana. Num. Theor., vol. 3(6), 2105, 165-169.

[19] R. Shah et al., *New common coupled fixed point results of integral type contraction in generalized metric spaces*, J. Ana. Num. Theor., vol. 4(2), 2106, 1-8.

[20] K. P. R. Rao, L. Bhanu, Z. Mustafa, *Fixed and related fixed point theorems for three maps in G-metric space*, J. Adv. Stud. Topol., vol. 3(4), 2012, pp. 12-19.

[21] W. Shatanawi, Z. Mustafa, *On coupled random fixed point results in partially ordered metric spaces*, Mat. Vesn., vol. 64, 2012, pp. 139-146.

[22] A. Zada, R. Shah, T. Li, *Integral Type Contraction and Coupled Coincidence Fixed Point Theorems for Two Pairs in G-metric Spaces*, Hacet. J. Math. Stat., vol. 45(5), 2016, pp. 1475-1484.