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Fractional Integral Inequalities via *s*-Convex Functions

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Abstract In this paper, we establish several inequalities for *s*-convex mappings that are connected with the Riemann-Liouville fractional integrals. Our results have some relationships with certain integral inequalities in the literature.

Keywords: *Hadamard's Inequality, Riemann-Liouville Fractional Integration, Hölder Inequality, s-convexity*

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1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}$$

is well known in the literature as Hadamard's inequality. Both inequalities hold in the reversed direction if *f* is concave.

On November 22, 1881, Hermite (1822-1901) sent a letter to the Journal Mathesis. This letter was published in Mathesis 3 (1883, p: 82) and in this letter an inequality presented which is well-known in the literature as Hermite-Hadamard integral inequality. Since its discovery in 1883, Hermite-Hadamard inequality has been considered the most useful inequality in mathematical analysis. Many uses of these inequalities have been discovered in a variety of settings. Moreover, many inequalities of special means can be obtained for a particular choice of the function *f*.

Let real function *f* be defined on some nonempty interval *I* of real line $\mathbb{R}$. The function *f* is said to be convex on *I* if inequality

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

In [3], *s*-convex functions defined by Orlicz as following.

Definition 1. *A function* $f : \mathbb{R}^+ \to \mathbb{R}$, *where* $\mathbb{R}^+ = [0,\infty)$, *is said to be s-convex in the first sense if*

$$f(\alpha x+\beta y) \le \alpha^s f(x)+\beta^s f(y)$$

for all $x, y \in [0,\infty)$, $\alpha, \beta \ge 0$ *with* $\alpha^s + \beta^s = 1$ *and for some fixed* $s \in (0,1]$. *We denote by* K_s^1 *the class of all s-convex functions.*

Definition 2. *A function* $f : \mathbb{R}^+ \to \mathbb{R}$, *where* $\mathbb{R}^+ = [0,\infty)$, *is said to be s-convex in the second sense if*

$$f(\alpha x+\beta y) \le \alpha^s f(x)+\beta^s f(y)$$

for all $x, y \in [0,\infty)$, $\alpha, \beta \ge 0$ *with* $\alpha + \beta = 1$ *and for some fixed* $s \in (0,1]$. *We denote by* K_s^2 *the class of all s-convex functions.*

Orlicz defined these class of functions in [3] and these definitions were used in the theory of Orlicz spaces in [4] and [5]. Obviously, one can see that if we choose *s* = 1, both definitions reduced to ordinary concept of convexity.

For several results related to above definitions we refer readers to [2,6,7] and [8].

In [6], Dragomir and Fitzpatrick proved a variant of Hermite-Hadamard inequality which holds for the *s*-convex functions.

Theorem 1. *Suppose that* $f : [0,\infty) \to [0,\infty)$ *is an s-convex function in the second sence, where* $s \in (0,1)$ *and let* $a, b \in [0,\infty)$, $a < b$. *If* $f \in L_1([a,b])$, *then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1}. \quad (1.1)$$

The constant $k = \frac{1}{s+1}$ *is the best possible in the second inequality in (1.1).*

In [7], Kırmacı et al. obtained Hadamard type inequalities which hold for s-convex functions in the second sence. It is given in the next theorem.

Theorem 2. *Let $f: I \to \mathbb{R}$, $I \subset [0,\infty)$, be differentiable function on I° such that $f' \in L_1([a,b])$, where $a,b \in I$, $a<b$. If $|f'|^q$ is s-convex on $[a,b]$ for some fixed $s \in (0,1)$ and $q \ge 1$, then:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2}\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left[\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right]^{\frac{1}{q}}\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}}. \tag{1.2}$$

In [1], Dragomir and Agarwal proved the following inequality.

Theorem 3. *Let $f: I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^{\circ}$ with $a<b$, and let $p>1$. If the new mapping $|f'|^{\frac{p}{p-1}}$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{1.3}$$

In [12], Set et al. proved the following Hadamard type inequality for *s*-convex functions in the second sense via Riemann-Liouville fractional integral.

Theorem 4. *Let $f:[a,b] \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f' \in L[a,b]$. If $|f'|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$ and $q \ge 1$, then the following inequality for fractional integrals holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right|$$
$$\le \frac{b-a}{2}\left[\frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\right]^{1-\frac{1}{q}}$$
$$\times\left[\begin{array}{l}\beta\left(\frac{1}{2},s+1,\alpha+1\right)-\beta\left(\frac{1}{2},\alpha+1,s+1\right)\\ +\dfrac{2^{\alpha+s}-1}{(\alpha+s+1)2^{\alpha+s}}\end{array}\right]^{\frac{1}{q}} \tag{1.4}$$
$$\times\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}.$$

Now, we give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper, see ([9]).

Definition 3. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha}f$ and $J_{b^-}^{\alpha}f$ of order $\alpha>0$ with $a \ge 0$ are defined by*

$$J_{a^+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ x>a$$

and

$$J_{b^-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\ x<b$$

respectively where $\Gamma(\alpha)=\int_0^{\infty}e^{-u}u^{\alpha-1}du$. Here is $J_{a^+}^{0}f(x)=J_{b^-}^{0}f(x)=f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. For some recent results connected with fractional integral inequalities see ([10-17]).

In order to prove our main theorems, we need the following lemma:

Lemma 1. *(see [18]) Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I with $a<r$, $a,r \in I$. If $f' \in L[a,r]$, then the following equality for fractional integral holds:*

$$\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-\alpha)^{\alpha}}\left[J_{a^+}^{\alpha}f(r)+J_{r^-}^{\alpha}f(a)\right]$$
$$=\frac{r-a}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'(r+(a-r)t)dt.$$

2. Main Results

Theorem 5. *Let $f:[a,b] \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a<r \le b$ such that $f' \in L[a,b]$. If $|f'|$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, then the following inequality for fractional integrals holds*

$$\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(r)+J_{r^-}^{\alpha}f(a)\right]\right|$$
$$\le \frac{r-a}{2}\left[\begin{array}{l}\beta\left(\frac{1}{2},s+1,\alpha+1\right)-\beta\left(\frac{1}{2},\alpha+1,s+1\right)\\ +\dfrac{2^{\alpha+s}-1}{(\alpha+s+1)2^{\alpha+s}}\end{array}\right]$$
$$\times\left[|f'(a)|+|f'(r)|\right].$$

Proof. From Lemma 1 and using the properties of modulus, we get

$$\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(r)+J_{r^-}^{\alpha}f(a)\right]\right|$$
$$\leq\frac{r-a}{2}\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(r+(a-r)t)\right|dt.$$

Since $|f'|$ is s-convex on $[a,b]$, we obtain the inequality

$$\left|f'(r+(a-r)t)\right|=\left|f'(ta-(1-t)r)\right|$$
$$\leq t^s\left|f'(a)\right|+(1-t)^s\left|f'(r)\right|,\quad t\in(0,1).$$

Hence,

$$\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(r)+J_{r^-}^{\alpha}f(a)\right]\right|$$
$$\leq\frac{r-a}{2}\left\{\int_0^{\frac{1}{2}}[(1-t)^{\alpha}-t^{\alpha}][t^s\left|f'(a)\right|+(1-t)^s\left|f'(r)\right|]dt\right.$$
$$\left.+\int_0^{\frac{1}{2}}[t^{\alpha}-(1-t)^{\alpha}][t^s\left|f'(a)\right|+(1-t)^s\left|f'(r)\right|]dt\right\}$$

and

$$\int_0^{\frac{1}{2}}t^s(1-t)^{\alpha}dt=\int_{\frac{1}{2}}^1(1-t)^s t^{\alpha}dt=\beta\left(\frac{1}{2};s+1,\alpha+1\right),$$

$$\int_0^{\frac{1}{2}}(1-t)^s t^{\alpha}dt=\int_{\frac{1}{2}}^1 t^s(1-t)^{\alpha}dt=\beta\left(\frac{1}{2};\alpha+1,s+1\right),$$

$$\int_0^{\frac{1}{2}}t^{s+\alpha}dt=\int_{\frac{1}{2}}^1(1-t)^{s+\alpha}dt=\frac{1}{2^{s+\alpha+1}(s+\alpha+1)},$$

$$\int_0^{\frac{1}{2}}(1-t)^{s+\alpha}dt=\int_{\frac{1}{2}}^1 t^{s+\alpha}dt$$
$$=\frac{1}{s+\alpha+1}-\frac{1}{2^{s+\alpha+1}(s+\alpha+1)}.$$

We obtain

$$\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(r)+J_{r^-}^{\alpha}f(a)\right]\right|$$
$$\leq\frac{r-a}{2}\left[\begin{array}{l}\beta\left(\frac{1}{2},s+1,\alpha+1\right)-\beta\left(\frac{1}{2},\alpha+1,s+1\right)\\+\frac{2^{\alpha+s}-1}{(\alpha+s+1)2^{\alpha+s}}\end{array}\right]$$
$$\times\left[\left|f'(a)\right|+\left|f'(r)\right|\right].$$

Theorem 6. *Let* $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b\leq b$ *such that* $f'\in L[a,b]$. *If* $|f'|^q$ *is s-convex in the second sense on* $[a,b]$ *for some fixed* $s\in(0,1]$ *and* $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality for fractional integrals holds*

$$\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(r)+J_{r^-}^{\alpha}f(a)\right]\right|$$
$$\leq\frac{r-a}{2}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+\left|f'(r)\right|^q}{s+1}\right)^{\frac{1}{q}}$$

where $\alpha\in[0,1]$.

Proof. By Lemma 1 and using Hölder inequality with the properties of modulus, we have

$$\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(r)+J_{r^-}^{\alpha}f(a)\right]\right|$$
$$\leq\frac{r-a}{2}\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(r+(a-r)t)\right|dt$$
$$\leq\frac{r-a}{2}\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(r+(a-r)t)\right|^q dt\right)^{\frac{1}{q}}.$$

We know that for $\alpha\in[0,1]$ and $\forall t_1,t_2\in[0,1]$,

$$\left|t_1^{\alpha}-t_2^{\alpha}\right|\leq\left|t_1-t_2\right|^{\alpha},$$

therefore

$$\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|dt\leq\int_0^1\left|1-2t\right|^{\alpha p}dt$$
$$=\int_0^{\frac{1}{2}}[1-2t]^{\alpha p}dt+\int_{\frac{1}{2}}^1[2t-1]^{\alpha p}dt$$
$$=\frac{1}{\alpha p+1}.$$

Since $|f'|^q$ is s-convex on $[a,b]$, we get

$$\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(r)+J_{r^-}^{\alpha}f(a)\right]\right|$$
$$\leq\frac{r-a}{2}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left[t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(r)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$=\frac{r-a}{2}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+\left|f'(r)\right|^q}{s+1}\right)^{\frac{1}{q}}$$

which completes the proof.

Corollary 1. *If in Theorem 6, we choose r = b, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right| \le \frac{b-a}{2}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}. \tag{2.1}$$

Remark 1. *If we choose* $\alpha=1$ *ve* $s=1$ *in Corollary 6, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p}{p-1}}$$

which is the inequality in (1.3).

Theorem 7. *Let* $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<r\le b$ *such that* $f'\in L[a,b]$. *If* $|f'|^q$ *is s-convex in the second sense on* $[a,b]$ *for some fixed* $s\in(0,1]$ *and* $q\ge 1$, *then the following inequality for fractional integrals holds*

$$\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(r)+J_{r^-}^{\alpha}f(a)\right]\right| \le \frac{r-a}{2}\left[\frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\right]^{1-\frac{1}{q}}$$
$$\times\left[\beta\left(\frac{1}{2},s+1,\alpha+1\right)-\beta\left(\frac{1}{2},\alpha+1,s+1\right)+\frac{2^{\alpha+s}-1}{(\alpha+s+1)2^{\alpha+s}}\right]^{\frac{1}{q}}\times\left(|f'(a)|^q+|f'(r)|^q\right)^{\frac{1}{q}}.$$

Proof. From Lemma 1 and using the well-known power mean inequality with the properties of modulus, we have

$$\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(r)+J_{r^-}^{\alpha}f(a)\right]\right| \le \frac{r-\alpha}{2}\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(r+(a-r)t)\right|dt$$
$$\le \frac{r-\alpha}{2}\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|dt\right)^{1-\frac{1}{q}}\times\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(r+(a-r)t)\right|^q dt\right)^{\frac{1}{q}}.$$

On the other hand, we have

$$\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|dt = \int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]dt+\int_{\frac{1}{2}}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]dt = \frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right).$$

Since $|f'|^q$ is *s*-convex on $[a,b]$, we obtain

$$\left|f'(r+(a-r)t)\right|^q=\left|f'(ta+(1-t)t)\right|^q \le t^s|f'(a)|^q+(1-t)^s|f'(r)|^q,\ t\in(0,1)$$

and

$$\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(r+(a-r)t)\right|^q dt \le \int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left[t^s|f'(a)|^q+(1-t)^s|f'(r)|^q\right]dt$$
$$=\int_0^{\frac{1}{2}}\left|(1-t)^{\alpha}-t^{\alpha}\right|\left[t^s|f'(a)|^q+(1-t)^s|f'(r)|^q\right]dt+\int_{\frac{1}{2}}^1\left|t^{\alpha}-(1-t)^{\alpha}\right|\left[t^s|f'(a)|^q+(1-t)^s|f'(r)|^q\right]dt.$$

Since

$$\int_0^{\frac{1}{2}}t^s(1-t)^{\alpha}dt=\int_{\frac{1}{2}}^1(1-t)^s t^{\alpha}dt=\beta\left(\frac{1}{2};s+1,\alpha+1\right),$$

$$\int_0^{\frac{1}{2}}(1-t)^s t^{\alpha}dt=\int_{\frac{1}{2}}^1 t^s(1-t)^{\alpha}dt=\beta\left(\frac{1}{2};\alpha+1,s+1\right),$$

$$\int_0^{\frac{1}{2}}t^{s+\alpha}dt=\int_{\frac{1}{2}}^1(1-t)^{s+\alpha}dt=\frac{1}{2^{s+\alpha+1}(s+\alpha+1)}$$

and

$$\int_0^{\frac{1}{2}}(1-t)^{s+\alpha}dt=\int_{\frac{1}{2}}^1 t^{s+\alpha}dt = \frac{1}{s+\alpha+1}-\frac{1}{2^{s+\alpha+1}(s+\alpha+1)}.$$

Therefore, we have

$$\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(r)+J_{r^-}^{\alpha}f(a)\right]\right| \le \frac{r-a}{2}\left[\frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\right]^{1-\frac{1}{q}}$$
$$\times\left\{\left[\beta\left(\frac{1}{2},s+1,\alpha+1\right)-\beta\left(\frac{1}{2},\alpha+1,s+1\right)-\frac{2^{\alpha+s}-1}{(\alpha+s+1)2^{\alpha+s}}\right]\left(|f'(a)|^q+|f'(r)|^q\right)\right\}^{\frac{1}{q}}.$$

Remark 2. *If we choose r = b in Theorem 7, we obtain the inequality in (1.4) of Theorem 4.*

Remark 3. *If we choose r = b and α = 1 in Theorem 7, we obtain the inequality in (1.2) of Theorem 2.*

Theorem 8. *Let* $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<r\le b$ *such that* $f'\in L[a,b]$. *If* $|f'|^q$ *is s-concave in the second sense on* $[a,b]$ *and* $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality for fractional integrals holds*

$$\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^\alpha}\left[J_{a^+}^\alpha f(r)+J_{r^-}^\alpha f(a)\right]\right|$$

$$\le\frac{r-a}{2^{\frac{2-s}{q}}}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{q}}\left|f'\left(\frac{a+r}{2}\right)\right|.$$

Proof. From Lemma 1 and using Hölder inequality, we have

$$\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^\alpha}\left[J_{a^+}^\alpha f(r)+J_{r^-}^\alpha f(a)\right]\right|$$

$$\le\frac{r-a}{2}\int_0^1\left|(1-t)^\alpha-t^\alpha\right|\left|f'(r+(a-r)t)\right|dt$$

$$\le\frac{r-a}{2}\left(\int_0^1\left|(1-t)^\alpha-t^\alpha\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(r+(a-r)t)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is *s*-concave on $[a,b]$, we get

$$\int_0^1\left|f'(r+(a-r)t)\right|^q dt\le 2^{-1}\left|f'\left(\frac{a+r}{2}\right)\right|^q,$$

so

$$\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^\alpha}\left[J_{a^+}^\alpha f(r)+J_{r^-}^\alpha f(a)\right]\right|$$

$$\le\frac{r-a}{2^{\frac{2-s}{q}}}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left|f'\left(\frac{a+r}{2}\right)\right|.$$

which completes the proof.

Corollary 2. *If we choose r = b in Theorem 8, we obtain*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]\right|$$

$$\le\frac{b-a}{2^{\frac{2-s}{q}}}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left|f'\left(\frac{a+b}{2}\right)\right|.$$

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