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A New Padé Approximant for the Appell Hypergeometric Function F1

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Abstract In this work, we present a simple method for computing the first Appell function $F_1(a,b,b';c;x,y)$, in some particular case. We use a new definition of the general multivariate Padé approximant which allows us to get the explicit expression of the denominator polynomial. Our approach seems to give a better precision than the Taylor's expansion, especially near the border of the convergence area.

Keywords: *hypergeometric functions, multivariate approximation.*

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1. Introduction

The four Appell functions [10] come from the natural extension of the well-known Gauss hypergeometric function $2F_1$ from one to two variables. At the same time, Appell functions are special cases of more general hypergeometric functions [11]. They have various applications in many branches of mathematics and physics, especially in quantum mechanics and field theory [8]. For example, as proved by Kniehl [2], they appear in the evaluation of some Feynman's integrals. These functions have many representations. Schlosser [9], in his survey, explains how they are related to some Euler's elliptic and double integrals, and solutions of some partial differential equations. In what follows, we will confine our work for the first Appell function F_1 in a particular case $(b'=1)$ and use the power series expansion as a starting definition. A truncated part of this Taylor's expansion will be taken as the standard estimation of $F_1(a,b,b';c;x,y)$. We will compare it with the value given by the Padé approximant defined by Baker [1].

2. Notations

For complex values of the parameters a,b,c and the variables x and y, the function we investigate is given by

$$F_1(a,b,1;c;x,y)=\sum_{i,j=0}^{+\infty} C_{ij}x^i y^j$$

where

$$C_{ij}:=\frac{(a)_{i+j}(b)_i}{(c)_{i+j}(1)_i},$$

and $(a)_i$ is the Pochhammer symbol defined by $(a)_i=0$ if $i<0$, $(a)_i=1$ if $i=0$ and $(a)_i=a(a+1)(a+2)...(a+i-1)$ if $i>0$. It is known that $\{(x,y)\in\mathbb{C}:|x|<1\, and\, |y|<1\}$ is the convergence region of this double series.

Here and in the following, let $\mathbb{C}$ and $\mathbb{N}$ be the sets of complex numbers and positive integers, respectively, and let $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. We generalize so, the results on $F_1(a,1,1;a+1;x,y)$ considered by Borwein [7], the case $F_1(1,1,1;2;x,y)$ examined by Cuyt [6], and those on some pseudo-multivariate functions investigated by Zhou [5]. To compute $F_1(a,b,1;c;x,y)$, we will consider the multivariate Padé approximation and compare it with the estimation given by Taylor's expansion, truncated at a certain order.

We introduce some basics, in order to define the multivariate Padé approximant.

For every $m,n\in\mathbb{N}_0$, we begin by construct the following subsets of $\mathbb{N}_0^2$:

$$D_m=\{0,1,2,...,m\},\, N_n=\{0,1,2,...,n\}$$
$$and\ E_{n+m}=\{0,1,2,...,n+m\}$$

as illustrated by Figure 1. The elements of these sets are classified with respect to a triangular numbering, since it is compatible with Taylor's expansion and allows us to express the error introduced by the truncation. So, each point (i,j) of $\mathbb{N}_0^2$ will take the rank given by $r_{i,j}=\frac{(i+j)(i+j+1)}{2}+j$. In the opposite sense, the r^{th} element will have the following coordinates:

$$j_r=r-\frac{d_r(d_r+1)}{2}$$

and

$$i_r = d_r - j_r$$

where d_r is the diagonal containing this point, given by

$$d_r = \left\lfloor \frac{-1+\sqrt{1+8r}}{2} \right\rfloor.$$

Our purpose is to find two polynomials

$$P_n(x,y) = \sum_{(i,j)\in N_n} p_{ij} x^i y^j \text{ and } Q_m(x,y) = \sum_{(\lambda,\mu)\in D_m} q_{\lambda\mu} x^\lambda y^\mu$$

which satisfy the following equation lattice adopted by Cuyt [4]:

$$P_n(x,y) - Q_m(x,y).F_1(a,b,1;c;x,y) = \sum_{(i,j)\in \mathbb{N}_0^2 \setminus E_{n+m}} d_{i,j} x^i y^j.$$

To determine $Q_m(x,y)$, one must resolve the homogeneous system

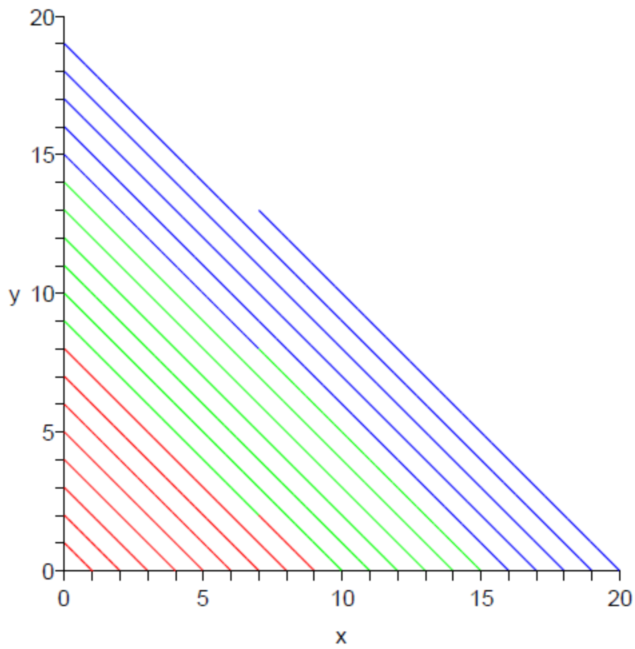

Figure 1. Sets defined by triangular numbering: $D_m = Red$; $N_n = R + G$; $E_{n+m} = R + G + B$

$$\sum_{(\lambda,\mu)\in D_m} C_{i-\lambda,j-\mu} q_{\lambda\mu} = 0$$

where (i,j) is a point of $E_{n+m} \setminus N_n$.

The coefficients of the numerator polynomial are then obtained using

$$p_{i,j} = \sum_{(\lambda,\mu)\in D_m} C_{i-\lambda,j-\mu} q_{\lambda\mu}$$

for each $(i,j)\in N_n$. The general multivariate Padé approximant will be the rational fraction

$$\frac{P_n(x,y)}{Q_m(x,y)}.$$

The fact that E_{n+m} satisfies the inclusion property, together with the imposed condition $Q_m(0,0)=1$, take care of the Padé approximation property, namely

$$Q_m(0,0)=1 \Rightarrow \left(F_1(a,b,1;c;x,y) - \frac{P_n(x,y)}{Q_m(x,y)} \right) = \sum_{\mathbb{N}_0^2 \setminus E_{n+m}} \tilde{d}_{ij} x^i y^j.$$

3. A New General Padé Approximant

To make the resolution of the homogeneous system above easier, we will define a new equation lattice with the following sets represented in Figure 2, in a similar way as in Golub [3].

$$D_{m,d} = \left\{ (\lambda,\mu) \in \mathbb{N}_0^2 \mid 0 \le \lambda \le J_1 \text{ and } 0 \le \mu \le J_2 \right\},$$

$$N_{n,d} = \left\{ (i,j) \in \mathbb{N}_0^2 \mid 0 \le i+j \le d-1 \right\},$$

and

$$E_d = \left\{ (i,j) \in \mathbb{N}_0^2 \mid 0 \le i+j \le d+\theta-1 \right\},$$

where $d = \left\lfloor \frac{-1+\sqrt{(9+8n)}}{2} \right\rfloor$ is the diagonal which contains the $(n+1)^{th}$ of E_d and θ is the number of diagonals of $E_d \setminus N_{n,d}$, also given by

$$\theta = \left\lfloor \frac{-1+\sqrt{(1+8(n+m))}}{2} \right\rfloor - \left\lfloor \frac{-1+\sqrt{(9+8n)}}{2} \right\rfloor + 1.$$

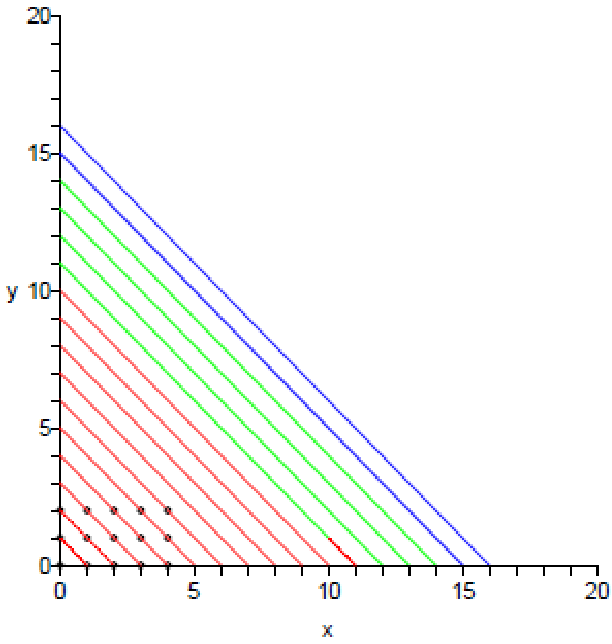

Figure 2. A solution with rectangular $D_{m,d}$

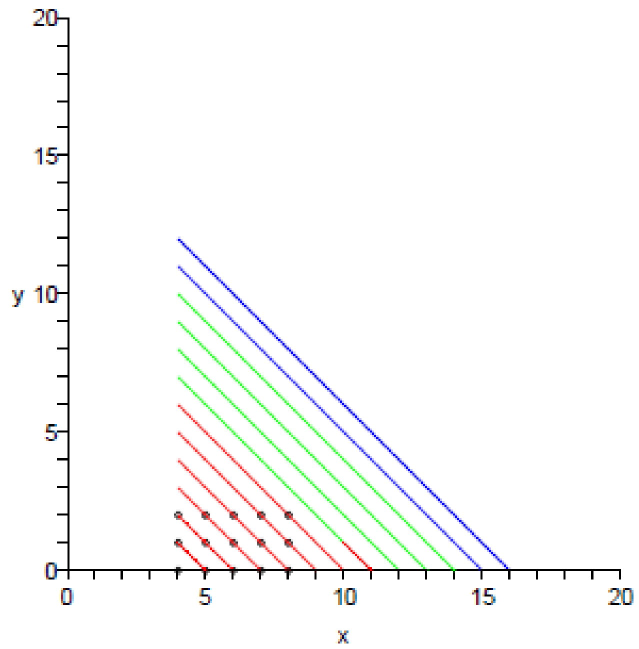

Figure 3. A solution with shifted $D_{m,d}$

Here we assume here that $m \geq 23$ such that the new set $D_{m,d}$ is inside D_m. We could already remark that for enough great values of n, θ takes only two values 1 or 2. The new problem is to find the parameters J_1 and J_2 which define the rectangular set D_m, by solving the equation:

$$(1+J_1)(1+J_2)-1=\theta(1+J_1)+\theta J_2,$$

which implies that the number of unknowns in the homogeneous system is exactly the number of independent equations

The new equation lattice to be satisfied is

$$P_{n,d}(x,y)-Q_{m,d}(x,y).F_1(a,b,1;c;x,y) = \sum_{(i,j)\in\mathbb{N}_0^2\setminus E_d} h_{i,j}x^i y^j.$$

Here, we have a simple solution of the corresponding homogeneous system, for relatively great values of *n*.

4. Asymptotic Case for Great Enough n

For *m* fixed, the precision of the approximation is better for great values of *n*. For instance, if $n \geq \frac{m(m-1)}{2}-1$, $E_d \setminus N_{n,d}$ will be contained in at most the two diagonals defined by $\left\{(i,j)\in\mathbb{N}_0^2 \mid d \leq i+j \leq d+1\right\}$. In what follows, we assume, without losing generality, that $\theta = 2$. With this hypothesis, we find $J_1 = 4$, $J_2 = 2$ and $D_m = \left\{(\lambda,\mu)\in\mathbb{N}_0^2 \mid 0 \leq \lambda \leq 4 \; and \; 0 \leq \mu \leq 2\right\}$. A solution of the homogeneous system, according to the new equation lattice is given by:

$$q_{\lambda\mu} = (-1)^{\lambda+\mu}\binom{4}{\lambda}\binom{2}{\mu}\frac{(a+d-\lambda-\mu)_{\lambda+\mu}(b+d-\lambda-1)_{\lambda}}{(c+1+d-\lambda-\mu)_{\lambda+\mu}(3+d-\lambda-1)_{\lambda}},$$

which defines the explicit expression of the denominator polynomial by

$$Q_{m,d}(x,y) = \sum_{(\lambda,\mu)\in D_{m,d}} q_{\lambda\mu}x^{\lambda}y^{\mu}$$

and the corresponding numerator polynomial $P_{n,d}(x,y)$ over $N_{n,d}$.

Let us remark that if $a = c+1$ and $b = 3$, all the coefficients $q_{\lambda\mu}$ are constant, which means that $F_1(a,3,1;a-1;x,y)$ is a rational fraction.

5. A Solution with Shifted Denominator

In this section, we will try to improve the results presented in previous sections, by reducing the number of operations we need to compute the coefficients of the numerator polynomial $P_{n,d}(x,y)$. For this, define the partial function $f_\delta(x,y)$ by: for each integer $\delta \geq 0$, $f_\delta(x,y) = \sum_{i+j=\delta} C_{ij}x^i y^i$ thus, we have $F_1(a,b,1;c;x,y) = \sum_{\delta\in\mathbb{N}_0} f_\delta(x,y)$.

Firstly, examine the error introduced by the truncation of Taylor's expansion.

$$F_1(a,b,1;c;x,y) = \sum_{\delta=0}^{d+1} f_\delta(x,y) + \sum_{\delta\geq d+2} f_\delta(x,y)$$

with

$$\text{error} = \sum_{\delta\geq d+2} f_\delta(x,y) = \frac{1}{(d+2)!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^{d+2} F_1\big|_{(\alpha x,\alpha y)}$$

α is a real such that the point $(\alpha x, \alpha y)$ is on the line segment joining $(0,0)$ and (x,y).

Thus, it is not easy to evaluate the error of truncation, since we have no idea about the real α.

We now try to express the error introduced by the Padé approximation.

$$Q_{m,d}(x,y)F_1(a,b,1;c;x,y) = Q_m(x,y)\sum_{\delta=0}^{d-7} f_\delta(x+y)$$

$$+\begin{pmatrix} q_{00}+xq_{10}+yq_{01}+x^2q_{20}+xyq_{11}+y^2q_{02} \\ +x^3q_{30}+x^2yq_{21}+xy^2q_{12}+x^4q_{40}+x^3yq_{31} \\ +x^2y^2q_{22}+x^4yq_{41}+x^3y^2q_{32} \end{pmatrix} f_{d-6}(x,y)$$

$$+\begin{pmatrix} q_{00}+xq_{10}+yq_{01}+x^2q_{20}+xyq_{11} \\ +y^2q_{02}+x^3q_{30}+x^2yq_{21}+xy^2q_{12} \\ +x^4q_{40}+x^3yq_{31}+x^2y^2q_{22} \end{pmatrix} f_{d-5}(x,y)$$

$$+\begin{pmatrix} q_{00}+xq_{10}+yq_{01}+x^2q_{20}+xyq_{11}+y^2q_{02} \\ +x^3q_{30}+x^2yq_{21}+xy^2q_{12} \end{pmatrix} f_{d-4}(x,y)$$

$$+\left(q_{00}+xq_{10}+yq_{01}+x^2q_{20}+xyq_{11}+y^2q_{02}\right) f_{d-3}(x,y)$$

$$+\left(q_{00}+xq_{10}+yq_{01}\right) f_{d-2}(x,y)+\left(q_{00}\right) f_{d-1}(x,y)$$

$$+\begin{pmatrix} xq_{10}+yq_{01}+x^2q_{20}+xyq_{11}+y^2q_{02}+x^3q_{30} \\ +x^2yq_{21}+xy^2q_{12}+x^4q_{40}+x^3yq_{31} \\ +x^2y^2q_{22}+x^4yq_{41}+x^3y^2q_{32}+x^4y^2q_{42} \end{pmatrix} f_{d+1}(x,y)$$

$$+\begin{pmatrix} x^2q_{20}+xyq_{11}+y^2q_{02}+x^3q_{30} \\ +x^2yq_{21}+xy^2q_{12}+x^4q_{40}+x^3yq_{31} \\ +x^2y^2q_{22}+x^4yq_{41}+x^3y^2q_{32}+x^4y^2q_{42} \end{pmatrix} f_{d}(x,y)$$

$$+\begin{pmatrix} x^3q_{30}+x^2yq_{21}+xy^2q_{12}+x^4q_{40}+x^3yq_{31} \\ +x^2y^2q_{22}+x^4yq_{41}+x^3y^2q_{32}+x^4y^2q_{42} \end{pmatrix} f_{d-1}(x,y)$$

$$+\begin{pmatrix} x^4q_{40}+x^3yq_{31}+x^2y^2q_{22}+x^4yq_{41} \\ +x^3y^2q_{32}+x^4y^2q_{42} \end{pmatrix} f_{d-2}(x,y)$$

$$+\left(x^4yq_{41}+x^3y^2q_{32}+x^4y^2q_{42}\right) f_{d-3}(x,y)$$

$$+\left(x^4y^2q_{42}\right) f_{d-4}(x,y)+Q_{m,d}(x,y)\sum_{\delta\geq d+2} f_\delta(x,y),$$

which can be written as follows:

$$Q_{m,d}(x,y)F_1(a,b,1;c;x,y)=P_{n,d}(x,y)$$

$$+\begin{pmatrix} xq_{10}+yq_{01}+x^2q_{20}+xyq_{11}+y^2q_{02}+x^3q_{30} \\ +x^2yq_{21}+xy^2q_{12}+x^4q_{40}+x^3yq_{31} \\ +x^2y^2q_{22}+x^4yq_{41}+x^3y^2q_{32}+x^4y^2q_{42} \end{pmatrix} f_{d+1}(x,y)$$

$$+\begin{pmatrix} x^2q_{20}+xyq_{11}+y^2q_{02}+x^3q_{30} \\ +x^2yq_{21}+xy^2q_{12}+x^4q_{40}+x^3yq_{31} \\ +x^2y^2q_{22}+x^4yq_{41}+x^3y^2q_{32}+x^4y^2q_{42} \end{pmatrix} f_{d}(x,y)$$

$$+\begin{pmatrix} x^3q_{30}+x^2yq_{21}+xy^2q_{12}+x^4q_{40}+x^3yq_{31} \\ +x^2y^2q_{22}+x^4yq_{41}+x^3y^2q_{32}+x^4y^2q_{42} \end{pmatrix} f_{d-1}(x,y)$$

$$+\begin{pmatrix} x^4q_{40}+x^3yq_{31}+x^2y^2q_{22}+x^4yq_{41} \\ +x^3y^2q_{32}+x^4y^2q_{42} \end{pmatrix} f_{d-2}(x,y)$$

$$+\left(x^4yq_{41}+x^3y^2q_{32}+x^4y^2q_{42}\right) f_{d-3}(x,y)$$

$$+\left(x^4y^2q_{42}\right) f_{d-4}(x,y)+Q_{m,d}(x,y)\sum_{\delta\geq d+2} f_\delta(x,y),$$

and finally,

$$Q_{m,d}(x,y)F_1(a,b,1;c;x,y)$$

$$=P_{n,d}(x,y)+\sum_{i+j\geq d+2}^{d+7} g_{ij}x^iy^i+Q_{m,d}(x,y)\sum_{\delta\geq d+2} f_\delta(x,y).$$

To reduce the time and number of operations we need for the computation, we will shift the rectangular set $D_{m,d}$ to the right side as far as possible (see Figure 3).

We write now the equation lattice just above after replacing d by $d-J+7$, where J is the diagonal containing the last point of D_m given by $J=\left\lfloor\frac{-1+\sqrt{1+8m}}{2}\right\rfloor$.

$$Q_{m,d-J+7}(x,y)F_1(a,b,1;c;x,y)$$

$$=P_{n,d-J+7}(x,y)+\sum_{i+j\geq d-J+9} h_{ij}x^iy^j.$$

But, if we multiply by x^{J-7} and translate the shift of x^{J-7}, we get

$$x^{J-7}Q_{m,d-J+7}(x,y)F_1(a,b,1;c;x,y)$$

$$=x^{J-7}P_{n,d-J+7}(x,y)+\sum_{i+j\geq d+2} \tilde{h}_{ij}x^iy^j,$$

which defines a new Padé approximant with respect to the equation lattice above expressed by

$$\frac{x^{J-7}P_{n,d-J+7}(x,y)}{x^{J-7}Q_{m,d-J+7}(x,y)}=\frac{P_{n,d-J+7}(x,y)}{Q_{m,d-J+7}(x,y)}.$$

This result shows that it is not necessary to compute the numerator polynomial over $N_{n,d}$, but only over the smaller set $N_{n,d-J+7}$. In the end, notice that this Padé approximant, together with Taylor's estimation, satisfies the same accuracy-through-order condition. We believe, our approach is more accurate than Taylor's development, since it needs less calculations and avoids errors introduced by the computer.

To compare these two approximations, we took the case of $F_1(1,1,1;2;x,y)$ for which we have the exact expression, namely $\frac{\ln(1-y)-\ln(1-x)}{x-y}$. The results are illustrated by Figure 4.

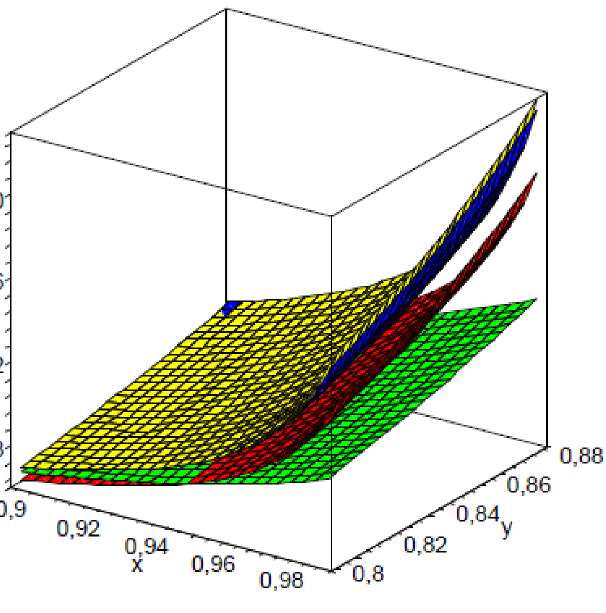

Figure 4. Padé (Blue and Red), Taylor (Green) and exact (Yellow)

6. Concluding Remarks

The approach considered here has an advantage of being simple. It seems to give better results than the classical Taylor's expansion as we can see in Figure 4. However, some difficulties occur in the evaluation of the error . To improve the precision of our method, one could consider the cases where the set E_d is contained in more than two diagonals $(\theta > 2)$, but a difficulty arises due to the fact that the homogeneous system to solve is bigger and contains more unknowns. Also, the solution can't be expressed with a simple formula like the one found with $\theta = 2$, presented in section 4 as an asymptotic case. To reduce the computation-time and the number of operations needed, we have shifted the denominator set, $D_{m,d}$ to the right side. There is also the possibility of shifting upward, along the $\overrightarrow{Oy}$ axis. There could be an optimal position of $D_{m,d}$ to find.

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